Problem 3. Shape created by foci points of a bouncing ball

Focus equation

To begin with the problem, it is necessary to find the foci in function of the problem data, the focus of a given parabola $y = a(x-x_0)^2 + b(x-x_0) + c$ can be found using a simple optics fact which says that a beam of light reflected in the insides of a parabola shaped mirror would end in the focus, for that derivation the next figure is necessary.

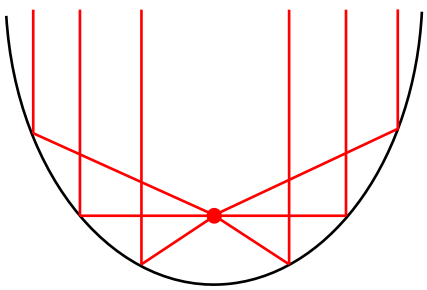

Figure 1. Parabolic mirror reflecting beams of light.

The derivative of the parabola is

$$\frac{dy}{dx} = \tan\theta = 2ax + b - 2ax_0$$

From the figure one can see that by symmetry the x coordinate of the focus is

$$x = -\frac{b}{2a} + x_0$$

The y coordinate of the focus is in the place in which the angle of the point is $\pm 45º$, so.

$$2ax + b - 2ax_0 = \pm 1$$

$$\Rightarrow x = \frac{\pm 1 - b}{2a} + x_0$$

Replacing the last relation into the parabola equation.

$$y = a\left(\frac{\pm 1 - b}{2a}\right)^2 + b\left(\frac{\pm 1 - b}{2a}\right) + c$$

$$\Rightarrow y = \frac{1 + b^2 \mp 2b}{4a} + \frac{\pm b - b^2}{2a} + c$$

$$y = \frac{1 - b^2}{4a} + c$$

We already know the parabola equation for a projectile is:

$$y = \frac{-g(x-x_0)^2}{2v_0^2\cos^2\theta} + (x-x_0)\tan\theta + y_0$$

Using the last relation to obtain the focus point

$$f = \frac{1}{2g}\left(2gx_0 + 2v_xv_y, 2gy_0 + v_y^2 - v_x^2\right) \quad (1)$$

Now we know the variables that define the foci, these variables v_x, v_y, x_0 and y_0 are n dependant where n is the number of bounces between the ball and the plane.

Variables in function of n

The problem can be simplified rotating the coordinate system.

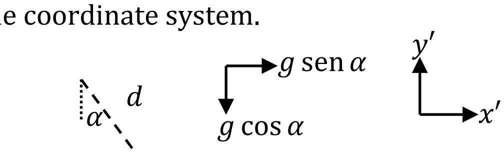

Figure 2. Rotated coordinate system.

Since the y' acceleration is constant trough the x' coordinate the amount of time Δt taken between each bounce is constant.

$$d\cos\alpha = \frac{g\cos\alpha}{2}\left(\frac{\Delta t}{2}\right)^2$$

$$\Rightarrow \Delta t = 2\sqrt{\frac{2d}{g}}$$

Using the last argument, the y' component of velocity right after colliding is the same in each bounce.

$$v_{y'} = g\frac{\Delta t}{2}\cos\alpha = \sqrt{2dg}\cos\alpha$$

The x' component of velocity increases in each bounce.

$$v_{x'} = g\Delta t\left(n - \frac{1}{2}\right)\sin\alpha$$

$$\Rightarrow v_{x'} = (2n-1)\sqrt{2gd}\sin\alpha$$

The x and y components of velocity can be derived geometrically from the previous relations, for the y direction.

$$v_y = v_{y'}\cos\alpha - v_{x'}\sin\alpha$$

$$\Rightarrow v_y = \sqrt{2dg}(1 - 2n\sin^2\alpha) \qquad (2)$$

The velocity in the x direction.

$$v_x = v_{y'}\sin\alpha + v_{x'}\cos\alpha$$

$$\Rightarrow v_x = 2n\sqrt{2dg}\cos\alpha\sin\alpha \qquad (3)$$

The distance in the x' direction can be obtained from the constant acceleration displacement equation.

$$s = \frac{g\sin\alpha}{2}\Delta t^2(n-1)^2 + v_{1x'}\Delta t(n-1)$$

$$\Rightarrow s = 4d(n-1)n\sin\alpha$$

The last is the component along the plane surface, its projection in y can be used to determine y_0 assuming that the point (0,0) is the one in which the ball started to fall from rest.

$$y_0 = -(s\sin\alpha + d)$$

$$y_0 = -d[4(n-1)n\sin^2\alpha + 1] \qquad (4)$$

Similarly, for x_0.

$$x_0 = 4d(n-1)n\sin\alpha\cos\alpha \qquad (5)$$

Replacing equations (2), (3), (4) and (5) in equation (1).

$$f = 2d\sin 2\alpha\,(n^2\cos 2\alpha\,, -n^2\sin 2\alpha)$$

To derive the shape, it is necessary to find n in terms of its x coordinate.

$$x = n^2\cos 2\alpha$$

$$n^2 = \frac{x}{\cos 2\alpha}$$

Replacing the last relation into y.

$$y = -n^2\sin 2\alpha$$

$$\boxed{y = -x\tan 2\alpha}$$

This result applies for the assumption of the origin (0,0) being in the initial moment when the ball was released from rest (the plane equation become $y = -x\tan\alpha - d$), the parameter d is implicit and the transformation used to move the equation of the plane should be the same as the one used to move the foci line since this shape must have the initial trajectory point. Even if the expression become undefined at $\alpha = 45º$ the limiting case stands for a vertical line $x = constant$. The shape is depicted in the next GeoGebra figure for $\alpha = 30º$, the scaling $y_s = \frac{y}{d}$ and $x_s = \frac{x}{d}$ for the plane function and for each parabola is used.

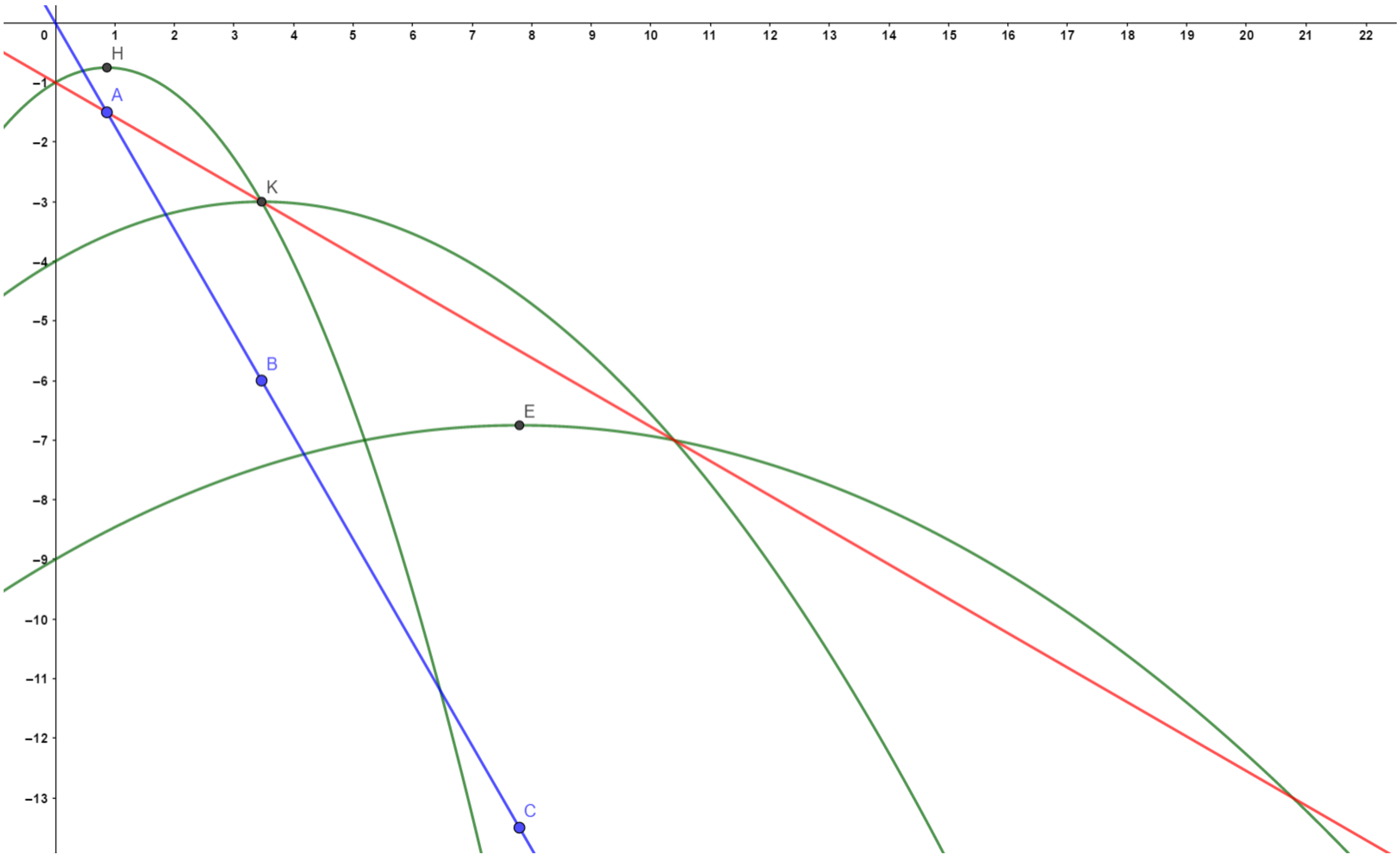

Figure 3. Ball trajectory for four bounces represented as the pieces of green parabola at the right side of the plane (red line), the blue line represents the line in which the foci of every parabola lies, the foci (blue dots) and each maximum (gray dots) are depicted too.