Solution to Problem 2

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1 First Process

First the fraction $\eta = \frac{n-1}{n}$ of the ice, whose temperature is not minimised is heated to T_0, while part of the water is frozen. Because the latter happens at constant temperature, the water can be treated as an ideal heat reservoir at constant temperature. It will be shown later that not all of the water is frozen. The small entropy increment of the ice is given by $dS = \frac{đQ}{T}$, but since $đQ = \eta m \alpha T dT$, this becomes:

$$dS = \eta m \alpha dT \tag{1}$$

The total change in entropy is obtained by integrating this expression:

$$\Delta S_1 = \int_{T_0 - t}^{T_0} \eta m \alpha \, \mathrm{d}T = \eta m \alpha t \tag{2}$$

The heat added during the heating process is found by integrating the heat capacity with respect to temperature:

$$\Delta Q_1 = \int_{T_0 - t}^{T_0} \eta m \alpha T \, \mathrm{d}T = \eta m \alpha \left(T_0 t - \frac{t^2}{2} \right) \tag{3}$$

Since the water has constant temperature T_0, its entropy change is simply:

$$\Delta S = \frac{\Delta Q_2}{T_0} \tag{4}$$

The efficiency is a maximum whence the total change in entropy vanishes:

$$\Delta S_1 + \Delta S_2 = 0 \tag{5}$$

Leading to:

$$\Delta Q_2 = \eta m \alpha T_0 t \tag{6}$$

The available work is

$$W = -\Delta Q_1 - \Delta Q_2 = \frac{\eta m \alpha}{2} t^2 \tag{7}$$

The energy that would be released by freezing all of the water is $\lambda m = 3.34 \cdot 10^5 Jkg^{-1}m$. Using $\alpha = 7.51 Jkg^{-1}$, $t \approx 10K$ (order of a few kelvins) and $\eta \approx 1$ (large n), the heat that is actually released $\Delta Q_2 \approx 2.05 \cdot 10^4 Jkg^{-1}m$. Since this is much smaller than than λm, some of the water always stays liquid and treating it as a heat reservoir at constant temperature is justified.

2 Second process

In the second process the fraction $\frac{1}{n}$ of the ice is cooled down to its minimum achievable temperature $T_{min} = T_0 - t'$ while the heat released as well as the work from the previous part are put into melting some of the remaining ice. Integrating (1), the entropy change due to cooling the ice is found to be:

$$\Delta S_3 = \frac{m\alpha}{n}(t - t') \tag{8}$$

Integrating the heat capacity, similarly to (3) gives:

$$\Delta Q_3 = \frac{m\alpha}{n}\left(T(t - t') + \frac{t'^2 - t^2}{2}\right) \tag{9}$$

As melting like freezing occurs at constant temperature, we simply have:

$$\Delta S_4 = \frac{W - \Delta Q_3}{T_0} \tag{10}$$

Similarly to the heat engine, the refrigerator has maximum efficiency when:

$$\Delta S_3 + \Delta S_4 = 0 \tag{11}$$

Which after some algebra simplifies to:

$$t' = \sqrt{n}t \tag{12}$$

So the minimum achievable temperature is given by:

$$\boxed{T_{min} = T_0 - \sqrt{n}t}$$

This is in fact the minimum achievable temperature, since the remaining ice and water are both at temperature T_0, so there is no temperature differential which would allow us to extract further work to be used for cooling. And since all steps were reversible, a different set of processes could not have lead to a lower temperature either.