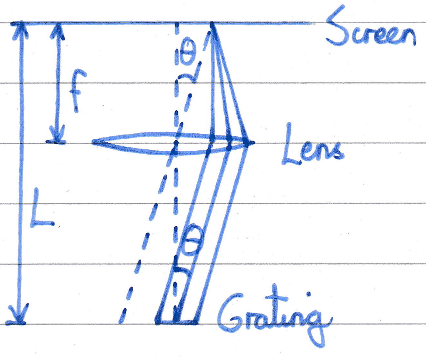
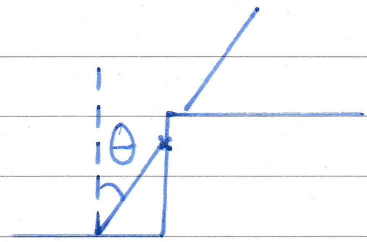
Physics Cup Problem 5

Method 1

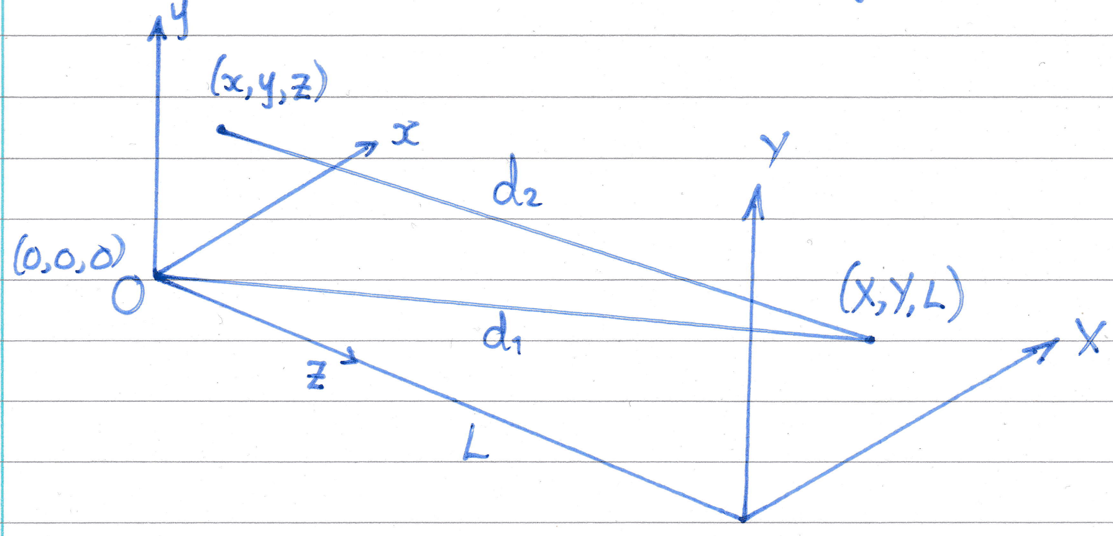


The effect of the lens is to map directions for which the reflected rays interfere constructively onto points in the focal plane. We therefore start by finding the angular intensity distribution.

We will assume that all the diffracted light makes a small angle to the z-axis, so the amount of reflected light blocked by the corners of the grating is negligible. This assumption will be justified later.



First, we need a formula for the path difference between a general point on the grating and the origin O, where $x = y = 0$.



Path difference before reflection: $(L - z) - L = -z$

Path difference after reflection: $d_2 - d_1$

$$= \sqrt{(X-x)^2 + (Y-y)^2 + (L-z)^2} - \sqrt{X^2 + Y^2 + Z^2}$$

$$\approx \sqrt{X^2+Y^2+L^2-2(xX+yY+zL)} - \sqrt{X^2+Y^2+L^2} \qquad (\text{as } x,y,z \ll L)$$

$$= \sqrt{X^2+Y^2+L^2}\left[\sqrt{1-2\left(\frac{xX+yY+zL}{X^2+Y^2+L^2}\right)} - 1\right]$$

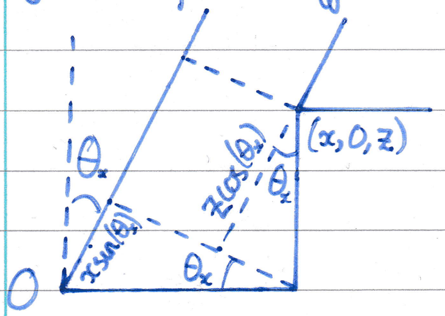
$$\approx -\left(\frac{xX+yY+zL}{\sqrt{X^2+Y^2+L^2}}\right) \qquad (\text{as } x,y,z,X,Y \ll L)$$

$$\approx -\left(\frac{xX}{L}+\frac{yY}{L}+z\right) \qquad (\text{as } X,Y \ll L)$$

$$\approx -(x\theta_x + y\theta_y + z), \qquad (\text{small-angle approximation})$$

where $\sin(\theta_x) = \dfrac{X}{\sqrt{X^2+L^2}} \approx \dfrac{X}{L}$ and $\sin(\theta_y) = \dfrac{Y}{\sqrt{Y^2+L^2}} \approx \dfrac{Y}{L}$.

By setting $\theta_y = 0$ and $y = 0$ we can see the geometrical meaning of this path difference:



Path difference after reflection $= -[x\sin(\theta_x) + z\cos(\theta_x)]$

$$\approx -(x\theta_x + z).$$

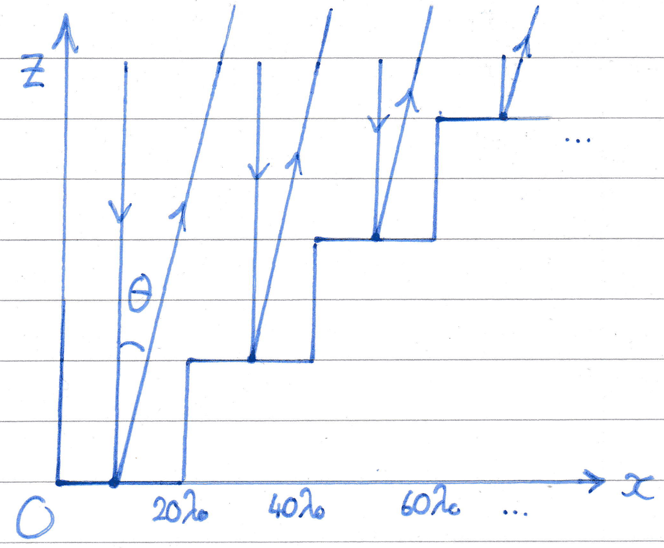
Here we have used the small-angle approximation. This will also allow us to neglect any amplitude variation due to spherical wavefront spreading, and lets us set to Fresnel obliquity factor to 1.

Total path difference $= -(x\theta_x + y\theta_y + 2z)$

Huygens' Principle gives:

$$a(\theta_x,\theta_y) \propto \int_{\text{grating}} e^{ik(x\theta_x + y\theta_y + 2z)}\, dx\, dy \qquad \text{where } k = \frac{2\pi}{\lambda}.$$

$$z = 20\lambda_0 \left\lfloor \frac{x}{20\lambda_0} \right\rfloor + \lambda_0 \left\lfloor \frac{y}{20\lambda_0} \right\rfloor$$



When x increases by $20\lambda_0$,
z increases by $20\lambda_0$.

When y increases by $20\lambda_0$,
z increases by λ_0.

So:

$$a \propto \sum_{n=-250}^{249} \sum_{m=-250}^{249} \int_{x=20\lambda_0 n}^{20\lambda_0(n+1)} \int_{y=20\lambda_0 m}^{20\lambda_0(m+1)} e^{ik(x\theta_x + y\theta_y)} e^{ik(40\lambda_0 n + 2\lambda_0 m)} \, dx \, dy$$

$$= \left(\sum_{n=-250}^{249} e^{i(40\lambda_0 k)n} \int_{20\lambda_0 n}^{20\lambda_0(n+1)} e^{ikx\theta_x} \, dx \right) \left(\sum_{m=-250}^{249} e^{i(2\lambda_0 k)m} \int_{20\lambda_0 m}^{20\lambda_0(m+1)} e^{iky\theta_y} \, dy \right)$$

We calculate the two integrals first:

$$\int_{20\lambda_0 n}^{20\lambda_0(n+1)} e^{ikx\theta_x} \, dx = \left[\frac{e^{ikx\theta_x}}{ik\theta_x} \right]_{20\lambda_0 n}^{20\lambda_0(n+1)}$$

$$= \frac{e^{ik\theta_x[20\lambda_0(n+1)]} - e^{ik\theta_x[20\lambda_0 n]}}{ik\theta_x}$$

$$= e^{i(20\lambda_0 k\theta_x)n} e^{i(10\lambda_0 k\theta_x)} \left[\frac{e^{i(10\lambda_0 k\theta_x)} - e^{-i(10\lambda_0 k\theta_x)}}{ik\theta_x} \right]$$

$$= 20\lambda_0 e^{i(20\lambda_0 k\theta_x)n} e^{i(10\lambda_0 k\theta_x)} \operatorname{sinc}(10\lambda_0 k\theta_x)$$

where $\operatorname{sinc}(x) = \frac{\sin(x)}{x}$. The y integral is calculated the same way, so:

$$a \propto e^{i(10\lambda_0 k\theta_x)} e^{i(10\lambda_0 k\theta_y)} \operatorname{sinc}(10\lambda_0 k\theta_x) \operatorname{sinc}(10\lambda_0 k\theta_y) \left(\sum_{n=-250}^{249} e^{i(40\lambda_0 k + 20\lambda_0 k\theta_x)n} \right) \times \hookleftarrow$$

$$\left(\sum_{m=-250}^{249} e^{i(2\lambda_0 k + 20\lambda_0 k\theta_y)m} \right).$$

4 We now need to be able to evaluate sums of the form:

$$\sum_{n=-250}^{249} e^{ni\varphi} = \sum_{m=0}^{499} e^{(m-250)i\varphi}$$

$$= e^{-250i\varphi} \sum_{m=0}^{499} e^{mi\varphi}$$

$$= e^{-250i\varphi} \left(\frac{e^{500i\varphi} - 1}{e^{i\varphi} - 1} \right)$$

$$= \left(\frac{e^{250i\varphi} - e^{-250i\varphi}}{2i} \right) \left(\frac{2i}{e^{i\varphi} - 1} \right)$$

$$= e^{-\frac{i\varphi}{2}} \frac{\sin\left(\frac{500}{2}\varphi\right)}{\sin\left(\frac{1}{2}\varphi\right)}.$$

Hence:

$$a \propto e^{-21i\lambda_0 k} \operatorname{sinc}(10\lambda_0 k\theta_x) \operatorname{sinc}(10\lambda_0 k\theta_x) \frac{\sin(500 \cdot 10\lambda_0 k[\theta_x + 2])}{\sin(10\lambda_0 k[\theta_x + 2])} \times$$

$$\frac{\sin(500\lambda_0 k[10\theta_y + 1])}{\sin(\lambda_0 k[10\theta_y + 1])}.$$

The intensity is:

$$I(\theta_x, \theta_y) \propto |a(\theta_x, \theta_y)|^2$$

$$\therefore I = \underbrace{K \operatorname{sinc}^2\left(20\pi \frac{\lambda_0}{\lambda}\theta_x\right) \operatorname{sinc}^2\left(20\pi \frac{\lambda_0}{\lambda}\theta_y\right)}_{=I_1} \underbrace{\frac{\sin^2\left(500 \cdot 20\pi \frac{\lambda_0}{\lambda}[\theta_x + 2]\right)}{500^2 \sin^2\left(20\pi \frac{\lambda_0}{\lambda}[\theta_x + 2]\right)}}_{=I_2} \underbrace{\frac{\sin^2\left(500 \cdot 2\pi \frac{\lambda_0}{\lambda}[10\theta_y + 1]\right)}{500^2 \sin^2\left(2\pi \frac{\lambda_0}{\lambda}[10\theta_y + 1]\right)}}_{=I_3}$$

In this expression, I_1 is the intensity pattern created by a square diffracting aperture with side length $20\lambda_0$. The product $I_2 I_3$ is the intensity pattern created by a square lattice of 500×500 holes spaced $20\lambda_0$ apart (this pattern is shifted, so it isn't centred at $\theta_x = \theta_y = 0$).

5 The $I_2 I_3$ term gives a lattice of very sharp bright peaks for each wavelength. The brightest maximum will be the peak which gives the largest value of I_1; normally this means the peak which is closest to $\theta_x = \theta_y = 0$, where I_1 has a broad central maximum.

To get a maximum of I_2 we need:

$$20\pi\frac{\lambda_0}{\lambda}(\theta_x + 2) = N\pi \implies \theta_x = \frac{N}{20}\Lambda - 2 \quad (1) \qquad \text{where } \Lambda = \frac{\lambda}{\lambda_0} \text{ and } N \in \mathbb{Z}.$$

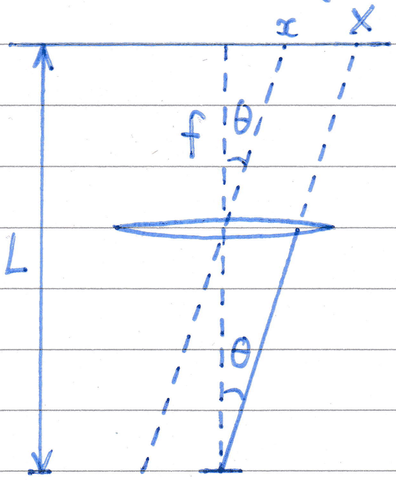
To get a maximum of I_3 we need:

$$2\pi\frac{\lambda_0}{\lambda}(10\theta_y + 1) = M\pi \implies \theta_y = \frac{M}{20}\Lambda - \frac{1}{10} \quad (2) \qquad \text{where } M \in \mathbb{Z}.$$

To find the locus of bright points as Λ varies, we eliminate Λ from these equations:

$$\left.\begin{aligned} M\theta_x - N\theta_y &= \frac{N}{10} - 2M \\ \Lambda &= \frac{(20\theta_y + 2)}{M} \end{aligned}\right\}$$

To convert from angles (θ_x, θ_y) to positions on the screen, we use the fact that the ray through the centre of the lens is undeviated.



Without the lens, a reflected ray would reach a point (X, Y) on the screen. Let (x, y) be the point reached by the ray when the lens is added. Using similar triangles:

$$\frac{x}{f} = \frac{X}{L} = \theta_x, \qquad \frac{y}{f} = \frac{Y}{L} = \theta_y.$$

If x and y are measured in cm, and $f = 20\,\text{cm}$:

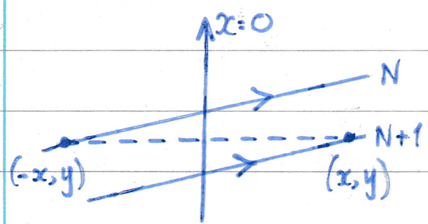
6 $\left.\begin{aligned} &Mx - Ny = 2N - 40M \\ &\Lambda = \frac{y+2}{M} \end{aligned}\right\}$

For each value of Λ in $[\frac{4}{5}, \frac{7}{5}]$, the peak with y-coordinate closest to zero is $M=2$. The only exception is $\Lambda = \frac{4}{5}$, which has $M=2$ peaks at $y = -\frac{1}{50}$ and $M=3$ peaks at $y = +\frac{1}{50}$. In reality the lower peaks with $M=2$ will be slightly brighter as no light is blocked by the corners of the grating. Setting $M=2$:

$$\left.\begin{aligned} &2x - Ny = 2N - 80 \\ &\Lambda = \tfrac{1}{2}y + 1 \end{aligned}\right\}$$

Since $\Lambda \in [\frac{4}{5}, \frac{7}{5}]$ we find: $-\frac{2}{5} \leq y \leq \frac{4}{5}$.

For a given value of Λ, there are infinitely many peaks with the same y-value but different x-values, each on a line with a different value of N. It will be useful to know which peak has x-coordinate closest to zero.



Suppose that for a particular value of Λ, the peak closest to zero lies on the line:

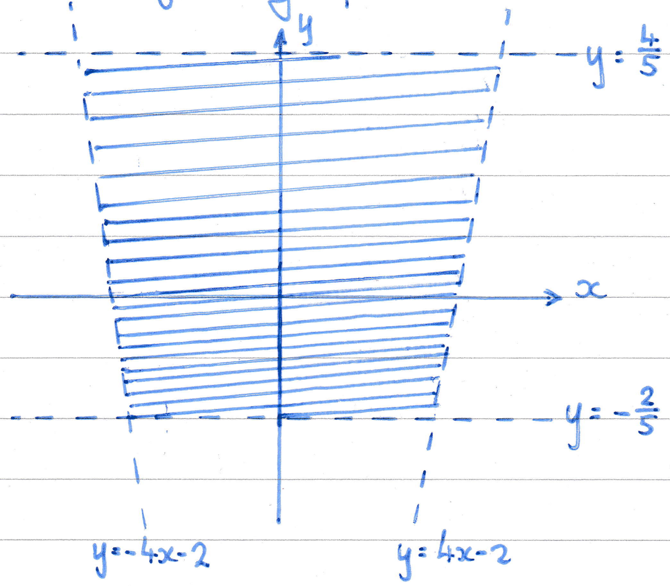
$$2x - (N+1)y = 2(N+1) - 80$$

As Λ increases, the closest peak will move upwards along this line until eventually the closest peak will change to the next line up: $2x - Ny = 2N - 80$.

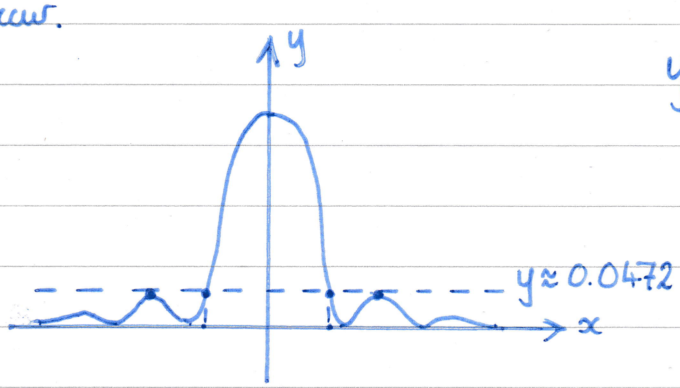
This occurs when the peaks on each line are equidistant from $x=0$, so their coordinates are (x,y) and $(-x,y)$ for some x. This gives:

$$\left.\begin{aligned} &2x - (N+1)y = 2(N+1) - 80 \\ &-2x - Ny = 2N - 80 \end{aligned}\right\} \implies 4x - y = 2$$

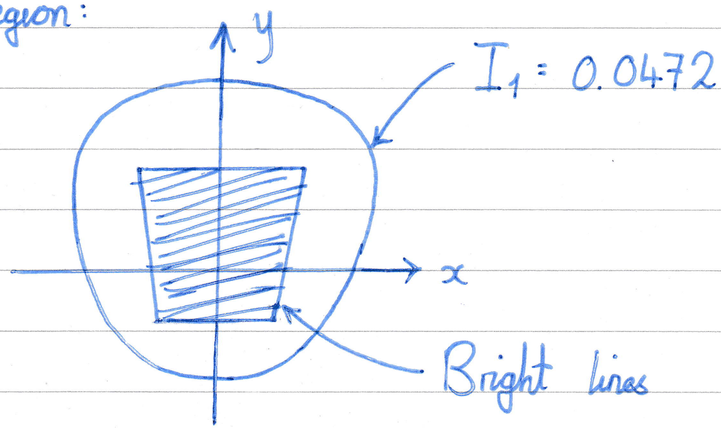
7 So far, we have worked out that the peaks closest to the origin form the following pattern:

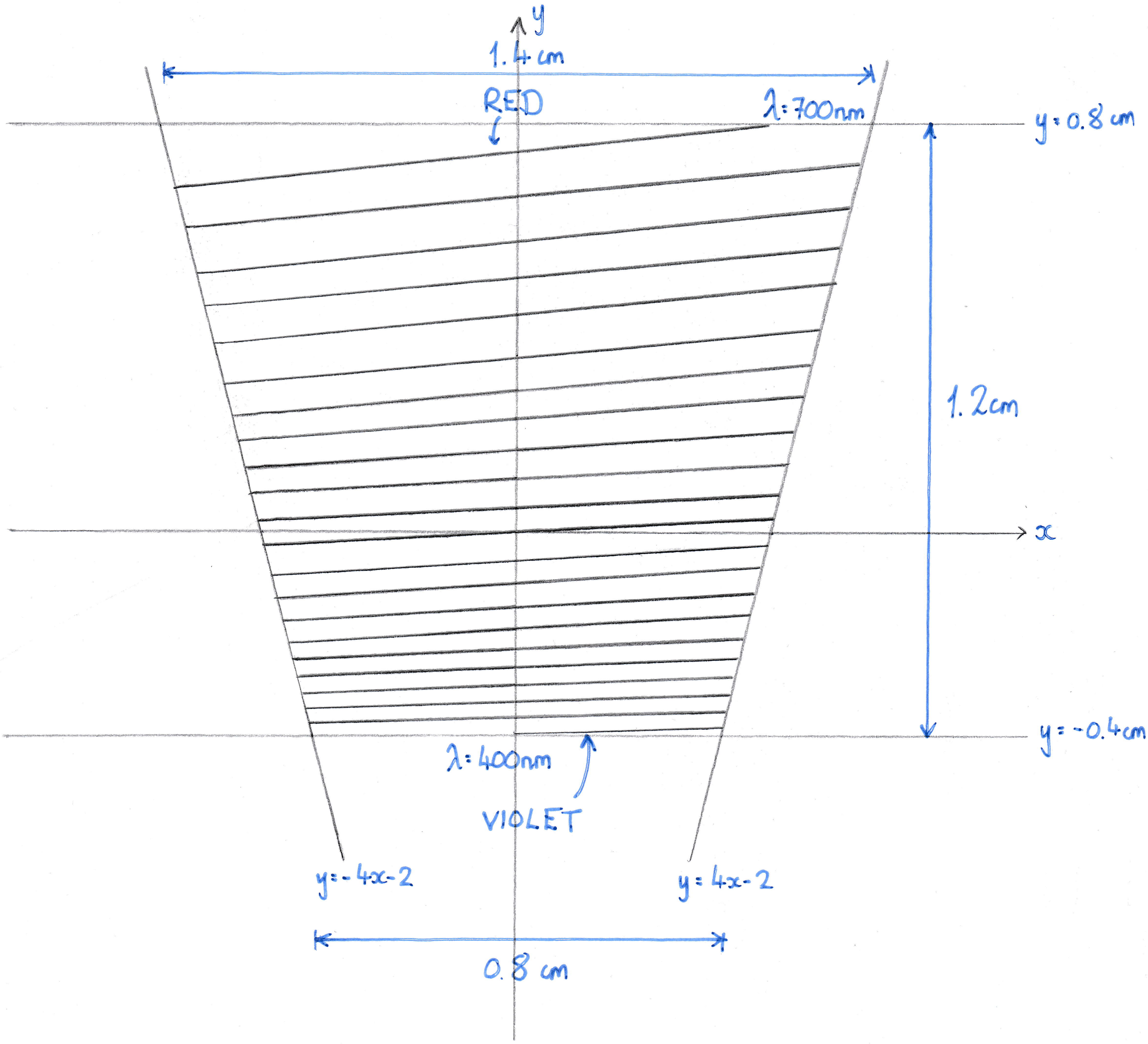


It is possible that these points may not be the brightest peaks for each wavelength - this would occur if they were far enough from the origin to be near a minimum of I_1. However, this does not occur.



As long as a peak falls inside the region in which $I_1 > 0.0472$, no peak outside this region can be brighter. It can be checked with a computer that the bright lines drawn above all fall within this region:




y
1.4 cm
RED
λ=700nm
y=0.8 cm
1.2cm
x
y=-0.4cm
λ=400nm
VIOLET
y=-4x-2
y=4x-2
0.8 cm

Method 2

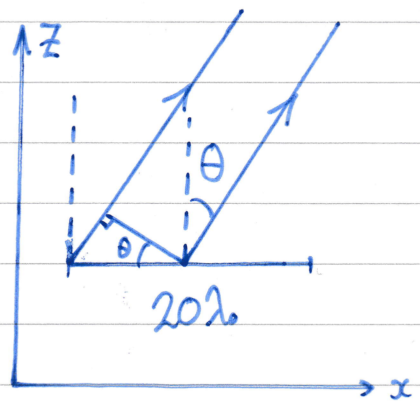
We can also derive the positions of the bright maxima without using integrals to calculate the full intensity distribution. As before, we assume that a negligible fraction of the reflected light is blocked by the corners of the grating.

There will be a bright maximum in a given direction if light from all parts of the grating interfere constructively.

We need light from each $20\lambda_0 \times 20\lambda_0$ square taken individually to interfere constructively. Otherwise, if a certain direction gives no light from each square, then that direction must give no light for the whole grating. This means that any bright maxima must lie within a roughly circular region of angles less than about:

$\theta = \sin^{-1}\left(\frac{\Lambda}{20}\right)$ where $\Lambda = \frac{\lambda}{\lambda_0}$.

This is the angle in the xz-plane or xy-plane which gives the first minimum, as light from each half of the square interferes destructively with light from the other half:



Destructive interference when:

Path difference $= \frac{1}{2}\lambda$

$\therefore\ 10\lambda_0 \sin(\theta) = \frac{1}{2}\lambda$

$\therefore\ \sin(\theta) = \frac{\Lambda}{20}$.

We also need light from each square to interfere constructively with light from the corresponding point on all the other squares. This amounts to the requirement that light reflecting off the following points interferes constructively:

10 $(x,y,z) = (20\lambda_0 n,\ 20\lambda_0 m,\ 20\lambda_0 n + \lambda_0 m)$ $\quad n, m \in \mathbb{Z}$

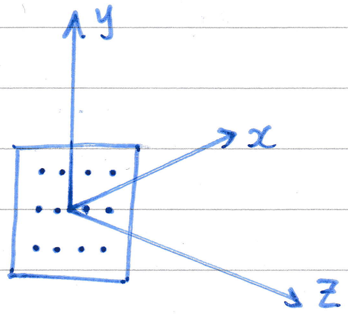
On p2 we showed that the path difference between these points, taking into account the path difference before and after reflection, is:

$$-(x\theta_x + y\theta_z + 2z)$$

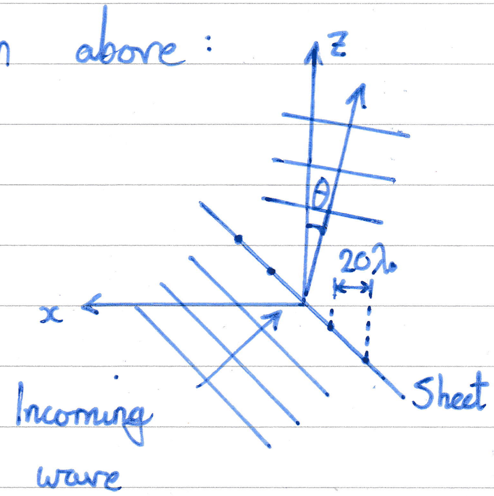
This is the same path difference that would result from light being emitted in phase from the points:

$(x,y,z) = (20\lambda_0 n,\ 20\lambda_0 m,\ 40\lambda_0 n + 2\lambda_0 m)$ $\quad n, m \in \mathbb{Z}$

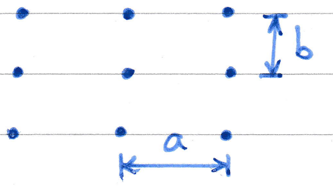
where the z-coordinate has been doubled. These points all lie on the plane $20x + y - 10z = 0$. This suggests we treat this as interference from a lattice of holes in a rotated sheet:



From above:



The spacing between holes is:



$$a = \sqrt{(20\lambda_0)^2 + (40\lambda_0)^2} = 20\sqrt{5}\,\lambda_0$$
$$b = \sqrt{(20\lambda_0)^2 + (2\lambda_0)^2} = \sqrt{404}\,\lambda_0 .$$

If the sheet wasn't rotated but was perpendicular to the z-axis, which directions would give maxima?

11 The path difference derived on p2 with $z=0$ and without the small-angle approximation is:

$$-[x\sin(\theta_x) + y\sin(\theta_y)].$$

This must be a multiple of λ for all x and y on the lattice:

$$x = an = 20\sqrt{5}\lambda_0 n, \qquad y = bm = \sqrt{404}\lambda_0 m, \qquad n, m \in \mathbb{Z}.$$

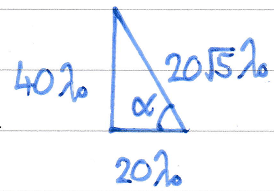
This means:

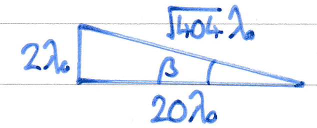
$$\sin(\theta_x) = \frac{N}{20\sqrt{5}}\Lambda, \qquad \sin(\theta_y) = \frac{M}{\sqrt{404}}\Lambda, \qquad N, M \in \mathbb{Z}.$$

When the sheet is rotated into the plane $20x + y - 10z = 0$, it is rotated through $-\alpha$ about the x-axis and $-\beta$ about the y-axis, where:

$$\tan(\alpha) = 2$$

$$\tan(\beta) = \frac{1}{10}.$$





So, the maxima will now be at:

$$\sin(\theta_x + \alpha) = \frac{N}{20\sqrt{5}}\Lambda \qquad \sin(\theta_y + \beta) = \frac{M}{\sqrt{404}}\Lambda$$

$$\therefore \frac{\sin(\theta_x)}{\sqrt{5}} + \frac{2\cos(\theta_x)}{\sqrt{5}} = \frac{N}{20\sqrt{5}}\Lambda \qquad \sin(\theta_y)\cdot\frac{20}{\sqrt{404}} + \cos(\theta_y)\cdot\frac{2}{\sqrt{404}} = \frac{M}{\sqrt{404}}\Lambda$$

$$\therefore \sin(\theta_x) + 2\cos(\theta_x) = \frac{N}{20}\Lambda \qquad \sin(\theta_y) + \frac{1}{10}\cos(\theta_y) = \frac{M}{20}\Lambda$$

Using the small-angle approximation gives:

12 $\theta_x = \frac{N}{20}\Lambda - 2$, $\theta_y = \frac{M}{20}\Lambda - \frac{1}{10}$.

These are equations ① and ② on p5. The remainder of the solution proceeds as before from this point on p5 onwards and leads to the sketch on p9.