Solution of Physics Cup 2019, Problem No 2

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The heat capacity of ice is up to a good approximation proportional to its temperature, namely $c_v = \alpha T_{ice}$. Therefore the energy needed to increase the temperature of ice with mass m is $\frac{\alpha}{2}m(T_f^2 - T_i^2)$, where T_i is the initial temperature and T_f the final temperature of the ice. Now let us consider the system described by the problem where an isolated system includes equal masses m of ice and water. The water is at temperature T_0 which can be assumed to be the melting point at the given pressure, while the ice is initially at a slightly lower temperature, $T_0 - t$. Since $T_0 = 273.15$ K and t is only a few kelvins, we can assume that $t << T_0$. If there is no heat engine, then the system will try to reach thermodynamics equilibrium. Some of the water will freeze to release heat and the ice will eventually have a higher temperature. The total heat needed to increase the temperature of the ice to temperature T_0 is $\Delta Q \approx m\alpha T_0 t$ which is much smaller compared to the heat that can be given by the water until the water completely freezes, λm because $\alpha T_0 t << \lambda$. This means that only a small portion of the water will need to freeze to increase the temperature of the ice to the melting point.

To decrease the temperature of the n-th part of the ice, heat must be released from it. However, by the second law of thermodynamics we know that it is impossible for heat to transfer spontaneously from a colder body to a hotter body. To achieve this, we need another source of energy to do the mechanical work such that the transfer becomes possible. This mechanical work is provided by the other heat engine in the system. To summarize the process then, a certain amount of heat must be released from the n-th part of the ice and given to the rest of the ice while some heat will also be given by the water to do the mechanical work and make the heat transfer possible. To make it simple, from this point, I will just write the ice to refer to the rest part of the ice. In this problem, we also assume that the engines are ideal and the heat capacity of the engines is negligible which means that it needs a very small amount of heat to increase (or decrease) the temperature of the engines. Now, using this assumption, there are some possible ways to refrigerate the n-th part of the ice. For example, we can use the water to be the one providing the heat for the first heat engine (the one that produces mechanical work) and the ice to be the one receiving the residual heat, while the heat taken from the n-th part of the ice can be given to either the ice or the water. However, this doesn't really matter, because the processes are basically the same (heat transfer from n-th part of the ice to the rest part of the ice with the help of mechanical work provided by the water). Hence, the result should be the same if we just choose one of the configurations.

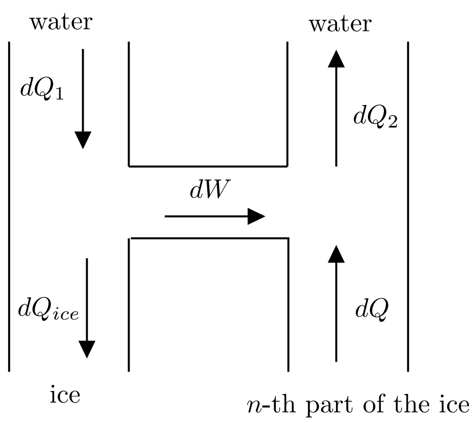

Figure 1: Schematic diagram of the engines, the left one is the heat engine producing mechanical work and the right one is the refrigerator

Here, the configuration where the water provides the heat in the first engine with the ice receiving the residual heat and the water to be the one receiving the heat in the second engine is chosen (see Figure 1). If we assume the engine to be operating infinitesimally small Carnot cycles with the substances being the heat reservoir (the heat capacity of the engine is very small), then for an infinitesimal amount of work done by the engine we have the amount of heat received by the ice is given by

$$dW = \frac{T_0 - T_{ice}}{T_{ice}} dQ_{ice}$$

and the relationship between the mechanical work and the heat released from the n-th part of the ice is

$$dW = -\frac{T_0 - T_1}{T_1} dQ$$

where T_1 is the temperature of the n-th part of the ice and the minus sign is because it is releasing heat. Equating both equations and using the fact that the heat capacity of ice is proportional to its temperature then we have

$$\left(\frac{T_0}{T_{ice}} - 1\right) m\alpha T_{ice} dT_{ice} = \left(1 - \frac{T_0}{T_1}\right) \frac{m}{n} \alpha T_1 dT_1$$

Simplifying and integrating both sides one will find

$$\int_{T_0 - t}^{T_0} n(T_0 - T_{ice})\, dT_{ice} = \int_{T_0 - t}^{T} (T_1 - T_0)\, dT_1$$

Note that the upper limit of the left hand integrand can be set to T_0 because we know that the amount of water is large enough to provide the heat to do the mechanical work. If there is a much smaller amount of water compared to the ice then the equilibrium might not be at T_0. Simplifying the equation and neglecting the term t^2 on the right hand side (there is also nt^2 term on the left hand side, this cannot be ignored), one will obtain a quadratic equation for the final temperature of the n-th part of the ice,

$$T^2 - 2T_0 T + T_0^2 - nt^2 = 0$$

Finally, after solving this equation and and using the fact that $T < T_0$, we have $T = T_0 - t\sqrt{n}$. This is the minimum temperature that can be given to the n-th part of the ice.