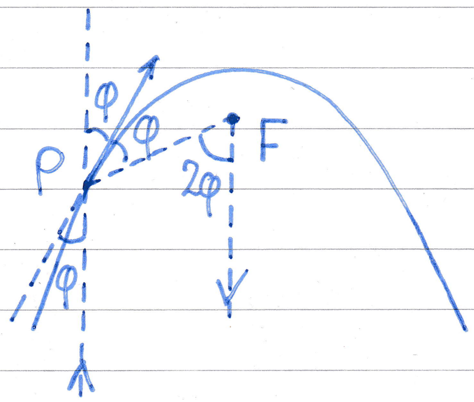
Physics Cup Problem 3



We need two properties of the parabolic trajectories of particles in a uniform gravitational field.

The field will always be assumed to act vertically downwards.

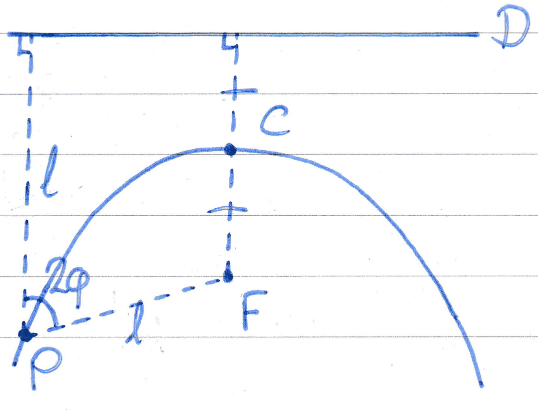
Claim 1: The velocity vector of a particle at point P makes equal angles φ to the vertical and to the line PF, where F is the focus of the parabola.

Proof:
This follows from the reflection property of parabolas. A ray of light incident on point P from below must reflect through F, so PF and the vertical must make equal angles to the normal at P. But this means they make equal angles with the velocity vector at P, which is tangential. ∎

Claim 2: If the particle's speed at point P is v, then the distance PF from P to the focus and the smallest distance from P to the directrix D are both:

$$l = \frac{v^2}{2g}.$$

Proof:



Let the highest point on the trajectory be C.

At P, let the particle's velocity make an angle φ with the vertical and θ with the horizontal, 2 so $\varphi = 90 - \theta$.

By the definition of a parabola, the distances from P to F and P to D are equal - call this distance l. The same is true at C.

The horizontal distance from P to C is:

$$x = l\sin(2\varphi) = l\sin(2\theta)$$

The vertical distance is:

$$y = \underbrace{\frac{l - l\cos(2\varphi)}{2}}_{=|CF|} + l\cos(2\varphi) = \frac{l[1+\cos(2\varphi)]}{2} = \frac{l[1-\cos(2\theta)]}{2} = l\sin^2(\theta)$$

Vertical distance from P to F

Using the trajectory equation:

$$y = \tan(\theta)x - \frac{gx^2}{2v^2\cos^2(\theta)}$$

$$\therefore \; l\sin^2(\theta) = l\tan(\theta)\sin(2\theta) - \frac{gl^2\sin^2(2\theta)}{2v^2\cos^2(\theta)}$$

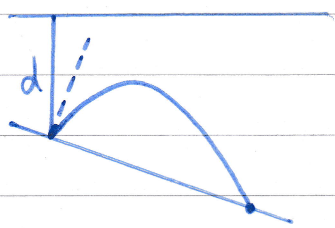
$$\therefore \; \sin^2(\theta) = 2\sin^2(\theta) - \frac{2gl\sin^2(\theta)}{v^2}$$

$$\therefore \; \frac{2gl}{v^2} = 1$$

$$\therefore \; l = \frac{v^2}{2g}. \quad \blacksquare$$

This result shows that the height of the directrix above the trajectory is proportional to the particle's energy at that point.

3 In this problem, all collisions are elastic so the parabolic arcs share a directrix.



After the first bounce, conservation of energy gives:

$$\frac{1}{2}v^2 = gd \Rightarrow l = \frac{v^2}{2g} = d.$$

The common directrix is therefore the horizontal line passing through the ball's initial point.

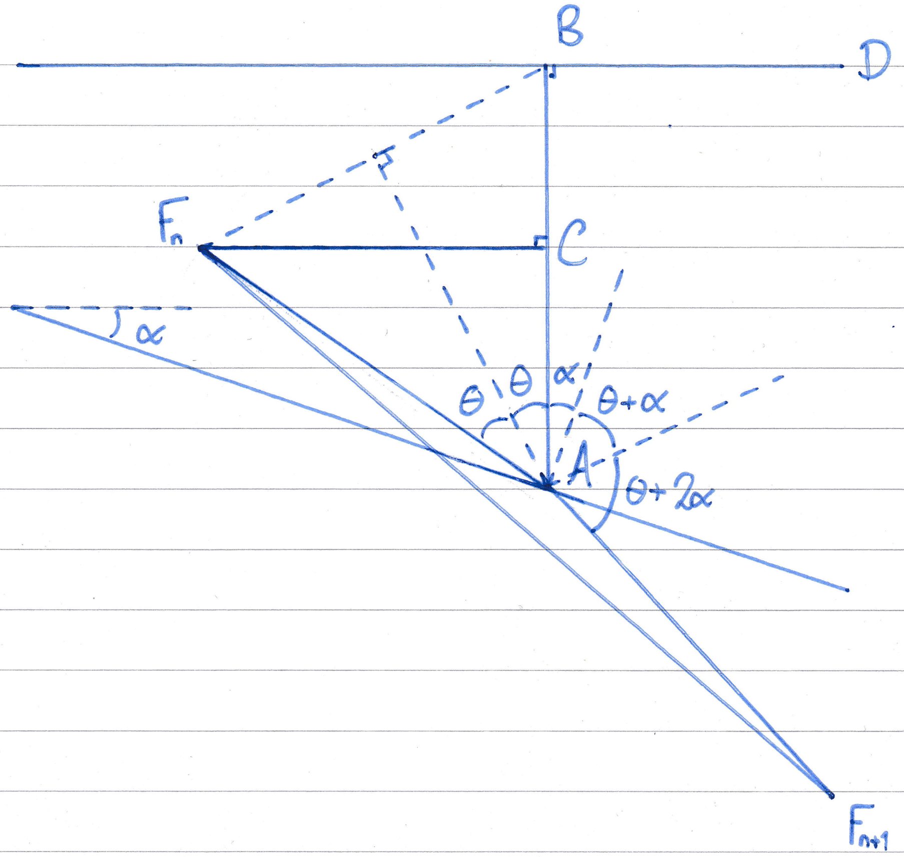
Now for the main result:

Claim 3: The foci of all the parabolic arcs lie on the straight line passing through the ball's initial point, inclined at an angle 2α below the horizontal (sloping downwards in the same direction as the frictionless plane).

Proof:
We use induction.

If we treat the ball's initial vertical fall as part of an infinitely thin parabola then its focus is at the ball's starting point. This lies on the claimed locus and is our base case.

Now suppose the focus of the n^{th} parabola lies on the locus (see next page for diagram). Call this focus F_n and let the lower point where the n^{th} parabola meets the inclined plane be A. Draw in the perpendicular from A to the directrix D – let the foot be B. Also draw the horizontal line through F_n and let its intersection with AB be C. Let the focus of the next parabolic arc (after the ball bounces at A) be F_{n+1}.



Suppose the ball's velocity just before the bounce at A makes an angle θ to both AF_n and AB (here we use Claim 1). Then the velocity after the bounce makes an angle $\theta+\alpha$ to the normal through A (law of reflection). This means $\angle BAF_{n+1} = 2\theta + 4\alpha$ (Claim 1 again).

$\angle F_n A F_{n+1} = 360 - \angle F_n AB - \angle BAF_{n+1} = 360 - 4\theta - 4\alpha$

Since the $(n+1)^{th}$ parabola has the same directrix D (Claim 2):

$|AF_{n+1}| = |AB| = |AF_n|$

so $\Delta F_n A F_{n+1}$ is isosceles. Hence:

$\angle AF_n F_{n+1} = 90 - \frac{1}{2}\angle F_n A F_{n+1} = 2\theta + 2\alpha - 90$

Finally:

$$\begin{aligned}\angle CF_n F_{n+1} &= \angle CF_n A + \angle AF_n F_{n+1} \\ &= 90 - \angle F_n AC + 2\theta + 2\alpha - 90\end{aligned}$$

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$$= -\angle F_nAB + 2\theta + 2\alpha$$
$$= 2\alpha .$$

So, F_nF_{n+1} is at an angle 2α below the horizontal, which means F_{n+1} lies on the claimed locus.

The claim is theregore true by induction. ∎