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1. *Electrostatics and hydrodynamics analogy*

Calculation of velocity vector field is complicated and requires solution of Laplace differential equation for potential stream. But we can use analogy between electrostatics and hydrodynamics. Let's consider Gauss theorem in differential form for electric field and charge density for metal polarized body:

$$\left(\nabla \cdot \vec{E}\right) = \frac{\rho_e}{\varepsilon_0},$$

where $\vec{E}$ is electric field, ρ_e is charge density and ε_0 is vacuum permittivity.

Volume around the body contains no charges, so $\rho = 0$ and our equation is:

$$\left(\nabla \cdot \vec{E}\right) = 0.$$

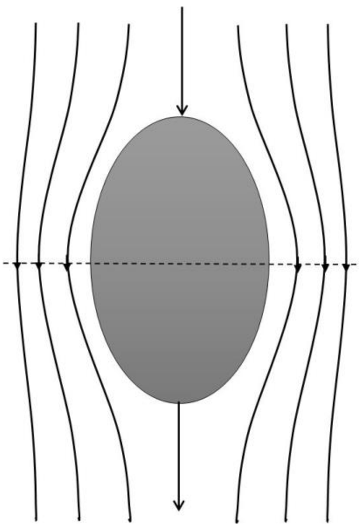

Fig. 1

If we consider the steady and laminar flow of incompressible liquid, we can apply the continuity equation of fluid flow

$$\frac{\partial \rho}{\partial t} + \left(\nabla \cdot (\rho \vec{v})\right) = 0.$$

For incompressible liquid $\rho = \text{const}$ and for steady current $\frac{\partial \rho}{\partial t} = 0$, as a result, we obtain

$$(\nabla \cdot \vec{v}) = 0,$$

where $\vec{v}$ is a velocity vector field. As we can see, velocity field and electric field satisfy the same equation. So, if boundary conditions are proportional in every point, solutions of the equations are proportional in every point (linearity) and we can find coefficient γ and equation $\gamma \vec{E} = \vec{v}$ satisfies in every point of space.

As we interested in kinetic energy of the liquid flow, it is convenient to denote γ as a coefficient that satisfy condition of equality of densities of kinetic energy of the flow (w_k) and electrostatic field (w_e):

$$w_k = w_e,$$

where $w_k = \rho \frac{v^2}{2}$ and $w_e = \varepsilon_0 \frac{E^2}{2}$. From equality, which is mentioned above, we can obtain γ by substitution of w_k and w_e and $\gamma = \sqrt{\frac{\varepsilon_0}{\rho}}$.

2. *Fluid flow near the surface*

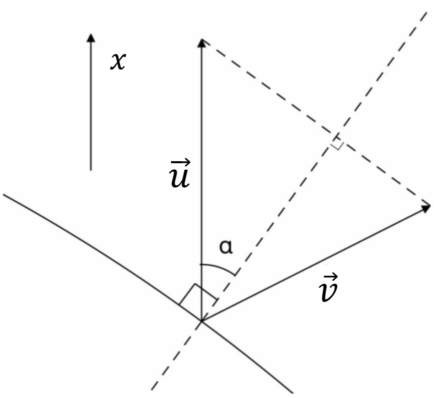

Fig. 2

Streamlines of liquid are shown on the Figure 1 in body reference frame. Let's consider liquid flow near the body surface in the liquid reference frame (reference frame, where liquid is in rest at infinitely far point from the body). Let's denote the velocity of body in this reference frame as $\vec{u}$ (Fig. 2). Liquid mustn't cross the surface, so, relative to the body, velocity of the liquid must be parallel to the surface of the body. Let's mark velocity of the liquid near the surface at some point as $\vec{v}$ (Fig. 2). Consequently, relative velocity is $\vec{v} - \vec{u}$. Let's denote normal vector to the surface at some point. So, if liquid flows parallel to the surface of the body, it is perpendicular to the normal vector and velocity of the liquid must satisfy $\vec{n} \cdot (\vec{v} - \vec{u}) = 0$.

$$(\vec{n} \cdot \vec{v}) = (\vec{n} \cdot \vec{u}) = u \cos \alpha.$$

3. *Electric field near the surface*

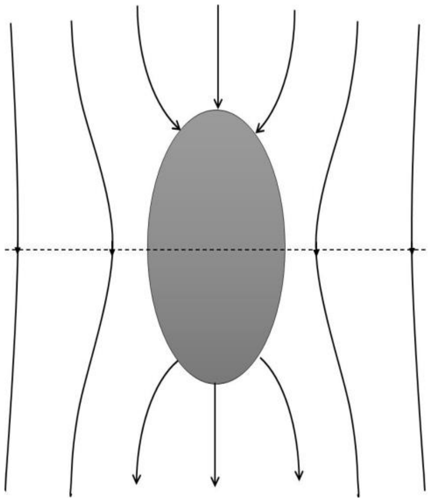

Fig. 3

Now let's consider electric field that is created by our concrete body, when it is made of metal and put in homogeneous electric field $\vec{E}_0$. Right near the surface of the metal body electric field must be perpendicular to the surface of the body due to metal's equipotential surface (Fig. 3). Let's denote resulting field near the surface as $\vec{E}_n$, external homogeneous field is $\vec{E}_0$ and electric field, which is created by induced charges on the surface of the metal body, is $\vec{E}_b$ (Fig. 4).

$$\vec{E}_n = \vec{E}_0 + \vec{E}_b.$$

Electrostatic induction on our particular homogeneous dielectric can be described with the model of two copies of body with constant charge density and oppositely charged, which are infinitesimally shifted relatively to each other. But, we know, if our shape is made of the dielectric material, then its polarization satisfies the special condition. Electric field in space, where two copies cross each other, is homogeneous. That means, that we can create a homogeneous field of strength $-\vec{E}_0$ inside our body. And if we add homogeneous external field $\vec{E}_0$ to this configuration, electric field in shape will vanish. That means, that model of two homogeneously charged bodies that infinitesimally shifted can give us solution for the same metal shape. Let's denote module homogeneous charge density of the bodies as ρ_e and shift as l. When l is much smaller than size of the body, we can describe charges, which aren't neutralized, as a surface charge density of body. Let's denote a thickness of charged layer as h. So, surface density σ is

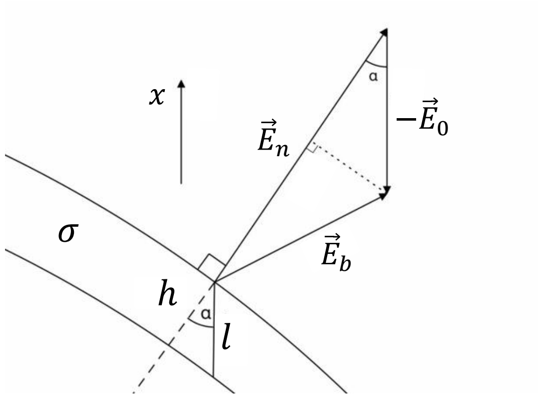

Fig. 4

$$\sigma = \rho_e h.$$

From geometric relation (Fig. 4) $h = l \cos \alpha$, as a result we can rewrite as $\sigma = \rho_e l \cos \alpha$.

Obviosly, $\vec{E}_n$ and $\vec{E}_b$ are dependent on the point on the surface, but some relations with liquid flow near the surface. Let's take dot product of both sides of the equality and $\vec{n}$:

$$\left(\vec{n} \cdot \vec{E}_n\right) = \left(\vec{n} \cdot \left(\vec{E}_0 + \vec{E}_b\right)\right) = \left(\vec{n} \cdot \vec{E}_0\right) + \left(\vec{n} \cdot \vec{E}_b\right)$$

$\vec{E}_n \,||\, \vec{n}$ and $|\vec{n}| = 1$ we obtain $\left(\vec{n} \cdot \vec{E}_n\right) = E_n$, $\left(\vec{n} \cdot \vec{E}_0\right) = E_0 \cos \alpha$ (Fig. 4). We can extract boundary conditions for $\vec{E}_b$:

$$\left(\vec{n}\cdot\vec{E}_b\right) = E_n - E_0\cos\alpha.$$

We can find E_n from the Gauss theorem in integral form. On the big scale our shape can be considered as a flat surface near the point, where we will apply Gauss theorem. So, let's take a small cylinder, which base is parallel to the surface, charged layer crosses the cylinder. Flux through the lateral surface is absent, because field is perpendicular to the surface of the body. Flux through the inner for body base is zero too, because there is no field in metal shape. So, there is the flux only through the outer for body surface. S – surface of the base of cylinder. By the Gauss theorem:

$$SE_n = \frac{\sigma S}{\varepsilon_0},$$

where SE_n – flux through the cylinder, σS – charge that covered by the cylinder.

$$E_n = \frac{\sigma}{\varepsilon_0} = \frac{\rho_e l\cos\alpha}{\varepsilon_0}.$$

Now let's define ρ_e and l through V (volume) and α (polarizability). As it was proven above, our polarized metal figure can be considered as two charged copies of body. We can consider this charge system as single dipole with the moment $p = l\rho_e V$. According to formal definition of dipole moment of the system of charges, after volume integration of dipole moment, we obtain that dipole moment is just equivalent to shift of one of all body. On the other hand $p = \alpha E_0$. As a result, we obtain

$$\alpha E_0 = l\rho_e V \;\; \rightarrow \;\; l\rho_e = \frac{\alpha E_0}{V}.$$

As a result, we can, finally, relate $\left(\vec{n}\cdot\vec{E}_b\right)$ and E_0:

$$\left(\vec{n}\cdot\vec{E}_b\right) = \left(\frac{\alpha}{V\varepsilon_0} - 1\right)E_0\cos\alpha.$$

As we can see, $\left(\vec{n}\cdot\vec{E}_b\right)$ is proportional to electric field E_0 and $\cos\alpha$. In the same way $(\vec{n}\cdot\vec{v})$ is proportional to u and $\cos\alpha$.

$$(\vec{n}\cdot\vec{v}) = u\cos\alpha.$$

So, boundary conditions are equivalent and differ on the dimension factor. As a result, we can state that $\gamma\vec{E} = \vec{v}$ in every point of space and $\gamma = \sqrt{\frac{\varepsilon_0}{\rho}}$.

From all mentioned above: $\gamma\vec{E}_b = \vec{v}$, consequently

$$\left(\frac{\alpha}{V\varepsilon_0} - 1\right)E_0\cos\alpha = \left(\vec{n}\cdot\vec{E}_b\right) = \frac{1}{\gamma}(\vec{n}\cdot\vec{v}) = \frac{1}{\gamma}u\cos\alpha,$$

$$\left(\frac{E_0}{u}\right)^2 = \frac{\rho\varepsilon_0 V^2}{(\alpha - V\varepsilon_0)^2}.$$

As we chose γ earlier, it is obvious that kinetic energy of water is equal to energy of outer electric field of our dipole. To find it we can firstly evaluate full dipole energy and then subtract inner energy.

4. *Energy of the dipole and flow*

To find full energy, let's evaluate full work, which we need to create this dipole with its electric field. Elementary work is equal

$$\delta A = F \cdot dx = Eq \cdot dx = dp \cdot E,$$

E – electric field, which corresponds to present p.

$$p = \alpha \cdot E \ \rightarrow \ dp = \alpha \cdot dE.$$

After integration we obtain:

$$A = \frac{\alpha}{2} {E_0}^2.$$

Next step is evaluating inner energy. If in homogeneous field E_0 in metal body field is zero, without external field, inner will be equal $-E_0$. Therefore $W_{in} = \frac{\varepsilon_0 {E_0}^2}{2} V$ (density of energy multiple its volume).

Thus

$$W_{out} = A - W_{in} = \frac{\alpha}{2} {E_0}^2 - \frac{\varepsilon_0 {E_0}^2}{2} V = \frac{{E_0}^2}{2} (\alpha - \varepsilon_0 V).$$

This expression should be equal K, kinetic energy of water. By definition $K = \frac{\Delta m u^2}{2}$. Consequently,

$$\frac{\Delta m u^2}{2} = \frac{{E_0}^2}{2} (\alpha - \varepsilon_0 V),$$

$$\Delta m = \frac{{E_0}^2}{u^2} (\alpha - \varepsilon_0 V).$$

Earlier we have obtained value of $\frac{{E_0}^2}{u^2} = \frac{\rho V^2 \varepsilon_0}{(\alpha - V\varepsilon_0)^2}$. Thus $\Delta m = \rho V \frac{V\varepsilon_0}{\alpha - V\varepsilon_0}$.

Answer:

$$\Delta m = \rho V \frac{V\varepsilon_0}{\alpha - V\varepsilon_0},$$

where ε_0 is vacuum permittivity.