Problem 4 - Physics Cup 2019

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1 Potential Difference Formula

Define the points on the grid $A = (0,0)$, $B = (1,1)$, $C = (n,n)$, $D = (n+1, n+1)$ where $n > 1$. Now let a current I flow from A to B. This current leads to a potential difference $U_{CD} \equiv U_C - U_D$ between C and D. Let us calculate this potential difference: The current from A to B can be represented as a superposition of a current I from A to infinity and a current $-I$ from B to infinity. Let the potential at A be equal to 0. If there is a current I into A and a current $-I$ into C, the potential at C is by Ohm's law $U_C = IR_{nn}$. By translational symmetry of the set up, both currents contribute equally to this, so a single current I into A leads to $U_C = \frac{I}{2}R_{nn}$. The resistance between A and B is R_{11} and that between B and C is R_{n-1n-1} so by similar reasoning a current $-I$ into B leads to $U_B = -U_{AB} = \frac{I}{2}R_{11}$ and $U_{CB} = -\frac{I}{2}R_{n-1n-1}$ implying $U_C = U_B + U_{CB} = \frac{I}{2}(R_{11} - R_{n-1n-1})$. Combining these results gives for the current distribution described above:

$$\begin{aligned} U_C &= \frac{I}{2}(R_{nn} - R_{n-1n-1} + R_{11}) \\ &= \frac{I\sqrt{Rr}}{\pi}\left(\sum_1^n \frac{1}{2k-1} - \sum_1^{n-1}\frac{1}{2k-1} + 1\right) \\ &= \frac{I\sqrt{Rr}}{\pi}\left(\frac{1}{2n-1} + 1\right) \\ &= \frac{I\sqrt{Rr}}{\pi}\left(\frac{2n}{2n-1}\right) \end{aligned} \tag{1}$$

Where the equation for R_{nn} given in the problem text was used. The potential at D is then obtained by making the replacement $n \to n+1$ in 1:

$$U_D = \frac{I\sqrt{Rr}}{\pi}\left(\frac{2n+2}{2n+1}\right) \tag{2}$$

This gives the potential difference between C and D as:

$$U_{CD} = \frac{I\sqrt{Rr}}{\pi}\left(\frac{2}{4n^2-1}\right) \tag{3}$$

2 Calculating the Result

Consider the current distribution described above with an additional current I' flowing into C and $-I'$ into D. Here I' is the current through the wire flowing from D to C. The resistance between C and D is $R_{11} = \frac{2\sqrt{Rr}}{\pi}$. Now let us determine I': If the points C and D are short-circuited, they are at the same potential. That means the potential differences due to I and I' cancel:

$$\begin{aligned} & I\frac{\sqrt{Rr}}{\pi}\left(\frac{2}{4n^2-1}\right) - I'\frac{2\sqrt{Rr}}{\pi} = 0 \\ \iff \quad & I' = \frac{I}{4n^2-1} \end{aligned} \tag{4}$$

Now, the contribution to the potential U_B due to I' gives the change in the potential at B that occurs when C and D are short-circuited while maintaining a constant current I. So

$$U_B = (R'_{11} - R'11)I \tag{5}$$

Fortunately, U_B can be found using equation 3. This is because the change $A, B, C, D \to D, C, B, A$ corresponds to a rotation of the grid that leaves it unchanged. Furthermore, we need to make the replacement $I \to -I'$, where the negative sign is used because I' flows into C rather than D. So combining 3 and 4 with these changes gives:

$$U_B = -\left(\frac{2\sqrt{Rr}}{\pi(4n^2-1)^2}\right)I \tag{6}$$

This can be compared with 5 to give the final result:

$$\boxed{R'_{11} - R_{11} = -\frac{2\sqrt{Rr}}{\pi(4n^2-1)^2}}$$