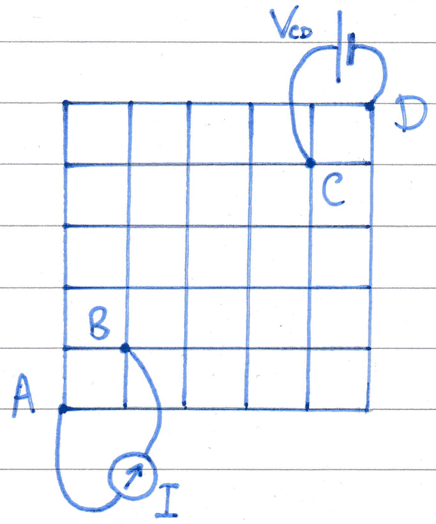
Physics Cup Problem 4



Horizontal resistances: R
Vertical resistances: r

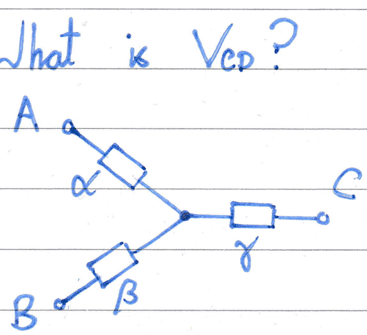
Coordinates:
$A: (0,0)$
$B: (1,1)$
$C: (n,n)$
$D: (n+1,n+1)$

Before nodes C and D are shorted together, the effective resistance between nodes A and B is R_{11}. This means that if a current source producing a current I is connected between A and B, the resulting voltage between A and B will be:

$$V_B - V_A = \mathcal{E} = IR_{11}.$$

In addition, nodes C and D will be raised to potentials V_C and V_D respectively. Hence, adding a battery with voltage $V_{CD} = V_C - V_D$ between C and D as shown will not change the distribution of currents and voltages in the circuit.

What is V_{CD}?



To work out V_C, we represent the circuit with terminals A, B and C as a Y-circuit as shown. We have:

$$\left.\begin{aligned} \alpha + \beta &= R_{11} \\ \alpha + \gamma &= R_{nn} \\ \beta + \gamma &= R_{n-1,n-1} \end{aligned}\right\}$$

$$\therefore \quad \alpha = \frac{1}{2}\left(R_{nn} + R_{11} - R_{n-1,n-1}\right)$$
$$\beta = \frac{1}{2}\left(R_{n-1,n-1} + R_{11} - R_{nn}\right)$$
$$\gamma = \frac{1}{2}\left(R_{nn} + R_{n-1,n-1} - R_{11}\right)$$

2 If the voltages across A and B is $\mathcal{E}$, and if A is grounded so $V_A = 0$, then:

$$V_C = \mathcal{E}\left(\frac{\alpha}{\alpha+\beta}\right) \qquad \text{(voltage divider formula)}$$

$$= \frac{\mathcal{E}\left(R_{nn} + R_{11} - R_{n-1,n-1}\right)}{2R_{11}}.$$

Similarly, by replacing $n \to n+1$ we get:

$$V_D = \frac{\mathcal{E}\left(R_{n+1,n+1} + R_{11} - R_{nn}\right)}{2R_{11}}$$

$$\therefore V_C - V_D = V_{CD} = \frac{\mathcal{E}\left(2R_{nn} - R_{n+1,n+1} - R_{n-1,n-1}\right)}{2R_{11}}.$$

The question tells us that

$$R_{nn} = \frac{2\sqrt{Rr}}{\pi}\sum_{k=1}^{n}\frac{1}{2k-1} = R_{11}\sum_{k=1}^{n}\frac{1}{2k-1}$$

so:

$$V_{CD} = \frac{\mathcal{E}\left[2R_{nn} - \left(R_{nn} + \frac{R_{11}}{2n+1}\right) - \left(R_{nn} - \frac{R_{11}}{2n-1}\right)\right]}{2R_{11}}$$

$$= \frac{\mathcal{E}}{2}\left(\frac{1}{2n-1} - \frac{1}{2n+1}\right)$$

$$= \frac{\mathcal{E}}{4n^2-1}.$$

Now, the superposition principle tells us that:

$$\mathcal{E} = \overbrace{\left(\begin{array}{c}\text{Voltage across AB when the battery} \\ \text{connecting C and D is shorted}\end{array}\right)}^{=\mathcal{E}_1} + \overbrace{\left(\begin{array}{c}\text{Voltage across AB when current} \\ \text{source is made open circuit}\end{array}\right)}^{=\mathcal{E}_2}$$

3 We want $\mathcal{E}_1$, as this will tell us the effective resistance between A and B when CD is shorted:

$$R_{11}' = \frac{\mathcal{E}_1}{I} = \frac{\mathcal{E} - \mathcal{E}_2}{I} = R_{11} - \frac{\mathcal{E}_2}{I} \implies R_{11}' - R_{11} = -\frac{\mathcal{E}_2}{I}.$$

To get $\mathcal{E}_2$, we use a symmetry argument; when a voltage of $\mathcal{E}$ is applied across AB, a potential difference of $\frac{\mathcal{E}}{4n^2-1}$ is created between C and D. Hence, by symmetry, when a voltage of V_{CD} is applied across CD, a potential difference of $\frac{V_{CD}}{4n^2-1}$ is created between A and B. So:

$$\mathcal{E}_2 = \frac{V_{CD}}{4n^2-1} = \frac{\mathcal{E}}{(4n^2-1)^2} \implies \boxed{R_{11}' - R_{11} = -\frac{R_{11}}{(4n^2-1)^2} = -\frac{2\sqrt{Rr}}{\pi(4n^2-1)^2}}.$$