Solution of Physics Cup 2019, Problem No 4

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In general, problems involving an infinite number of resistors can be very interesting, as usually ideas which make use of the symmetry of the system can be used. In this problem, we have an infinite square grid of resistors where each horizontal resistor has the same resistance R and each vertical resistor has resistance r. For convenience, let us denote every node using the coordinate system as given in the problem. The effective resistance between two nodes at $(0,0)$ and (n,n) is given by

$$R_{n,n} = \frac{2\sqrt{Rr}}{\pi}\sum_{k=1}^{n}\frac{1}{2k-1}$$

Therefore, the effective resistance between the node $(0,0)$ and $(1,1)$ is simply $R_{1,1} = \frac{2}{\pi}\sqrt{Rr}$. Now, if the node (n,n) and $(n+1,n+1)$ is short-circuited (i.e. connected with a wire of zero resistance), the effective resistance between node $(0,0)$ and node $(1,1)$ will change because electric current can go through the wire. The change of resistance due to this wire is what we are looking for.

To obtain the new effective resistance, we first need to know the potential difference between node (n,n) and $(n+1,n+1)$ when we drive a current I through node $(0,0)$ and $(1,1)$. To do this we can use two different ways, one way is by using the reciprocity theorem and another one is by using superposition principle. The first way is by doing the following steps. First, let us drive a current I into the node $(0,0)$ and take it out of node (n,n) (by connecting the two nodes to a current source for example). The potential difference between the two nodes is simply $\epsilon = IR_{n,n}$, because we can see the two nodes as having effective resistance $R_{n,n}$. Now, this configuration can also be obtained by making use of superposition of two other systems: one which we drive current I through node $(0,0)$ and $(1,1)$ and another system which we drive the current in and out of node $(1,1)$ and (n,n). See figure 1. In the former case the potential difference between node $(0,0)$ and (n,n) is given by $IR_{1,1}+V_{1n,01}$, where $V_{1n,01}$ is the potential difference between node $(1,1)$ and (n,n) due to current I from node $(0,0)$ to $(1,1)$. Similarly, for the latter case, the potential difference between node $(0,0)$ and (n,n) is $IR_{n-1,n-1}+V_{01,1n}$, where $V_{01,1n}$ is the potential difference between node $(0,0)$ and $(1,1)$ due to current I from $(1,1)$ to (n,n). Now, when we add the two systems we will have the initial system where I is going into node $(0,0)$ and out of node (n,n) (the currents at node $(1,1)$ from the two configurations cancel each other). So, we must have

$$IR_{n,n} = IR_{1,1} + IR_{n-1,n-1} + V_{01,1n} + V_{1n,01}$$

By using the reciprocity theorem[1], we can see that $V_{01,1n} = V_{1n,01} = V$. Therefore, one will obtain V to be

$$|V| = \frac{\sqrt{Rr}}{\pi}I\left(\frac{2(n-1)}{2n-1}\right) \tag{1}$$

This is the potential difference between node $(1,1)$ and (n,n) when a current I is going into node $(0,0)$ and out of node $(1,1)$. The potential difference between node $(1,1)$ and $(n+1,n+1)$ can be obtained by changing n in equation (1) to be $n+1$. After that, we can easily obtain the potential difference between node $(n+1,n+1)$ and (n,n) by subtracting the two results,

$$\Delta V = \frac{\sqrt{Rr}}{\pi}I\left(\frac{2n}{2n+1}-\frac{2(n-1)}{2n-1}\right) = \frac{\sqrt{Rr}}{\pi}\left(\frac{2}{4n^2-1}\right)I$$

Another way without using the reciprocity theorem can also be used to get the same result. Let us consider when a current I is driven into node $(0,0)$ (and goes to infinity). By symmetry the potential of a node (n,n) with respect to node $(0,0)$ is the same as node $(-n,-n)$. Let us denote this potential as $\phi(n)$. Now, if we reverse the current direction (let's say we pull a current I out of node (n,n) from infinity), then the potential of node (m,m) with respect to node (n,n) must be $-\phi(|n-m|)$ (this is because the "distance" between the two nodes is $|n-m|$ and the minus sign is because of the reversed direction of the current). If we take $m=0$ and we add the two configurations, (this means that there will be a current I going into node $(0,0)$ and out of node (n,n)), then the potential difference between

[1] Reciprocity theorem tells us that if we have a current I driven into A and out of B such that the potential difference between C and D is V, then when a current I is driven into C and out of D, the potential difference between A and B will also be V. A nice proof for this theorem is given in the appendix 2 of the circuits booklet by J. Kalda

node $(0,0)$ and (n,n) must be $-2\phi(n) = IR_{n,n}$ (the sign actually doesn't really matter here, as we are only interested in the potential difference). So, using this result, the potential difference between node (n,n) and node $(0,0)$ will be $-\phi(n)+\phi(n-1)-\phi(1)$ when we drive a current I into node $(0,0)$ and out of node $(1,1)$. Hence, we can obtain ΔV, the potential difference between node (n,n) and $(n+1,n+1)$.

$$\Delta V = \phi(n+1) - 2\phi(n) + \phi(n-1)$$
$$\Delta V = \frac{\sqrt{Rr}}{\pi}\left(\frac{2}{4n^2-1}\right) I$$

The result agrees with the earlier derivation.

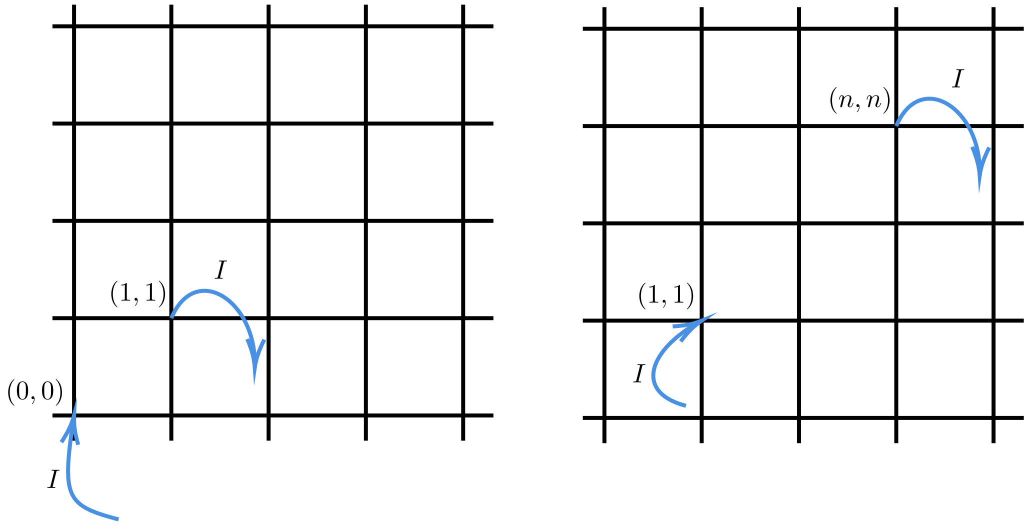

Figure 1: The superposition of the two configurations will result in a current I goes into node $(0,0)$ and out of node (n,n). On the figure, I take $n=3$ for simplicity in drawing, but this also holds for any n.

Having known the above result, let us consider what happen when node (n,n) and $(n+1,n+1)$ are short-circuited. Since the wire has a negligible resistance, the new potential difference between the two nodes must be zero. This exact configuration can be seen as a superposition of the original system (i.e. before the two nodes are short-circuited) and another system (see figure 2). For convenience, I will refer this other system as the second configuration. The potential difference between (n,n) and $(n+1,n+1)$ in the original configuration is $\Delta V = \frac{\sqrt{Rr}}{\pi}\left(\frac{2}{4n^2-1}\right) I$, where the potential of node $(n+1,n+1)$ should be higher. Since the effective resistance between node (n,n) and $(n+1,n+1)$ is also $R_{1,1}$, then if a current $I' = \frac{\Delta V}{R_{1,1}}$ is driven from node (n,n) to node $(n+1,n+1)$ in the second configuration, the potential difference between (n,n) and $(n+1,n+1)$ will be $-\Delta V$. And therefore, when we add the two configurations, we will get a configuration which is exactly the same as if we short-circuit node (n,n) and $(n+1,n+1)$. The current I' is, evidently, the current that goes through the wire short-circuiting the two nodes. Now, after we add the second configuration to the original configuration, the potential difference between node $(0,0)$ and $(1,1)$ is no longer $IR_{1,1}$, because there is also the contribution of potential difference coming from the second configuration. Due to symmetry of the system, the potential difference between node $(0,0)$ and $(1,1)$ brought by the second configuration is simply, $\Delta V' = \frac{I'}{I}\Delta V$. So, in the final configuration, the potential difference between node $(0,0)$ and $(1,1)$ is

$$V' = IR'_{1,1} = IR_{11} - \Delta V'$$

And finally,

$$R'_{1,1} - R_{1,1} = -\frac{1}{R_{1,1}}\left(\frac{\Delta V}{I}\right)^2 = -\frac{2\sqrt{Rr}}{\pi(4n^2-1)^2}$$

Note that when $n=0$ we can see that $R'_{1,1}$ is zero as the wire is now connecting node $(0,0)$ and $(1,1)$, and when $n\to\infty$, $R'_{1,1} = R_{1,1}$.

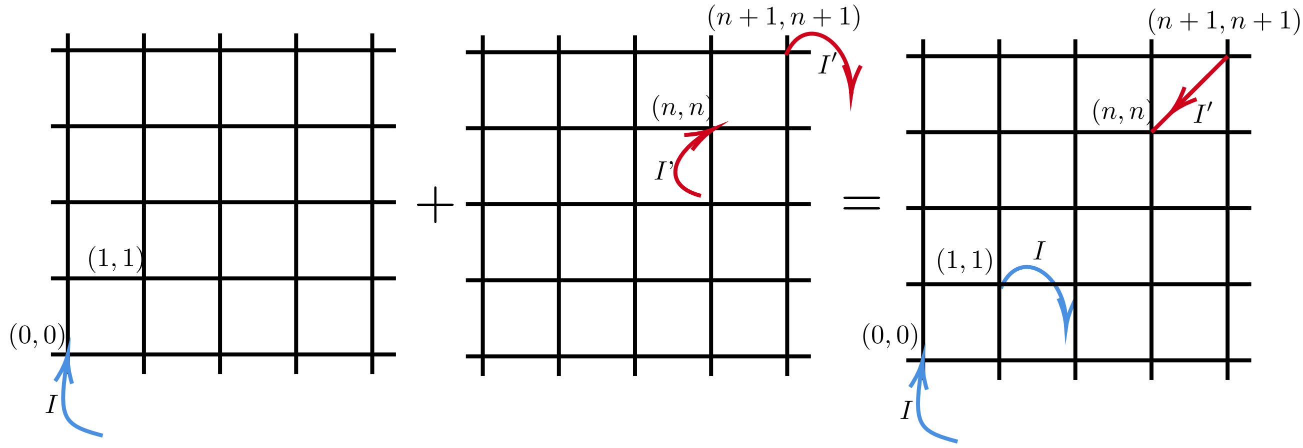

Figure 2: To obtain the configuration which the potential difference between node (n,n) and $(n+1,n+1)$, the superposition of the original system and a new one can be done. The left figure is the original system, while the right one is the one we use to obtain the final configuration.