Problem No 5

Without losing generality we assume that light beams perpendicular to OXY plane, passing through a lens are focused at $x,y=0$ (on the screen, so $z=50\,\text{cm}$). Light of each wavelength can be analyzed separately. Let's consider wavelength λ . Let's choose point on the screen with coordinates (x,y,z) , where $z=50\,\text{cm}$. Let $\hat{v}$ denote unit vector defined below:

$$\hat{v}=\frac{1}{\sqrt{x^2+y^2+z^2}}\begin{bmatrix}x\\y\\z\end{bmatrix}$$

After reflection of light beam from the grating, we can treat it as superposition of spherical waves (according to Huygens principle). Spherical waves can be treated as light beams propagating in many directions. Light beams from considered spherical waves reach point (x,y,z) on the screen, if they are parallel to $\hat{v}$. Let ϕ_1 denote phase shift between light beams parallel to $\hat{v}$ (which is measured when they meet on the screen), that come from points on the grating with coordinates $A=(0,0,0)$ and $B=(-20\lambda_0,0,-20\lambda_0)$. Let $\vec{s_1}$ denote vector $\overrightarrow{AB}$. Phase shift can be calculated by founding difference of optical paths length. We don't know exact optical path length in lens, but we can use Fermat principle. Let Ω denote plane perpendicular to $\hat{v}$, containing point $(0,0,0)$. From Fermat rule, we know, that optical path length of two considered beams measured from intersection with Ω to the point (x,y,z) on the screen is the same. Then we can easily find, that:

$$\phi_1=\frac{2\pi}{\lambda}\left(-\hat{v}\cdot\vec{s_1}+20\lambda_0\right)=2\pi\frac{20\lambda_0}{\lambda}\left(\frac{x+z}{\sqrt{x^2+y^2+z^2}}+1\right)$$

Similarly we can express ϕ_2 . It denotes phase shift between light beams parallel to $\hat{v}$ (which is measured when they meet on the screen), that come from points on the grating with coordinates $A=(0,0,0)$ and $C=(0,-20\lambda_0,-\lambda_0)$:

$$\phi_2=2\pi\frac{20\lambda_0}{\lambda}\left(\frac{y+\frac{1}{20}z}{\sqrt{x^2+y^2+z^2}}+\frac{1}{20}\right)$$

Consider case $|x|,|y|\le\frac{\lambda}{40\lambda_0}z$ (it applies also in reasoning below), than with good accuracy we can write:

$$\phi_1=2\pi\frac{20\lambda_0}{\lambda}\left(\frac{x}{z}+2\right),\phi_2=2\pi\frac{20\lambda_0}{\lambda}\left(\frac{y}{z}+\frac{2}{20}\right)$$

Consider all light beams parallel to $\hat{v}$. When x or y is positive, some of these beams are absorbed (by vertical sides). But if $|x|,|y| \le \frac{\lambda}{40\lambda_0} z$, absorption is negligible. Light beams coming from a single square on the grating, can be replaced by single effective light beam, which has intensity $A\, sinc^2\left(\pi \frac{20\lambda_0 x}{\lambda z}\right) \sin^2\left(\pi \frac{20\lambda_0 y}{\lambda z}\right)$, where A is constant and $sinc\, t = \begin{cases} \sin t / t \text{ for } t \ne 0 \\ 1 \text{ for } t = 0 \end{cases}$. This fact can be easily derived using phasor arithmetic. Superposition of mentioned effective light beams from all squares on the grating has intensity:

$$I(x,y) = A\, sinc^2\left(\pi \frac{20\lambda_0 x}{\lambda z}\right) \sin^2\left(\pi \frac{20\lambda_0 y}{\lambda z}\right) \frac{\sin^2\left(\frac{N\phi_1}{2}\right)}{\sin^2\left(\frac{\phi_1}{2}\right)} \frac{\sin^2\left(\frac{N\phi_2}{2}\right)}{\sin^2\left(\frac{\phi_2}{2}\right)},$$

where $N=500$ is number of squares in each column on the grating. That result also can be derived using phasor arithmetic. Coefficient $\frac{\sin^2\left(\frac{N\phi_1}{2}\right)}{\sin^2\left(\frac{\phi_1}{2}\right)}$ is equal to $\left|\sum_{n=0}^{N-1} \exp(¿\phi_1)\right|^2$.

$$I(x,y) = A\, sinc^2\left(\pi \frac{20\lambda_0 x}{\lambda z}\right) \sin^2\left(\pi \frac{20\lambda_0 y}{\lambda z}\right) \frac{\sin^2\left(\pi N \frac{20\lambda_0}{\lambda}\left(\frac{x}{z}+2\right)\right)}{\sin^2\left(\pi \frac{20\lambda_0}{\lambda}\left(\frac{x}{z}+2\right)\right)} \frac{\sin^2\left(\pi N \frac{20\lambda_0}{\lambda}\left(\frac{y}{z}+\frac{2}{20}\right)\right)}{\sin^2\left(\pi \frac{20\lambda_0}{\lambda}\left(\frac{y}{z}+\frac{2}{20}\right)\right)}$$

In points where function I is undefined, we can defined it as proper limit. Thanks that I becomes continuous. Let Σ denote set of points on the screen with coordinates defined :

$$x = \frac{\lambda z}{20\lambda_0}\left(n_1 - \frac{40\lambda_0}{\lambda}\right), y = \frac{\lambda z}{20\lambda_0}\left(n_2 - \frac{2\lambda_0}{\lambda}\right),$$

for integers n_1 and n_2 . At these points functions $\frac{\sin^2\left(\pi N \frac{20\lambda_0}{\lambda}\left(\frac{x}{z}+2\right)\right)}{\sin^2\left(\pi \frac{20\lambda_0}{\lambda}\left(\frac{x}{z}+2\right)\right)}$ and $\frac{\sin^2\left(\pi N \frac{20\lambda_0}{\lambda}\left(\frac{y}{z}+\frac{2}{20}\right)\right)}{\sin^2\left(\pi \frac{20\lambda_0}{\lambda}\left(\frac{y}{z}+\frac{2}{20}\right)\right)}$ have global maximums. Let $d(x,y)$ denote distance (but measured in maximum norm) between point (x,y) and the closest (according to maximum norm) element of set Σ . If we plot functions $\dfrac{\sin^2\left(\pi N \frac{20\lambda_0}{\lambda}\left(\frac{x}{z}+2\right)\right)}{\sin^2\left(\pi \frac{20\lambda_0}{\lambda}\left(\frac{x}{z}+2\right)\right)}$ and $\dfrac{\sin^2\left(\pi N \frac{20\lambda_0}{\lambda}\left(\frac{y}{z}+\frac{2}{20}\right)\right)}{\sin^2\left(\pi \frac{20\lambda_0}{\lambda}\left(\frac{y}{z}+\frac{2}{20}\right)\right)}$, we can easily see, that points for which $d(x,y)>\frac{\lambda z}{20\lambda_0}\frac{1}{N}$, satisfies $\dfrac{\sin^2\left(\pi N \frac{20\lambda_0}{\lambda}\left(\frac{x}{z}+2\right)\right)}{\sin^2\left(\pi \frac{20\lambda_0}{\lambda}\left(\frac{x}{z}+2\right)\right)}\dfrac{\sin^2\left(\pi N \frac{20\lambda_0}{\lambda}\left(\frac{y}{z}+\frac{2}{20}\right)\right)}{\sin^2\left(\pi \frac{20\lambda_0}{\lambda}\left(\frac{y}{z}+\frac{2}{20}\right)\right)}<N^4\left(\frac{2}{\pi}\right)^4$. It means, that for $d(x,y)>\frac{\lambda z}{20\lambda_0}\frac{1}{N}$, we have $I(x,y)<A N^4\left(\frac{2}{\pi}\right)^4$.

Consider point on the screen with coordinates showed below (note that this point is an element of Σ):

$$x_m=\frac{\lambda z}{20\lambda_0}\left(\left[\frac{40\lambda_0}{\lambda}\right]-\frac{40\lambda_0}{\lambda}\right), y_m=\frac{\lambda z}{20\lambda_0}\left(\left[\frac{2\lambda_0}{\lambda}\right]-\frac{2\lambda_0}{\lambda}\right)$$

$$[t]=\begin{cases}\lfloor t\rfloor \text{ for } t-\lfloor t\rfloor<0.5\\ \lceil t\rceil \text{ for } t-\lfloor t\rfloor\geq 0.5\end{cases}$$

We can calculate that:

$$I(x_m, y_m)\geq A N^4\left(\frac{2}{\pi}\right)^4$$

It means that maximum of function I is placed at point (x,y) for which $d(x,y)\leq\frac{\lambda z}{20\lambda_0}\frac{1}{N}$. We consider only $|x|,|y|\leq\frac{\lambda}{40\lambda_0}z$, so value of $sinc^2\left(\pi\frac{20\lambda_0 x}{\lambda z}\right)\sin^2\left(\pi\frac{20\lambda_0 y}{\lambda z}\right)$ increases when $|x|,|y|$ decreases. Point x_m, y_m is element of Σ with lowest $|x|,|y|$. Let x_M, y_M denote maximum of function I (we still consider only $|x|,|y|\leq\frac{\lambda}{40\lambda_0}z$). $\frac{\lambda z}{20\lambda_0}\frac{1}{N}$ is very small, so we can write:

$$x_M=\frac{\lambda z}{20\lambda_0}\left(\left[\frac{40\lambda_0}{\lambda}\right]-\frac{40\lambda_0}{\lambda}\right)\pm\frac{\lambda z}{20\lambda_0}\frac{1}{N}, y_M=\frac{\lambda z}{20\lambda_0}\left(\left[\frac{2\lambda_0}{\lambda}\right]-\frac{2\lambda_0}{\lambda}\right)\pm\frac{\lambda z}{20\lambda_0}\frac{1}{N}$$

We haven't considered $|x|,|y|>\frac{\lambda}{40\lambda_0}z$ yet. In this case (as before) light beams coming from a single square on the grating, can be replaced by single effective light beam, which has intensity smaller than $4A\left(\frac{\lambda}{2\pi 20\lambda_0 \frac{x}{z}}\right)^2\left(\frac{\lambda}{2\pi 20\lambda_0 \frac{y}{z}}\right)^2$. Changing area of the square (by changing its shape), $4A\left(\frac{\lambda}{2\pi 20\lambda_0 \frac{x}{z}}\right)^2\left(\frac{\lambda}{2\pi 20\lambda_0 \frac{y}{z}}\right)^2$ is the greatest possible intensity, we can receive. We used inequality instead of exact value because of absorption (which changes effective shape of square). Using $|x|,|y|>\frac{\lambda}{40\lambda_0}z$ we can write:

$$4A\left(\frac{\lambda}{2\pi 20\lambda_0 \frac{x}{z}}\right)^2\left(\frac{\lambda}{2\pi 20\lambda_0 \frac{y}{z}}\right)^2<\frac{4A}{\pi^4}$$

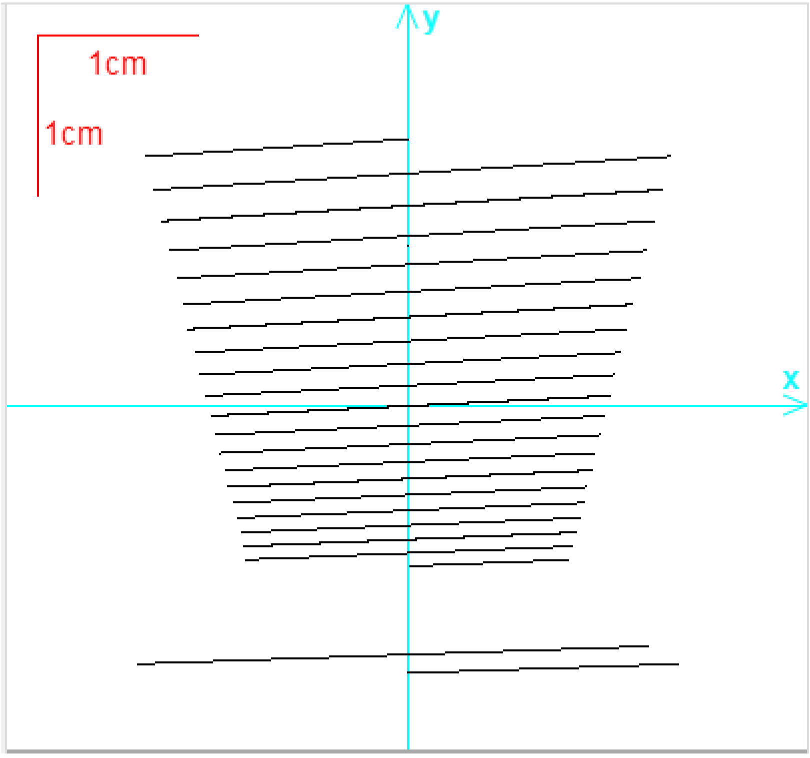
So intensity coming from every square is for sure lower than $N^4\frac{4A}{\pi^4}<AN^4\left(\frac{2}{\pi}\right)^4$. So for $|x|,|y|>\frac{\lambda}{40\lambda_0}z$ there is no global maximum.

Finally, the brightest spot on the screen (for wavelength λ) has coordinates:

$$x_M=\frac{\lambda z}{20\lambda_0}\left(\left[\frac{40\lambda_0}{\lambda}\right]-\frac{40\lambda_0}{\lambda}\right), y_M=\frac{\lambda z}{20\lambda_0}\left(\left[\frac{2\lambda_0}{\lambda}\right]-\frac{2\lambda_0}{\lambda}\right)$$

Warning: There might be two brightest spots (for example if $\frac{2\lambda_0}{\lambda}-\lfloor\frac{2\lambda_0}{\lambda}\rfloor=0.5$). Formula above shows only one of them.

On webpage https://pl.khanacademy.org/computer-programming/problem-5-physics-cup/6331198839816192, there is computer program written by me, which marks brightest spots on the screen. Using it we receive:



Black lines show brightest points on the screen (they look like stairs because of low quality). Red lines in top-left corner show dimensions of the picture.