Solution of Physics Cup 2019, Problem No 3

by Johanes Suhardjo

As it is well known, the trajectory of a particle moving in a uniform gravitational field is a parabola. Therefore, when a ball is dropped on an inclined plane, it will bounce many times from the plane and the trajectory will consist of many parabola arcs. In this problem, we assume that the collisions are perfectly elastic and the size of the ball is negligible, meaning that the energy is conserved and rotational motion can also be neglected. Since there is also no friction, then the acceleration of the ball in the direction parallel to the plane is only due to gravity.

Now, the problem claims that the foci of these parabola arcs lie on a well-known shape. To show this, we must then determine the positions of these foci. There are many ways to do this. One can obviously do the brute force way to obtain the answers, but there are also relatively easier ways using the properties of a parabola. First, let us consider a mirror in the shape of a parabola. If there are rays of light coming parallel to the symmetry axis of the parabola, then all these rays will be reflected towards one point, which is the focus of the parabola (I believe this is why it is called the focus point of the parabola). To determine the focus, therefore it is enough to use 2 light rays and look for the position where they intersect each other. Two methods will be discussed in this solution, but both are actually using this same idea.

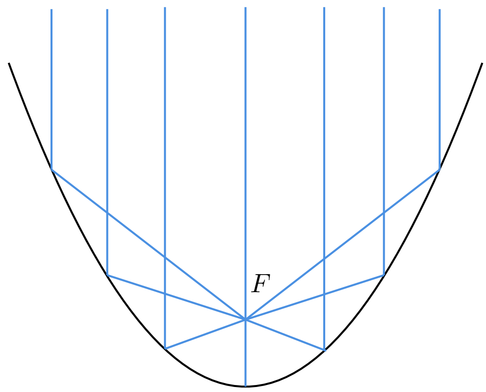

Figure 1: Light rays which are parallel to the symmetry axis of the parabola will be focused on point F which is the focus of the parabola

First Method

For convenience, let us choose our coordinate such that x-axis is parallel to the plane and y-axis is perpendicular to the plane. The acceleration of the ball due to gravity is therefore $-g\cos\alpha$ to the y direction and $g\sin\alpha$ to the x direction. Next, the movement of the ball can be seen as two separated motions along x- and y-axis. Along the x direction the ball is uniformly accelerated with acceleration $g\sin\alpha$, while on the y direction it is bouncing back and forth. If the initial distance of the ball from the plane is d, then this is the maximum distance from the plane that the ball will reach along the y direction. At this maximum distance, the velocity of the ball is parallel to the plane and suppose that its magnitude is u_n when the ball reaches this maximum distance for the n-th time, then it has travelled over $s_n = \frac{u_n}{2}t_n$ along the x direction, where $t_n = \frac{u_n}{g\sin\alpha}$ is the time when the ball reaches this point. This is true because the ball is accelerated with constant acceleration along the x direction so it can be seen as it is moving with constant average speed $\frac{u_n}{2}$. We will denote this point as point P_n and the initial position of the ball as point O (see Figure 2). Now, consider an imaginary vertical light ray which is reflected by the parabola at this point. The reflected light will make an angle 2α with the vertical line. The focus of the parabola must then be at one point along this line.

We need another light ray to determine the position of the focus. If the ray is reflected when the velocity of the ball is making an angle θ with horizontal, then the reflected ray will make an angle

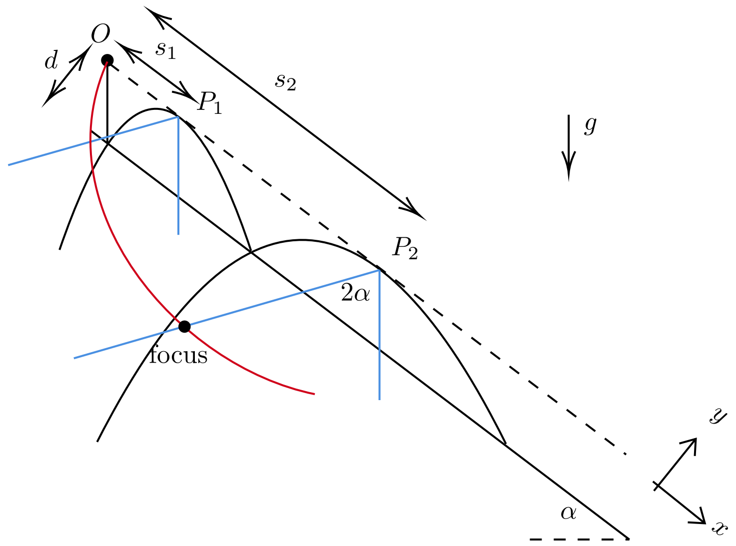

Figure 2: The blue line is the ray light while the red line is the line connecting the foci (the shape which we are looking for). Obviously, the ball only goes through the parabola arcs which are above the inclined plane, but on the figure above I draw the parabola arcs to be symmetrical so that we can imagine the mirror better.

2θ with the vertical line. For convenience, we can choose $\theta = \frac{\pi}{4}$, such that the reflected light will be horizontal. This is achieved when the magnitude of the horizontal component and the vertical component of the velocity of the ball are the same (even if the ball actually never has this velocity along one of the parabola arcs, we can imagine as if the ball is moving without bouncing back). During its motion along one of the parabola arc the horizontal component of the velocity is constant and equals to $u_n \cos\alpha$. Using the equation along the vertical direction then we have,

$$u_n^2 \cos^2\alpha = u_n^2 \sin^2\alpha + 2gh_n$$

where h_n is the vertical distance between point P_n and when the ball is making an angle $\frac{\pi}{4}$ with the horizontal. Since the second reflected light is horizontal, it will intersect the first reflected light at the focus and the vertical distance between the focus and point P_n is evidently also h_n (see Figure 3). From simple geometry it is clear that the distance between point P_n and the focus is $l_n = h_n/\cos(2\alpha)$, so we have

$$l_n = \frac{h_n}{\cos(2\alpha)} = \frac{u_n^2(\cos^2\alpha - \sin^2\alpha)}{2g\cos(2\alpha)} = \frac{u_n^2}{2g} \tag{1}$$

This result is interesting because we also have

$$s_n = \frac{u_n}{2}t_n = \frac{u_n^2}{2g\sin\alpha} \tag{2}$$

We can clearly see that $l_n = s_n \sin(\alpha)$ and therefore point O, P_n and the focus are making a right angle triangle. If we denote the focus point to be F_n we can also see that $\angle F_nOP_n = \alpha$ and $\angle OF_nP_n = \frac{\pi}{2}$. This is true for all n. So, finally, we have shown that the foci lie on a straight line which makes an angle α with the inclined plane (or 2α with the horizontal) and going through the initial position of the ball (actually the shape must be going through point O, because initially the trajectory of the ball before bouncing for the first time is a straight line which is the special case of a parabola with point O being its focus).

Second Method

This method is actually a little bit longer, but it is still relatively simple and reasonable to be done. We will use analytical approach rather than geometrical one in this method, and although it is perhaps less elegant, it definitely can give a variation to the problem.

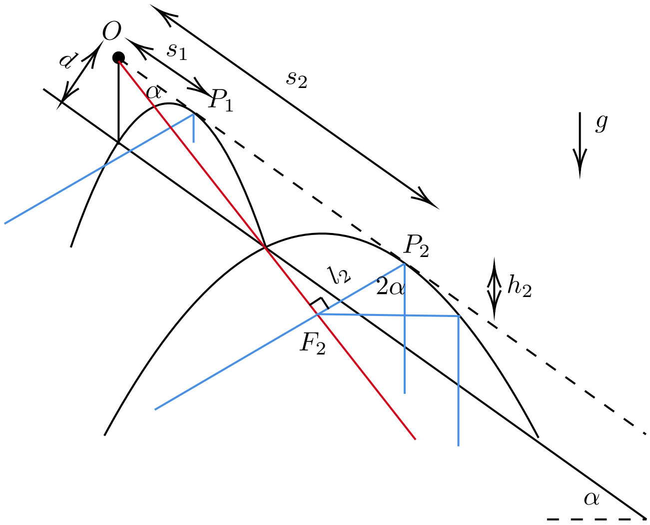

Figure 3: The foci lie on a straight line which make an angle α with the incline plane and going through point O (red line). Note that this figure is not perfect (I tried my best to draw it as good as possible). The focus should also lie at the symmetry axis of each parabola.

First, let us consider the impulse given by the inclined plane to the ball. Suppose that the ball hit the plane with velocity v_0 for the first time, the velocity component perpendicular to the plane is $v_0 \cos\alpha$, so the impulse given by the plane is $\Delta p = 2mv_0 \cos\alpha$, where m is the mass of the ball. This is true for every time the ball is bouncing back from the incline plane not just the first one. Now, let us choose our coordinate system such that point O (the initial position of the ball) is the origin, x-axis is horizontal to the right and y-axis to be positive downward (see Figure 4). The impulse given by the plane to the x direction is simply $\Delta p \sin\alpha = 2mv_0 \cos\alpha \sin\alpha$. So, after bouncing back for the n-th time, the horizontal velocity of the ball must be $v_{x,n} = 2nv_0 \cos\alpha \sin\alpha$. Using the same reasoning, it is easy to show that the vertical component velocity of the ball is $v_{y,n} = v_0 - 2nv_0 \cos\alpha \cos\alpha = v_0(1 - 2n\cos^2\alpha)$. To know the positions of the foci we can therefore use the fact that the focus of a parabola lies on the symmetry axis of the parabola. This symmetry axis position is achieved when the vertical component velocity of the ball is zero. We know that the horizontal distance between this point and where the ball bounced back is

$$\Delta x = v_{x,n}\frac{v_{y,n}}{g} = \frac{v_0^2 n \sin(2\alpha)(1 - 2n\cos^2(\alpha))}{g}$$

The horizontal distance of the n-th focus point from point O is therefore $x_f = \Delta x + d_n \cos\alpha$, where d_n is the distance between the first bouncing point to the n-th bouncing point. The time difference between the n-th and the $(n+1)$-th hit is simply $\Delta t = \frac{2v_0 \cos\alpha}{g\cos\alpha} = \frac{2v_0}{g}$. Therefore, we have

$$d_n = v_0 \sin(\alpha) n\Delta t + \frac{g\sin(\alpha)}{2} n^2 \Delta t^2 = \frac{2v_0^2 \sin(\alpha)}{g} n(n+1)$$

Simplifying the result, one will obtain

$$x_f = \frac{v_0^2 \sin(2\alpha)\cos(2\alpha)}{g}(n+1)^2 \tag{3}$$

To obtain the y position of the focus, we know that a vertical light ray which hit the parabola when it is making an angle $\frac{\pi}{4}$ will be reflected horizontal. Therefore, we can look for the height when the velocity of the ball is making an angle $\frac{\pi}{4}$, since this height is the same as for the focus point. Using energy conservation and the value of horizontal component velocity we have obtained before we have

$$mgy_f = \frac{1}{2}m(v_{x,n}^2 + v_{x,n}^2)$$

Note that the vertical component of the velocity is the same with its horizontal component because it is making an angle $\frac{\pi}{4}$. So,

$$y_f = \frac{v_0^2 \sin^2(2\alpha)}{g}(n+1)^2 \tag{4}$$

Finally, after eliminating the n in equation (3) and (4), we obtain the equation for the line connecting the foci which is $y_f = x_f \tan(2\alpha)$.

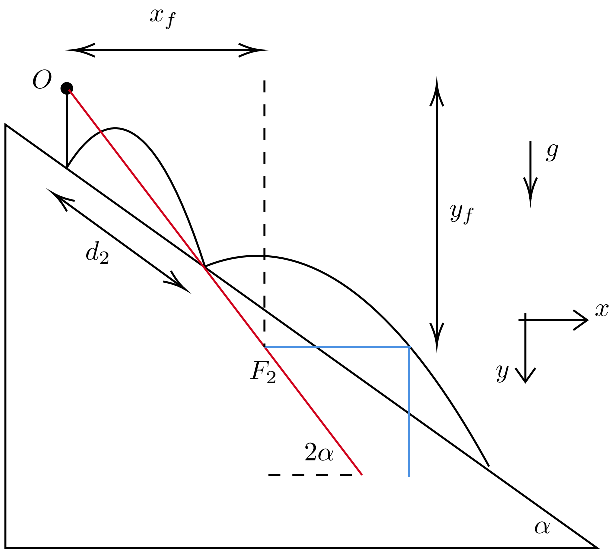

Figure 4: Diagram for the second method