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Problem 2

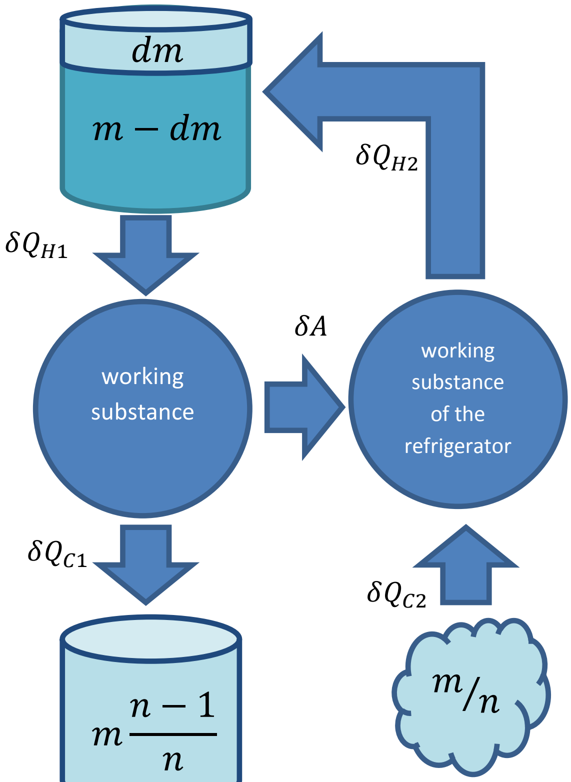

Figure 1.

Scheme of the engine and description

By the text of the question, we were told that our system consists of two masses m of water and ice. We separate small piece of ice of mass $m_2 = {}^{m}/_{n}$, and try to cool it as much as we can with our system. All quantities, which are connected to this small piece of ice we denote by sub-index 2. All quantities, which are connected to another piece of ice of mass $m_1 = m - m_2$we denote by sub-index 1. All quantities, which are connected to water, we denote with no special index.

So, to additionally cool the ice of mass m_2 amount of work under one of the engines, which may be used as cooler, must be done. We can obtain this energy only as a work of another heat engine, which may be used as an "heater". In order to set to work heater engine, we have to connect it to masses of water and ice. By considering all that was mentioned above, we can conclude, that only two schemes of such engine exist and they are shown on Fig. 1 and Fig. 2. Difference between them consists in "heater" source, which is used by each scheme. Heat flows are shown by arrows. In an appendix to the solution, we show that both two cases are equivalent and give the same answer by considering Carnot cycles in each.

In the main part of the solution, we consider the essential equations, initial and final conditions that lead to an answer without consideration of the exact scheme of heat machine.

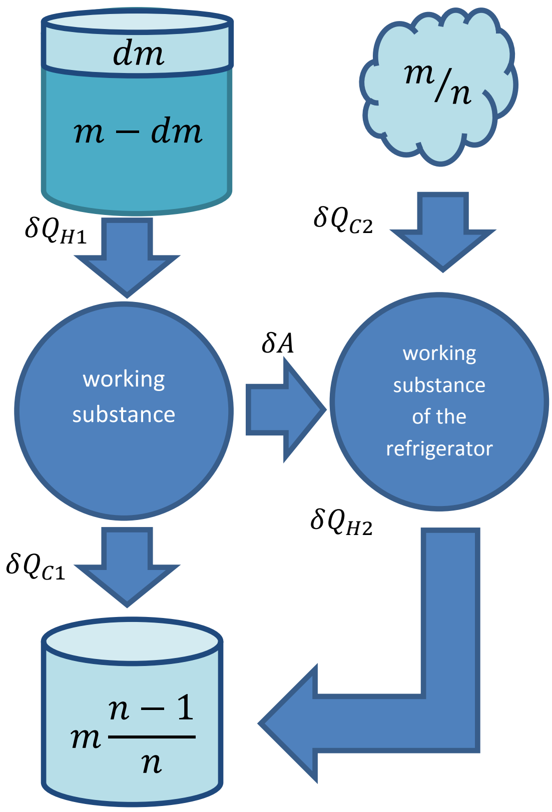

Figure 2.

Initial and final conditions.

Let us consider the initial conditions and possible states of the system when it reaches the thermodynamic equilibrium state. Obviously, the entire system stops if the temperature of the water mass is equal to the temperature of the ice mass, which we use as a cooler in the first stage cycle. Water loses energy when first stage system works. Water freezes and becomes ice because it was initially on the melting point T_0. As a result, we can strictly bound amount of water, which turns into ice after all process has been done. We were told from the text of the problem that $t \sim$ few Kelvines, so let's consider $t < 10\,K$. Energy, which is obtained from freezing part of water is

$$\Delta\tilde{U} = \Delta m\lambda,$$

and energy, that is demanded to heat ice up from $T_0 - t$ to T_0 can be bounded from above as ($\frac{\partial c_V}{\partial T} = \alpha > 0$)

$$\Delta\widetilde{U}' \approx m\, c_V(T_0)t = m\alpha T_0 t.$$

From this we can estimate the part of frozen water:

$$\Delta\widetilde{U}' \approx \Delta\widetilde{U}$$

$$\frac{\Delta m}{m} = \frac{\alpha T_0 t}{\lambda} \sim 10^{-2} \ll 1.$$

Now it is clear that only a small amount of water freezes in a thermodynamic process. Thermodynamic equilibrium is reached when the mass of ice is heated to the melting point and part of the water freezes.

Essential equations

For the solution of the problem, we may need two thermodynamic equations.

Our thermodynamic system consists of ice, water, and ideal reversible heat engines. From this, we can conclude, that system is conservative (inner energy of the system conserves) and reversible (entropy of the system conserves).

The energy conservation law:

$$dU_1 + dU_2 + dU = 0.$$

The entropy conservation law (components of the system, except heat engines with neglectable heat capacity, don't do any work, so $\delta Q = dU$ for all of the mentioned components):

$$\frac{dU_1}{T_1} + \frac{dU_2}{T_2} + \frac{dU}{T} = 0.$$

Inner energy

As water loses energy and freezes, we can express a small amount of energy, that is loosed by water, as

$$dU = -\lambda\, dm,$$

where dm is a mass of the small amount of frozen water.

Energy variation of ice can be written as next (suggested model):

$$dU_1 = m_1 c_V(T_1)\, dT_1 = m_1\alpha\, T_1 dT_1.$$

Identically:

$$dU_2 = m_2 c_V(T_2)\, dT_2 = m_2\alpha\, T_2 dT_2.$$

Final system of equations

As a result, we obtain a system of equations:

$$\begin{cases} m_1\alpha\, T_1 dT_1 + m_2\alpha\, T_2 dT_2 - \lambda\, dm = 0 \\ m_1\alpha\, dT_1 + m_2\alpha\, dT_2 - \lambda\, \dfrac{dm}{T} = 0 \end{cases}$$

We can integrate each equation separately and take into account that initial temperatures are

$$T_1^{(in)} = T_2^{(in)} = T_0 - t, \quad T^{(in)} = T_0,$$

the final temperatures are

$$T_1^{(fin)} = T_0, \qquad T_2^{(fin)} = T_{min}, \quad T^{(fin)} = T_0,$$

the temperature of the water T remains constant in process of all heat machine work (water just freezes). Δm is a mass of all frozen water that was initially liquid.

$$\begin{cases} m_1\alpha(2T_0t - t^2) + m_2\alpha\left({T_{min}}^2 - (T_0 - t)^2\right) - 2\lambda\,\Delta m = 0 \\ m_1\alpha T_0 t + m_2\alpha(T_{min} - T_0 + t)T_0 - \lambda\,\Delta m = 0 \end{cases}$$

By multiplying by 2 the second equation, subtracting it from the first and canceling out similar expressions, we obtain

$$-m_1t^2 + m_2\left({T_{min}}^2 - 2T_{min}T_0 + {T_0}^2 - t^2\right) = 0.$$

Now let us use our notation for $m_2 = {}^{m}/_{n}$ and $m_1 = m - m_2$. By substituting, we obtain a quadratic equation

$${T_{min}}^2 - 2T_{min}T_0 + {T_0}^2 - n\,t^2 = 0.$$

The equation can be solved in a next way:

$$(T_{min} - T_0)^2 = n\,t^2 \;\; \rightarrow \;\; T_{min} = T_0 \pm t\sqrt{n}.$$

We chose a negative sign because the final temperature T_{min} should not be higher than initial temperature.

Answer: $T_{min} = T_0 - t\sqrt{n}$

Appendix

Ideal work of the reversible heat engine can be modeled as an infinite sequence of infinitesimal Carnot cycles. Let’s check for both cases that equations are the same. We consider the small period of time and temperatures change for a negligibly small amount.

The case on the Fig. 1

Water gives δQ_{H1} to the first stage of the engine, a big part of ice receives δQ_{C1}, for other cycle transfers δA.

From the energy conversation law, we can obtain: $\delta Q_{H1} - \delta Q_{C1} - \delta A = 0$

For the Carnot cycle: $\frac{\delta Q_{H1}}{\delta Q_{C1}} = \frac{T}{T_1}$.

The second cycle receives energy and cools a small piece of ice. Now the small piece of ice gives δQ_{C2} and water receives δQ_{H2}. For this, we use the δA amount of energy.

From the energy conversation law, we can obtain: $\delta Q_{H2} - \delta Q_{C2} - \delta A = 0$

For the Carnot cycle: $\frac{\delta Q_{H2}}{\delta Q_{C2}} = \frac{T}{T_2}$.

From these equations we can obtain:

$$\begin{cases} \delta Q_{H1} - \delta Q_{C1} - \delta Q_{H2} + \delta Q_{C2} = 0 \\ \delta Q_{H1} = \dfrac{T}{T_1}\delta Q_{C1} \\ \delta Q_{H2} = \dfrac{T}{T_2}\delta Q_{C2} \end{cases}$$

If we change variables to δQ, δQ_1 and δQ_2, where $\delta Q = \delta Q_{H1} - \delta Q_{H2}$, $\delta Q_1 = -\delta Q_{C1}$, $\delta Q_2 = \delta Q_{C2}$ (these variables mean how much energy each part of our system give), we can obtain:

$$\begin{cases} \delta Q + \delta Q_1 + \delta Q_2 = 0 \\ \delta Q = -\dfrac{T}{T_1}\delta Q_1 - \dfrac{T}{T_2}\delta Q_2 \end{cases}$$

$$\begin{cases} \delta Q + \delta Q_1 + \delta Q_2 = 0 \\ \dfrac{\delta Q}{T} + \dfrac{\delta Q_2}{T_2} + \dfrac{\delta Q_1}{T_1} = 0 \end{cases}$$

This set of the equations is the same as one that we used for the solution of the problem, if we take into account that $\delta Q = dU$, $\delta Q_1 = dU_1$, $\delta Q_2 = dU_2$.

The case on the Fig. 2

Water gives δQ_{H1} to the first stage of the engine, the big part of ice receives δQ_{C1}, for other cycle transfers δA.

The energy conversation law implies: $\delta Q_{H1} - \delta Q_{C1} - \delta A = 0$. For the Carnot cycle: $\frac{\delta Q_{H1}}{\delta Q_{C1}} = \frac{T}{T_1}$.

The second cycle receives energy and cools a small piece of ice. The small piece of ice gives δQ_{C2} and big piece receives δQ_{H2}. For this cycle, we use the δA amount of energy.

From the energy conversation law, we can obtain: $\delta Q_{H2} - \delta Q_{C2} - \delta A = 0$

For the Carnot cycle: $\frac{\delta Q_{H2}}{\delta Q_{C2}} = \frac{T_1}{T_2}$.

From these equations we can obtain:

$$\begin{cases} \delta Q_{H1} - \delta Q_{C1} - \delta Q_{H2} + \delta Q_{C2} = 0 \\ \delta Q_{C1} = \dfrac{T_1}{T}\delta Q_{H1} \\ \delta Q_{H2} = \dfrac{T_1}{T_2}\delta Q_{C2} \end{cases}$$

If we change variables to δQ, δQ_1 and δQ_2, where $\delta Q = \delta Q_{H1}$, $\delta Q_1 = -\delta Q_{C1} - \delta Q_{H2}$, $\delta Q_2 = \delta Q_{C2}$ (these variables mean how much energy each part of our system give), we can obtain:

$$\begin{cases} \delta Q + \delta Q_1 + \delta Q_2 = 0 \\ -\delta Q_1 = \dfrac{T_1}{T}\delta Q + \dfrac{T_1}{T_2}\delta Q_2 \end{cases}$$

And finally:

$$\begin{cases} \delta Q + \delta Q_1 + \delta Q_2 = 0 \\ \dfrac{\delta Q}{T} + \dfrac{\delta Q_2}{T_2} + \dfrac{\delta Q_1}{T_1} = 0 \end{cases}$$

This set of the equations is the same as in the previous case and in the entropy approach. We just show that two mentioned cases are equivalent. Moreover, we can conclude, that we can apply particularly these two cases in every order and we obtain the same final result.