Solution of Physics Cup 2019, Problem No 1

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As it has been mentioned in the problem, a body moving in liquid will cause the liquid to be moving too. This increases the total kinetic energy of the system, and generally it will be difficult to be calculated with the present of vortices and for arbitrary shape of the body. The problem, however, asked the *added mass* for an object having special symmetry and at a time when vortices have not emerged.

Because initially there is no vortex, the velocity fulfills $\oint \vec{v}.d\vec{r} = 0$. This has the same property as electric field in electrostatic case. The kinetic energy density for liquid moving with velocity $\vec{v}$ is $k = \frac{\rho}{2}v^2$, which also has a similarity with the energy density of electric field, $u = \frac{\epsilon}{2}E^2$. Therefore, there is a parallel relationship between the two vectors. We also need to realize that the polarizability is a constant determined only by the geometrical shape of the body.

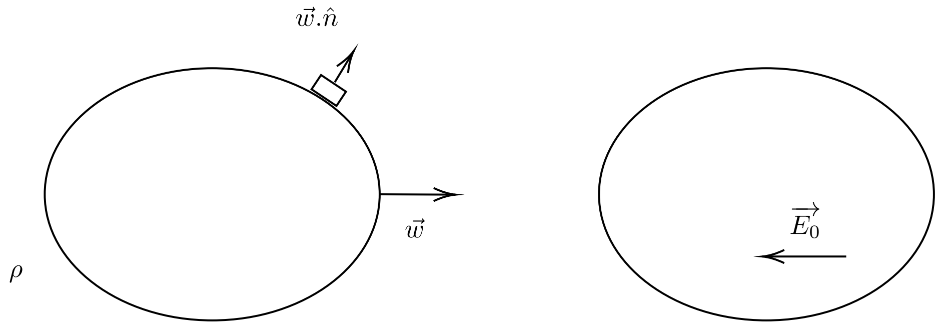

Figure 1: Left figure shows the system at which the body is moving in the liquid with velocity $\vec{w}$, while the right figure shows the analogy of the system in electrostatic case. Note that the direction of $\vec{E_0}$ must be the opposite of $\vec{w}$ so that the induced electric field outside will have the same direction as the movement of the liquid.

To solve the problem let us think the analogy of the system in electrostatic case. As initially the liquid is not moving, we can assume an empty space with permeability ϵ_0 for the analogy. When the body starts to move, the liquid will start to move as well. The added kinetic energy of the system is clearly, $\Delta K = \frac{\rho}{2}\int v^2\,dV$, where $\vec{v}$ is the induced velocity of the liquid due to the movement of the body and the integration is for volume outside the body. Suppose that the body is moving with velocity $\vec{w}$ to the x-direction, then the velocity of the liquid perpendicular to the body must be $\vec{w}.\hat{n}$, where $\hat{n}$ is the unit vector pointing out of the body and perpendicular to its surface. This must be true because we assume the liquid to be incompressible. Another thing which is important to be considered is that the liquid has zero velocity far from the body, as the influence of the movement of the body is negligible there. Now, let us consider a certain charge distribution such that there is uniform electric field $\vec{E_0}$ inside volume V of the body. The total dipole moment is then $\vec{p} = -\alpha\vec{E_0}$. The minus sign is because if we have a metallic body with the same shape and we put it in a uniform external electric field $\vec{E}$ pointing to the x-axis, then the electrons will move such that the induced electric field inside the body is $-\vec{E}$, and the total electric field becomes zero. Because of the symmetry of the shape of the body, we can also take it as there is a uniform distribution of dipole moment throughout volume V, with dipole moment density $-\alpha\vec{E_0}/V$. Therefore, the charge per unit area on the surface of the body is $\sigma = -\frac{\alpha}{V}\vec{E_0}.\hat{n}$ (we can think it as if the body is made of uniform dielectric material and there is uniform polarization, this is possible because the symmetry of the body). The perpendicular component of the electric field just outside the body can be easily obtained using Gauss' Law. The perpendicular field inside the body is $\vec{E_0}.\hat{n}$, so using Gauss' Law, the perpendicular field outside the body is $E_\perp = \vec{E_0}.\hat{n} + \sigma/\epsilon_0 = \vec{E_0}.\hat{n}\,(1 - \alpha/\epsilon_0 V)$. We can see now that both systems have similarities. The two vectors ($\vec{E}$ and $\vec{v}$) go to zero far from the body and also are described by the same equations (the line integration through a closed loop outside the body is zero and there is no source, except of course at the boundary where the liquid is force to move). The velocity of the body, $\vec{w}$, is evidently analog to $(1 - \frac{\alpha}{\epsilon_0 V})\vec{E_0}$, because these two give the same perpendicular field.

To calculate the energy of the moving liquid, we can exploit the analog of the two systems. For the electric system, the total energy can be calculated this way: the interaction energy of an infinitesimal dipole moment due to the electric field of the others is $dU = -\vec{E_0}\cdot\frac{\vec{p}}{V}\,dV$. Integrating this equation, we can easily obtain $-\vec{E}_0.\vec{p}$, but this counts the interaction energy of each infinitesimal dipole pair twice, so the total energy must be $U = -\frac{1}{2}\vec{E}_0.\vec{p} = \frac{\alpha}{2}E_0^2$. When we think of the electric field, this energy comes from 2 contributions, the first contribution is from the uniform electric field inside volume V and the second contribution is from the electric field outside. So, we have

$$\frac{\alpha}{2}E_0^2 = \frac{\epsilon_0}{2}E_0^2 V + \frac{\epsilon_0}{2}\int E^2\, dV$$

The energy of the moving liquid can then be calculated using the analog between $\vec{w}$ and $(1 - \frac{\alpha}{\epsilon_0 V})\vec{E}_0$

$$\Delta K = \frac{\rho}{2}\int v^2\, dV = \frac{\rho}{2}\frac{w^2}{(1-\frac{\alpha}{\epsilon_0 V})^2 E_0^2}\int E^2\, dV$$

Substituting the value one will get

$$\Delta K = \frac{w^2}{2}\frac{\rho V}{(\frac{\alpha}{\epsilon_0 V} - 1)}$$

Finally, we can clearly see that the *added mass* is $M_{added} = \rho V/(\alpha/\epsilon_0 V - 1)$.

Another method

There is another slightly different way to obtain the *added mass*. We know that the total kinetic energy of the system can be written as $K = \frac{1}{2}(M_{body} + M_{added})w^2$. The system can be seen, then, as having an effective mass of $M_{body} + M_{added}$. This means that the momentum of the system at the moment when the body starts to move is $\vec{p} = (M_{body} + M_{added})\vec{w}$. Instead of looking for the energy we can, therefore calculate the momentum of the liquid and determine the added mass from this result.

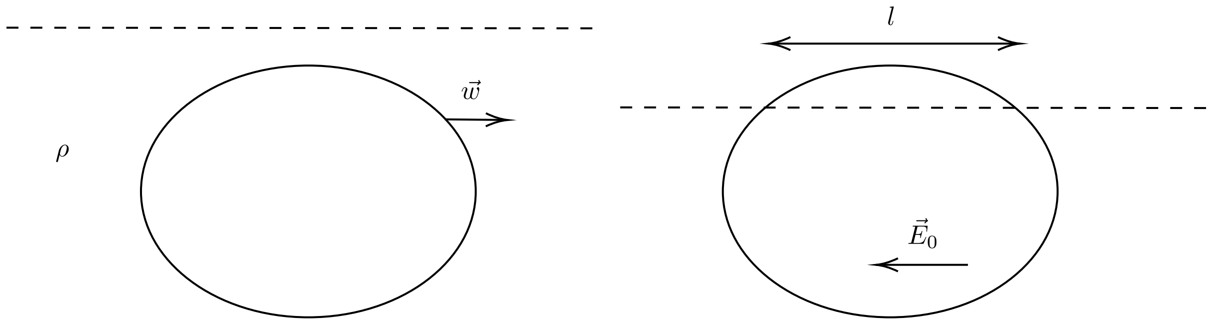

Figure 2: The dotted line is the integration path of the amperian loop. This amperian loop can also be thought as a rectangular loop, but the other 3 sides are very far away thus the contribution of the velocity is effectively zero.

First, let us consider an amperian loop in the shape of an infinitely long straight line parallel to x-axis. If this line does not go through the body (see Figure 2 left), then the integration of v_x along this line must be zero, because $\oint \vec{v}.\, d\vec{r} = 0$. The net momentum inside this part of the liquid is consequently also zero. Now, we use the analogy of the system in electrostatic case. For the line which goes through the body we have, $\int E_x\, dx - E_0 l = \oint \vec{E}.\, d\vec{r} = 0$, where l is the length of the line which is inside the body (See Figure 2 right). Thus, we can see that $\int E_x\, dV = E_0 V$. Using this result and the fact that $\vec{w}$ is analogous to $(1 - \frac{\alpha}{\epsilon_0 V})\vec{E}_0$, we can calculate the total momentum of the liquid

$$\int v_x\, dV = \frac{w}{(\frac{\alpha}{\epsilon_0 V} - 1)E_0}\int E_x\, dV = \frac{w}{(\frac{\alpha}{\epsilon_0 V} - 1)}V$$

In deriving the last equation we need to realize that in order to have the correct direction of the velocity we must have $\vec{E}_0$ pointing to the opposite direction of $\vec{w}$, so w (the magnitude of $\vec{w}$) is analogous to $(\frac{\alpha}{\epsilon_0 V} - 1)E_0$. Finally, the total momentum of the liquid is

$$p_{added} = \rho\int v_x\, dV = \rho\frac{V}{(\frac{\alpha}{\epsilon_0 V} - 1)}w$$

and the *added mass* is $M_{added} = \rho V/(\alpha/\epsilon_0 V - 1)$.