Problem 1

Physics Cup

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Consider the velocity field $\vec{v}(\vec{r},t)$ defined in the problem specifications at time t=0 with $\vec{v}$ inside the object defined as the velocity of the object v_0.
Since the following equations are true at any point in the vector field outside the object
The liquid is incompressible:

$$\nabla \cdot \vec{v} = 0 \tag{1}$$

The liquid is vortex-free:

$$\nabla \times \vec{v} = 0 \tag{2}$$

and since the component of the $\vec{v}$-field on the surface of the object orthogonal to the surface of the object is equal to the orthogonal component of the object velocity (a boundary condition for the field that arises from the fact that water cannot flow into the object) we can define an electric displacement field $\vec{D} = \sqrt{\epsilon\rho}\vec{v}$ with zero bound free density at all points and $\epsilon = \epsilon_0$ at any point outside the object (we will find ϵ inside the object later). We also define a corresponding $\vec{E}$-field as $\vec{E} = \frac{\vec{D}}{\epsilon}$.

The reason we define $\vec{D}$ in this way is because the energy densities u_v and u_D of the D and v fields are calculated by

$$u_v = \frac{1}{2}\rho\vec{v} \cdot \vec{v} \tag{3}$$

$$u_D = \frac{1}{2\epsilon}\vec{D} \cdot \vec{D} \tag{4}$$

Thus by defining $\vec{D}$ in this way the energy in the v-field for some region of space R will be the same as the energy in the D-field in the same region R
Inside the object we have $\vec{D}_{in} = -D_0\hat{x}$ and infinitely far away from the object we have $\vec{D}^{\infty}_{out} = 0$

We will now add a vector $\vec{D'} = D'\hat{x}$ to the field at every point (this corresponds to a change of inertial frame.
Thus:

$$D_{in} = D' - D_0 \tag{5}$$

$$D^{\infty}_{out} = D' \tag{6}$$

Because of the second condition in the problem statement this is equivalent to placing a dielectric of some relative permiability ϵ_r in a homogeneous D-field of value $\vec{D} = D'\hat{x}$. This fact can be proven through uniqueness of solution and the fact that the dielectric in the homogeneous field satisfies the boundary conditions of the D-field.

To calculate ϵ_r we consider the dipole moment density $\vec{p}$ of the dielectric in

the field.
Per definition

$$p = (\epsilon_r - 1)\epsilon_0 E_{in} \tag{7}$$

Where:
E_{in} is the strength of the homogeneous E-field inside the dipole.
Additionally a metallic object of the same shape as the dielectric has

$$p = \frac{\alpha}{V}\Delta E \tag{8}$$

Where:
ΔE is the change in strength of the electric field inside the field due to the charge distribution on the metal surface.

Through uniqueness of solution we find that the dipole moment density of the dielectric will have the same form

$$p = \frac{\alpha}{V}(E' - E_{in}) \tag{9}$$

Furthermore we know that for any E-field

$$\vec{E}(\vec{r}) = \frac{\vec{D}(\vec{r})}{\epsilon_0 \epsilon_r} \tag{10}$$

Combining the results in equations 5, 7, 9, 10 we can derive

$$\epsilon_r = \frac{\frac{\alpha}{V} - \epsilon_0}{\frac{\alpha}{V}D' - \epsilon_0 D_{in}} D_{in} \tag{11}$$

We can now derive a formula for the change of energy in the D-field outside the object due to the dielectric. We have three energy changes to consider due to the adding of the dielectric. ΔU_{in}, the change of energy of the field inside the region covered by the dielectric, ΔU_{out} the change of energy of the field inside the region not covered by the dielectric, and ΔU_p, the change in potential energy of the dipole or rather the change in potential energy due to the new charge distribution inside the dipole. Due to conservation of energy we get

$$\Delta U_{in} + \Delta U_{out} + \Delta U_p = 0 \tag{12}$$

Since $p \propto D'$

$$\Delta U_p = -\frac{1}{2\epsilon_0} p D' V \tag{13}$$

Furthermore ΔU_{in} i can be calculated by multiplying the volume of the object with the energy density of the field in the object before and after adding the dielectric.

$$\Delta U_{in} = \frac{V}{2\epsilon_0}(\frac{D_{in}^2}{\epsilon_r} - D'^2) \tag{14}$$

Using equation 11 and 14 we can calculate ΔU_{in} to be

$$\Delta U_{in} = -\frac{V}{2\epsilon_0(\frac{\alpha}{V} - \epsilon_0)}(\epsilon_0 D_0^2 - (\frac{\alpha}{V} + 2\epsilon_0)D_0 D' - \frac{\alpha}{V}D'^2) \tag{15}$$

Finally we want to find ΔU_{out} when $D' = 0$. Since $D' = 0$ the change of energy in the outside field will also be the total energy of the outside field. At $D' = 0$, ΔU_p and many terms in the expression for ΔU_{in} will be zero. Using equations 10, 12, 13, 15 we get:

$$U_{out} = \Delta U_{out} = \frac{V\epsilon_0 D_0^2}{2\epsilon_0(\frac{\alpha}{V} - \epsilon_0)} \tag{16}$$

Using the definition of $\vec{D}$, $\vec{D} = \sqrt{\epsilon\rho}\vec{v}$, we get

$$E_k = U_{out} = \frac{V\rho\epsilon_0 v_0^2}{2(\frac{\alpha}{V} - \epsilon_0)} \tag{17}$$

Thus, using the definition of added mass $m_{add} = \frac{2E_k}{v^2}$ we can derive our final result

$$m_{add} = \frac{\epsilon_0 V\rho}{(\frac{\alpha}{V} - \epsilon_0)} \tag{18}$$