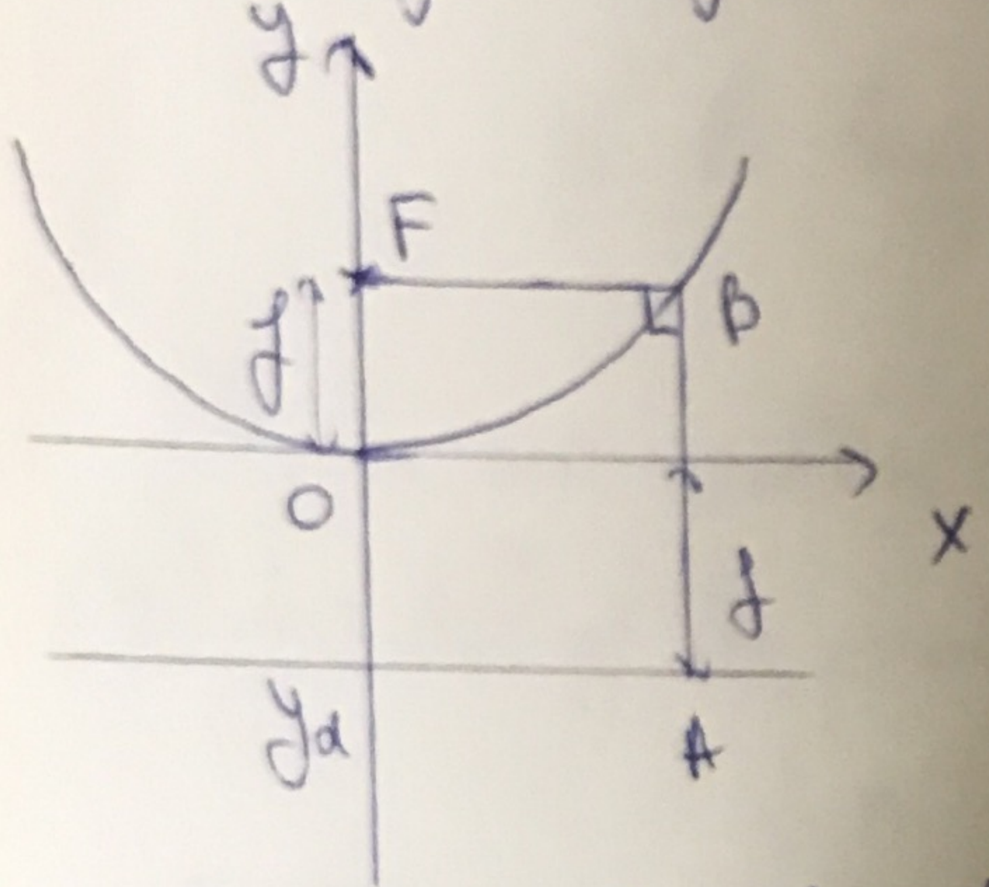
Physics Cup 2019
Problem 3

Let's analyse the position of the directrix for a simple parabola described by the equation $y(x) = ax^2$

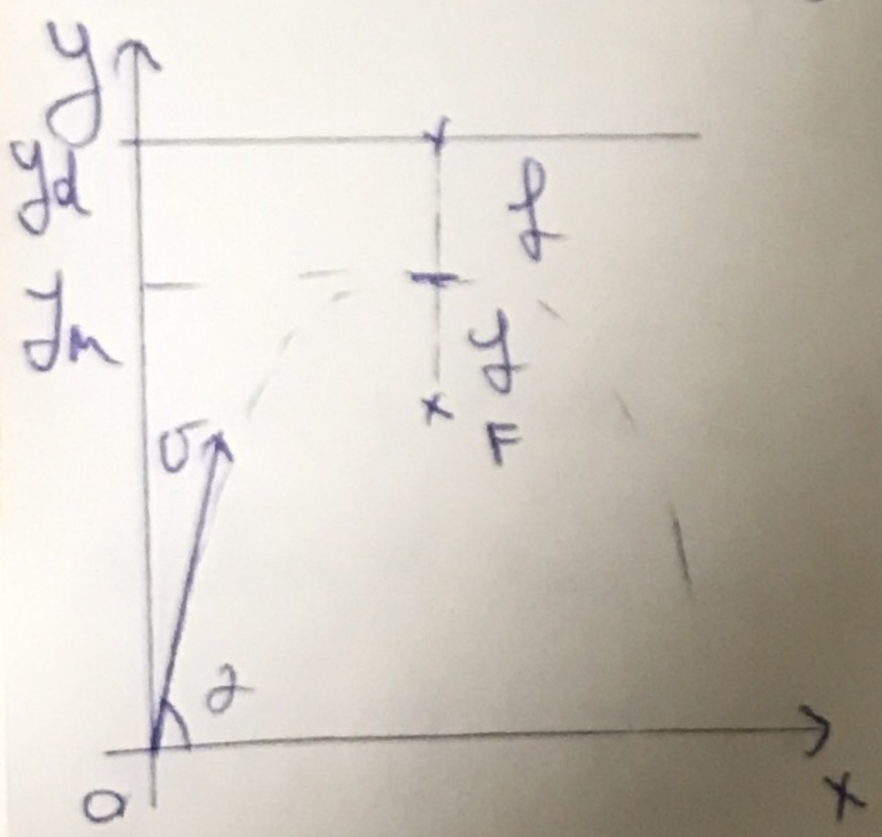



For $y(x) = f$, $x = \sqrt{\frac{f}{a}}$

$FB = BA \Rightarrow 2f = \sqrt{\frac{f}{a}}$

$$f = \frac{1}{4a} \quad (1)$$

For the motion of a body in a constant gravitational field:




$$\left.\begin{aligned} x(t) &= v\cos\alpha\, t \\ y(t) &= v\sin\alpha\, t - g\frac{t^2}{2} \end{aligned}\right\} \Rightarrow y(x) = x\,tg\alpha - \frac{gx^2}{2v^2}(tg^2\alpha + 1) \quad (2)$$

$$\left.\begin{matrix}(1)\\(2)\end{matrix}\right\} \Rightarrow f = \frac{1}{4 \cdot \frac{g}{2v^2}\frac{1}{\cos^2\alpha}} = \frac{v^2}{2g}\cos^2\alpha \quad (3)$$

$$\left.\begin{aligned} y_M &= \frac{(v\sin\alpha)^2}{2g} \\ y_d &= y_M + f \end{aligned}\right\} \Rightarrow y_d = \frac{v^2}{2g} \quad (4)$$

(4) is the main idea we will use to solve the problem. If we ~~ch~~ change the position of the launching point O by $\Delta y$ on $O_y$

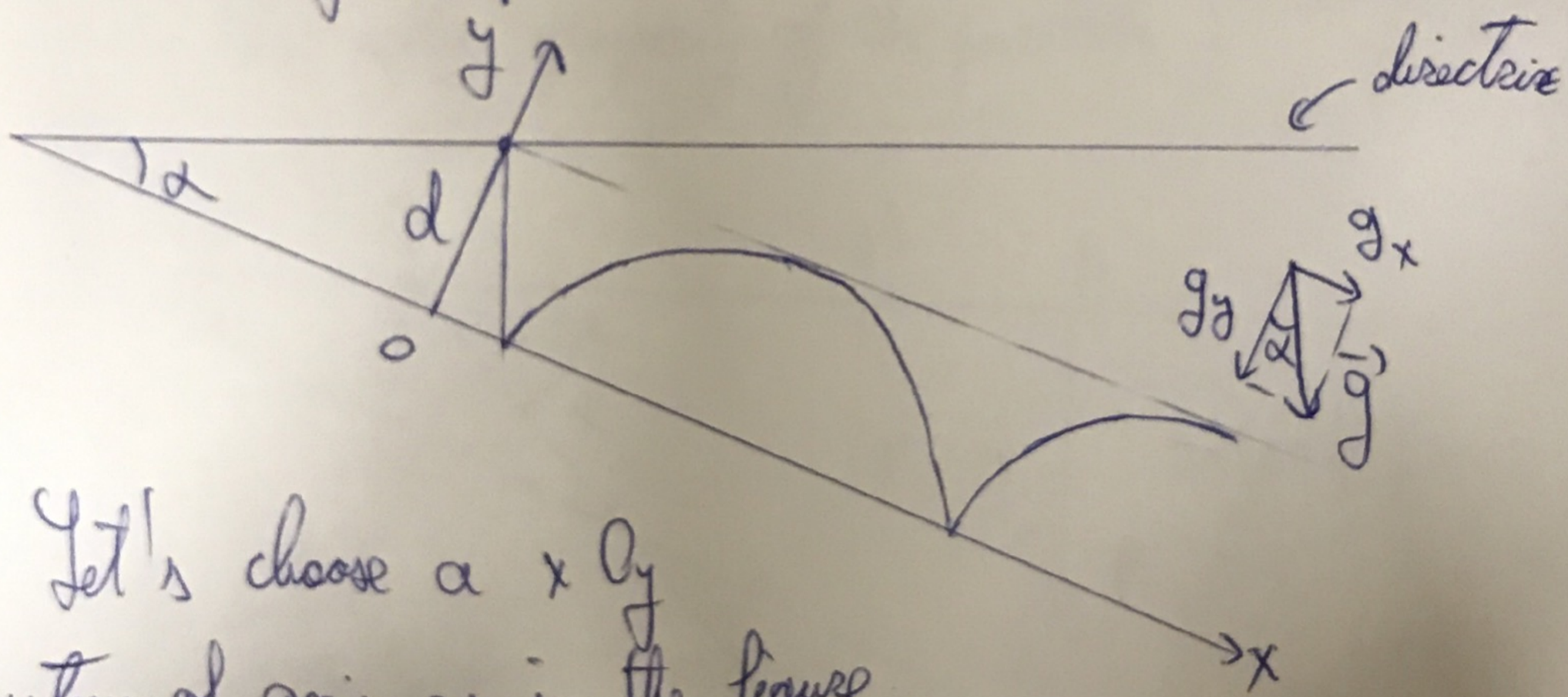
and maintain the launching speed $v$, the directrix will move upward by $\Delta y$.

$$y_d' = \frac{v^2}{2g} + \Delta y = \frac{E_c + E_{pg}}{2g} = \frac{E_{tot}}{2g} \quad (5)$$

(5) $\Leftrightarrow$ The position of the directrix depends only of the total energy of the body.

In our problem the total mechanical energy is conserved.

(5) $\rightarrow$ All of the parabolas have the same directrix,



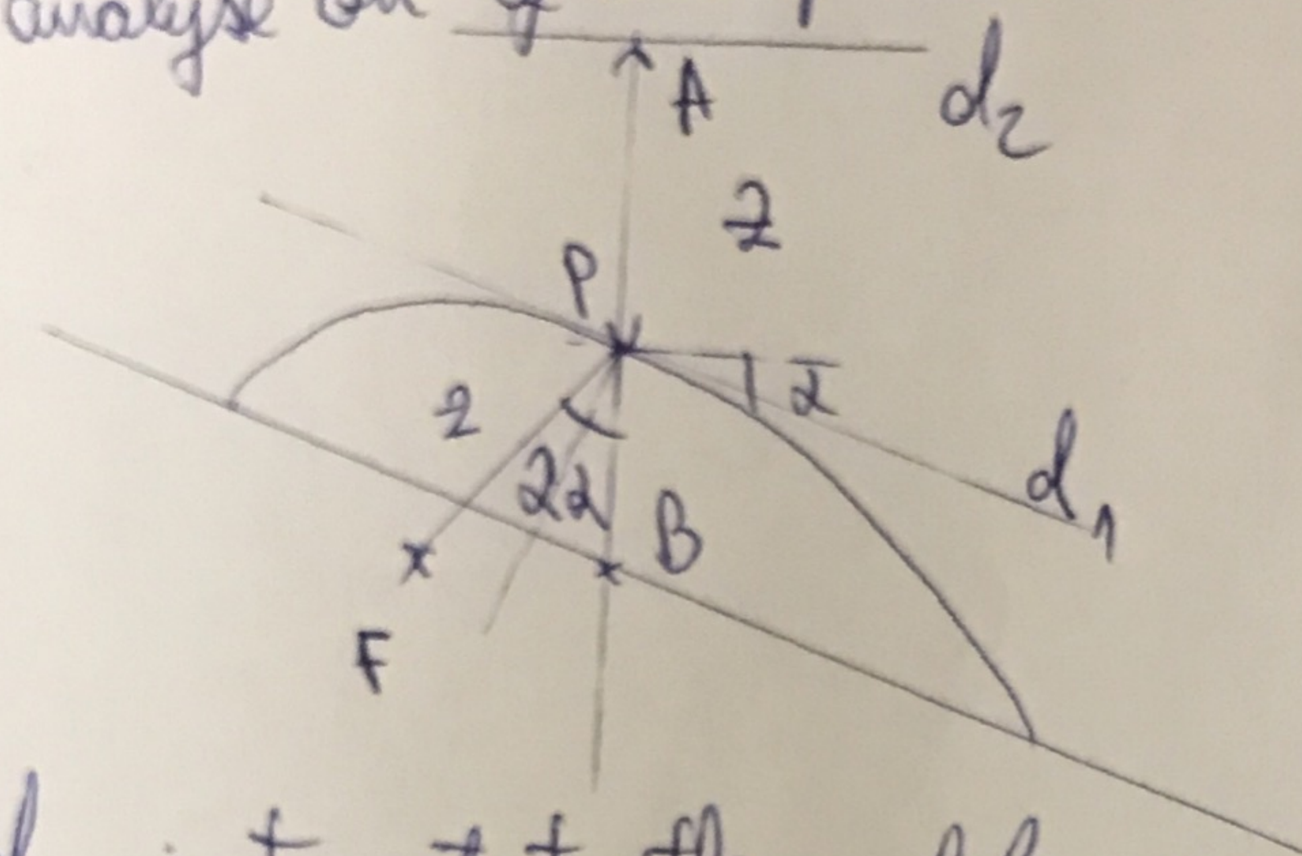

Let's choose a $xOy$ system of axis as in the figure.

$$y(t) = d - g_y \frac{t^2}{2} = d - \frac{g t^2}{2} \cos\alpha, \text{ for } t \leq t_1 \text{ for the first collision with the incline)}$$

Because the colisions are perfectly elastic the motion of ball will be periodic in $y(t)$, reaching its maximum ~~for~~ at $d = y_{max}$ for every segment of parabola.

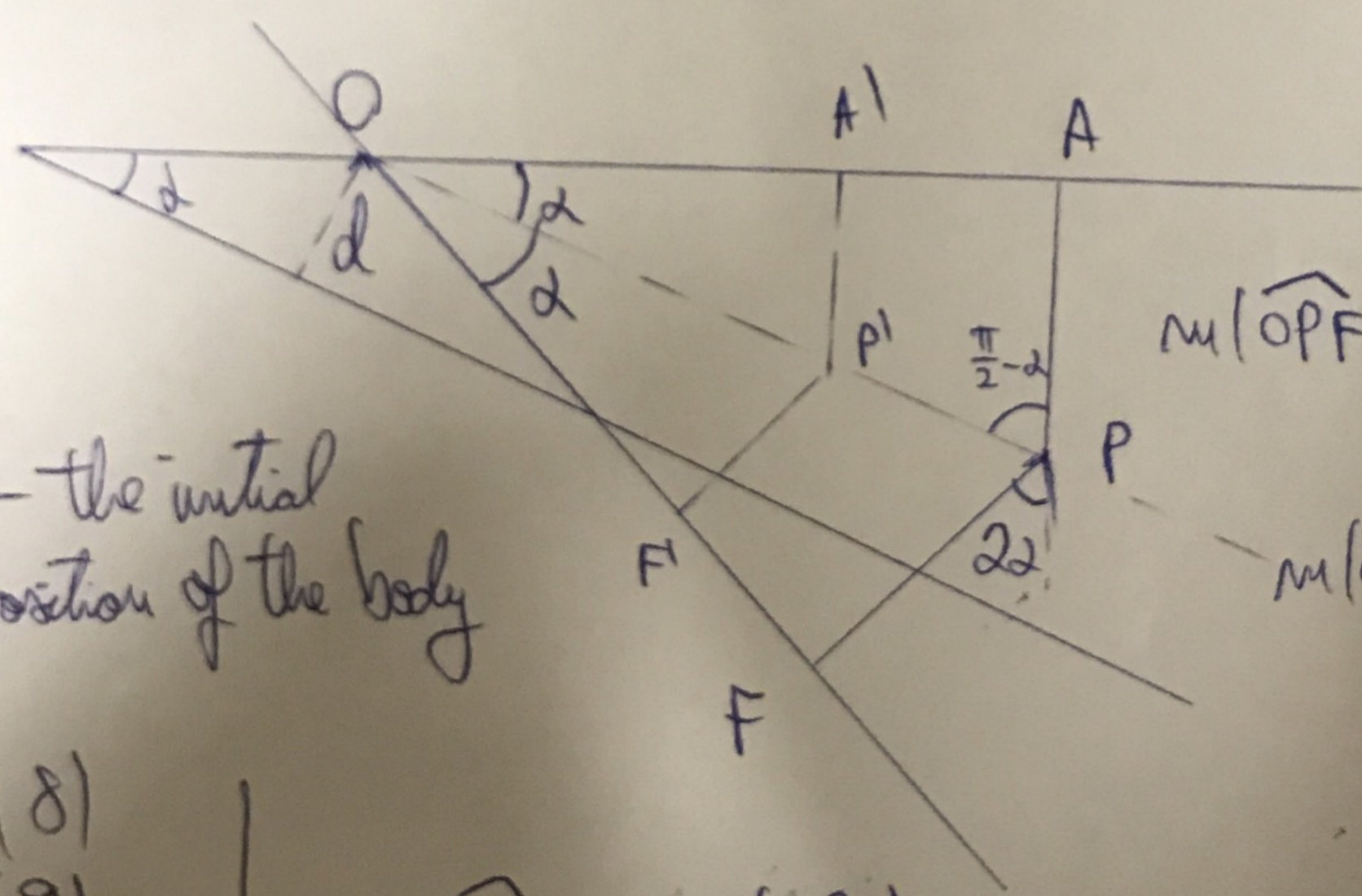
In those points, $\frac{dy}{dt} = 0$ (6)

Let's analyse one of those points.




$AP = FP$ (8)

(6) => $d_1$ is tangent to the parabola
The optic property of the parabola $\Big\}$ => $m(\widehat{FPB}) = 2\alpha$ (7)




$m(\widehat{OPF}) = \pi - 2\alpha - (\frac{\pi}{2} - \alpha)$
$= \frac{\pi}{2} - \alpha \Rightarrow$
$m(\widehat{OPF}) = m(\widehat{OPA}) \rightarrow$ (9)

O – the initial position of the body

(8)
(9)
OP = OP...
$\Big\}$ => $m(\widehat{FOP}) = m(\widehat{POA}) = \alpha$

Because any F' focus can be determined by two points, A' and P' as shown, we conclude that all of the foci ~~the~~ are on a line of slope $2\alpha$, as in the figure.