Physics Cup - TalTech 2019, Felix Bekir Christensen

Problem 1

As the liquid is initially vortex free, it is vortex free at all times. $\oint_C \vec{v}\,d\vec{r} = 0 \;\; \forall C \Leftrightarrow \nabla \times \vec{v} = 0$ By Stokes' theorem.

Furthermore $\nabla \cdot \vec{v} = 0$ as the fluid is incompressible.

As no liquid can flow into or out of the body, $\vec{v}$ is purely tangential to its surface S, ie. $(\vec{v}-\vec{u})\cdot\hat{n} = 0$ on S, where $\vec{u}$ is the velocity of the body and $\hat{n}$ is the unit normal to its surface.
Finally, $\vec{v} \to 0$ very far away from the body, as otherwise the total energy of the liquid $T = \int \frac{1}{2}\rho|\vec{v}|^2 dV$ would not be finite. (As it is clearly finite initially, it has to stay finite at all times.)
In summary: (All for a fixed moment in time, t)

$$\nabla \cdot \vec{v} = 0 \quad (1)$$

$$\nabla \times \vec{v} = 0 \quad (2)$$

$$(\vec{v}-\vec{u})\cdot\hat{n} = 0 \text{ on } S \quad (3)$$

$$\vec{v} \to 0 \text{ far from } S \quad (4)$$

$$\frac{dT}{dV} = \frac{1}{2}\rho|\vec{v}|^2 \quad (5)$$

Of course (1) - (5) are only valid outside of the body.
Now consider putting the body in a constant E-field $\vec{E}_0$ and then "freezing" its polarisation and removing the field. For simplicity, let $\vec{E}_0$ and $\vec{u}$ have the same direction, the positive x-direction. Note that the total charge on the sphere is zero and hence all surface charges are bound charges. Let us examine the resulting field. (We assume that there is no B-field)

We are given the information that if the body were made of a dielectric and placed in a homogeneous E-field, the field inside it would be homogeneous as well. As inside a dielectric, $\vec{P} = \chi \varepsilon_0 \vec{E}$ where $\vec{P}$ is the polarisation density, it follows, that if the body is polarised, $\vec{P}$ is homogeneous. So we can simply write $\vec{P} = \frac{\vec{p}}{V}$ where (lower case) $\vec{p}$ is the dipole moment. But $\vec{p} = \alpha \vec{E}_0$, so simply $\vec{P} = \frac{\alpha}{V}\vec{E}_0$ everywhere inside the body. (6)

The E-field inside a conductor is always zero, so the field inside the polarised body is $-\vec{E}_0$. This lets us calculate the electric displacement field inside the body. As $\underbrace{\vec{D} = \varepsilon_0 \vec{E} + \vec{P}}_{\text{generally}}$, we get:

$$\vec{D}_0 = -\varepsilon_0 \vec{E}_0 + \frac{\alpha}{V}\vec{E}_0$$

$$\Leftrightarrow \vec{D}_0 = \left(\frac{\alpha}{V} - \varepsilon_0\right)\vec{E}_0 \qquad (7)$$

Outside the body, we have by Gauss' and Faraday's laws:

$\nabla \cdot \vec{E} = 0$ (8) (no charge)

$\nabla \times \vec{E} = 0$ (9) (no time-dependent B-field)

~~Ats~~ At the air-body boundary, notice that there are no free charges, so the normal component of $\vec{D}$ ~~stays~~ is continuous. ~~to~~ Outsinde ~~the~~ of the body, $\vec{D} = \varepsilon_0 \vec{E}$, so:

$$\varepsilon_0 \vec{E} \cdot \hat{n} = \left(\frac{\alpha}{V} - \varepsilon_0\right)\vec{E}_0 \cdot \hat{n}$$

$$\Leftrightarrow \left(\vec{E} - \left(\frac{\alpha}{\varepsilon_0 V} - 1\right)\vec{E}_0\right) \cdot \hat{n} = 0 \qquad (10) \quad \text{(on } S\text{)}$$

Far away from all charges, the E-field vanishes, ie.

$\vec{E} \to 0$ far from S (11)

Finally, the energy density of an electric field is

$$\frac{dU}{dV} = \frac{1}{2}\varepsilon_0 |\vec{E}|^2 \qquad (12)$$

Comparing eqs. (1)-(5) with (8)-(12) we see that we have successfully restated the original fluid dynamics problem as an electrostatics one.
Let us now find the energy of the surface charge configuration:

The surface charge density is given by:

$$\sigma = \vec{P}\cdot\hat{n}$$

$$\Leftrightarrow \sigma = \frac{\alpha}{V}\vec{E}_0\cdot\hat{n} \qquad (13)$$

Let the potential be zero at some point P inside the body.
Then, the potential at Q inside the body is simply $\phi = x|\vec{E}_0|$ (14)
where x is the difference of the x-coordinates of Q and P, $x = (\vec{q}-\vec{P})\cdot\hat{x}$.
(Recall that the direction of the field inside the body is the negative x-direction as it is $-\vec{E}_0$)
The potential energy of a charge distribution is $U = \int \frac{1}{2}\rho\varphi\, dV$
hence for our surface charge configuration $U = \frac{1}{2}\oint_S \sigma\varphi\, dA$ | use (13) and (14)

$$\Rightarrow U = \frac{\alpha|\vec{E}_0|}{2V}\oint_S \vec{E}_0\cdot\hat{n}\, x\, dA \qquad | \ \vec{E}_0 = |\vec{E}_0|\hat{x}$$

$$\Leftrightarrow U = \frac{\alpha|\vec{E}_0|^2}{2V}\oint_S x\,\hat{x}\cdot\hat{n}\, dA \qquad | \text{ use Gauss' law with } \nabla\cdot(x\hat{x}) = 1$$

$$\Leftrightarrow U = \frac{\alpha|\vec{E}_0|^2}{2V}\int 1\, dV$$

$$\Leftrightarrow U = \frac{\alpha|\vec{E}_0|^2}{2V}\cdot V$$

$$\Leftrightarrow U = \frac{\alpha|\vec{E}_0|^2}{2} \qquad (15)$$

But also $U = \int_{\text{all space}} \frac{1}{2}\varepsilon_0|\vec{E}|^2 dV = \int_{\text{inside body}} \frac{1}{2}\varepsilon_0|\vec{E}|^2 dV + \int_{\text{outside body}} \frac{1}{2}\varepsilon_0|\vec{E}|^2 dV$

As inside the body, $\vec{E} = -\vec{E}_0$ we get $\int_{\text{inside body}} \frac{1}{2}\varepsilon_0|\vec{E}|^2 dV = \frac{1}{2}\varepsilon_0|\vec{E}_0|^2 V$

so

$$\int_{\text{outside body}} \frac{1}{2}\varepsilon_0|\vec{E}|^2 dV = U - \frac{1}{2}\varepsilon_0|\vec{E}_0|^2 V$$

$$\Leftrightarrow \int_{\text{outside body}} \frac{1}{2}\varepsilon_0 |\vec{E}|^2 dV = (\alpha - \varepsilon_0 V)\frac{|\vec{E}_0|^2}{2} \qquad (16)$$

Now compare eqs. (5) and (12). The velocity field of the fluid and the E-field have the same Energy density whence

$$\frac{1}{2}\rho|\vec{v}|^2 = \frac{1}{2}\varepsilon_0 |\vec{E}|^2$$

$$\Leftrightarrow |\vec{E}|^2 = \frac{\rho}{\varepsilon_0}|\vec{v}|^2 \qquad (17)$$

Scaling the boundary condition (3), (10) similarly gives:

$$\left(\frac{\alpha}{\varepsilon_0 V} - 1\right)^2 |\vec{E}_0|^2 = \frac{\rho}{\varepsilon_0}|\vec{u}|^2$$

$$\Leftrightarrow |\vec{E}_0|^2 = \frac{\rho|\vec{u}|^2}{\varepsilon_0\left(\frac{\alpha}{\varepsilon_0 V} - 1\right)^2} \qquad (18)$$

The kinetic energy of the liquid is just its energy density integrated over its volume (that is, outside of the body):

$$T = \int_{\text{outside body}} \frac{1}{2}\rho|\vec{v}|^2 dV \qquad (19)$$

But this equals the integral in (16) if (18) is satisfied:

$$\Rightarrow T = (\alpha - \varepsilon_0 V)\frac{|\vec{E}_0|^2}{2}$$

$$\Rightarrow T \overset{(18)}{=} \frac{1}{2}(\alpha - \varepsilon_0 V)\frac{\rho|\vec{u}|^2}{\varepsilon_0\left(\frac{\alpha}{\varepsilon_0 V} - 1\right)^2}$$

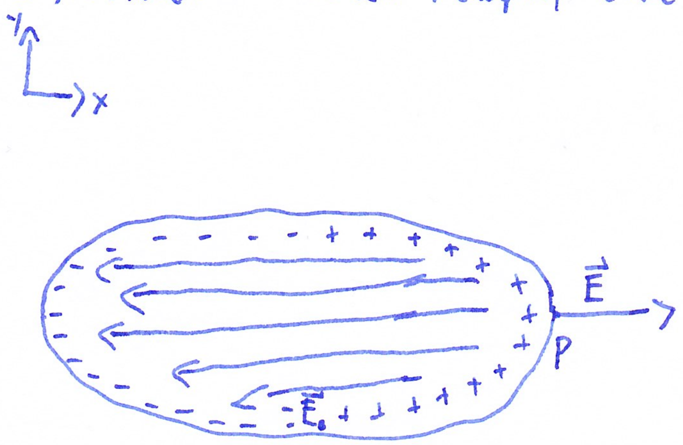
$$\Leftrightarrow T = \frac{1}{2}\cdot\frac{\varepsilon_0 V^2 \rho}{\alpha - \varepsilon_0 V}|\vec{u}|^2 \qquad (20)$$

Finally giving an expression resembling the well-known K.e. formula $T = \frac{1}{2}m|\vec{u}|^2$. Comparing this to (20) then gives the added mass:

$$\boxed{m_{add} = \frac{\varepsilon_0 V^2 \rho}{\alpha - \varepsilon_0 V}}$$

One might wonder about the possibility of this expression being negative, which would be unphysical. But the inequality $\alpha - \varepsilon_0 V > 0$ which implies $m_{add} > 0$ can be motivated as follows:

The charge distribution looks roughly like this:



The direction of $\vec{E}$ at P is the positive x-direction as the positive charges are much closer to P than the negative ones. As $\vec{D} = \varepsilon_0 \vec{E}$, the direction of $\vec{D}$ at P is also the positive x-direction. Recall that $\vec{D} \cdot \hat{n}$ is continuous at the boundary, so $\vec{D}_0$ must point in the ~~posit~~ same direction. But by (7) this is the case iff

$$\frac{\alpha}{V} - \varepsilon_0 > 0 \qquad (\text{As } \vec{E}_0 \text{ points in the positive x-direction})$$

$$\Leftrightarrow \boxed{\alpha - \varepsilon_0 V > 0}$$