ELECTRICAL CIRCUITS

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New: idea 49, fact 8, appendix 9, and problems 89, 106–111; updated ideas 47,52

1 Circuits with resistors, batteries, ammeters and voltmeters

The fundamental physics of circuits of resistors, batteries, ammeters and voltmeters is really simple, and is essentially covered with just four laws: the two Kirchoff's laws, Ohm's law and Joule's law[1] — formulated here as "facts". First, the Kirchoff's laws:

fact 1: The sum of electrical currents flowing into a node[2] of a circuit is zero.

Mathematically,

$$\sum_{\text{wires connected to the } i\text{-th node}} I_\nu = 0,$$

where $I_{i\nu}$ stands for the current in the ν-th wire. This assumes that I_ν is taken with a '+' sign if it flows into the i-th node and with a '−' sign otherwise. We can also say that the sum of in-flowing currents equals to the sum of out-flowing currents.

Since the electrical current is defined as the charge flowing through a wire's cross-section per unit time, this law is essentially the **continuity law** for electrical charges, combined with the fact that typically, the capacitance of any wire and any circuit node is negligible [3] (hence, the charges residing on the nodes and wires can be neglected).

For those who are not yet developed good intuition with electrical currents, the analogy with water flow in branching rivers or water pipes might be useful: the sum of the water fluxes (measured in cubic meters of water per second) equals to the water flux in the main stream. Note that the continuity law plays an important role for many physical processes (with gas- fluid or granular flows, but also e.g. for traffic fluxes).

fact 2: Along a closed loop of an electrical circuit, the sum of voltage drops on the circuit elements (resistors, diodes, capacitors, etc) equals to the sum of the electromotive forces (of batteries and inductors).

Mathematically,

$$\sum_{\text{wires forming a closed loop}} V_\nu = 0,$$

where the voltage drop on ν-th wire is taken with '+' sign if the voltage of the destination node is lower than that of the departure node, unless the wire includes an electromotive force (emf.): the voltage drop caused by an emf. is taken with the opposite sign.

This law simply states that electrostatic field is a potential field; using a mountain-hiking-analogy, if you walk so that you end at the same point from where you started, you ascended exactly as many meters as you descended. The electromotive force of a battery performs work on charge carriers by using chemical energy (in the case of magnetohydrodynamic generators and inductors/inductor based dynamos, the nature of emf. is somewhat different but for the time being, the details are not important: practical application of the Kirchoff's laws remains the same). With the mountain-hiking-analogy, an electromotive force can be considered as a ski-elevator which lifts you upwards and performs a certain work on you each time you use it.

While the mountain-hiking-analogy works only for the Kirhoff's voltage law, the channel-network-analogy can be extended to all the direct current phenomena. More specifically, we consider a closed system of water channels; in a channel, the water flows only downhill, but there are also pumps which raise the water uphill. Then, there are the following matching pairs: (a) electrical charge Q — mass of water m; (b) electrical current I in a wire, defined as the charge flow rate Q/t, where Q is the charge flowing through a cross-section of a wire during a time interval t — mass flow rate of water μ in a channel, defined as m/t, where m is the water mass flowing through a cross-section of a channel during a time interval t; (c) a battery of electromotive force $\mathcal{E}$ which performs work $\mathcal{E}Q$ on charge Q (which passes through the battery) — a pump which pumps water uphill, to an height h, and performs work hm on a pumped water mass m[4]. Then, obviously, for a closed loop of channels and pumps, the total pumping height (i.e. the sum of the contributions of all the pumps) equals to the total downhill descending height in channels (i.e. to the sum of downhill displacements of all the channels).

Next, the Ohm's law:

fact 3: Typically, the voltage V between the input- and output leads (also referred to as the *ports*)[5] of a piece of electrically conducting material can be considered to be proportional to the current I through it; the coefficient of proportionality

$$R = V/I$$

is referred to as its resistance, and the circuit elements of a non-negligible resistance are called resistors.

Let us try to interpret this using the pipe-flow analogy. Consider a straight pipe connecting two water reservoirs at different height. Let us assume that the drag force F between a unit volume of the flowing water and the pipe's walls is proportional to the speed v of the flow[6]: $F = kv$. Then, the water speed is established by the balance between the drag $F = kv$ and pressure $\rho_w gh$, where h is the height difference, ρ_w — the water density, and g — the free fall acceleration. Therefore, v will be proportional to h, which, according to the analogy, corresponds to the voltage. Now, let us recall that the current I corresponds to the water flux, which equals to the product of the water speed v and pipe's cross-sectional area S, and is therefore also proportional to h (the counterpart of the voltage V). Such a proportionality is exactly what is stated by the Ohm's law.

[1] G. Kirchoff 1845, G.S. Ohm 1827, and J.P. Joule 1841, respectively

[2] *node* (=*vertex* in graph theory) — a point where different wires meet

[3] In the case of very fast or high-frequency processes, this approximation is not valid; then, an equivalent circuit can be used, with ideal wiring and equivalent capacitors and inductors representing the capacitances and inductances of the real wires.

[4] Here we have put the free fall acceleration $g = 1$ which can be done if an appropriate system of units is used

[5] *Ports (input- and output leads)* — points where the current can enter and exit; often just the endpoints of a wire.

[6] This is valid for sufficiently thin pipes for which viscous drag dominates over the turbulent one

For water flow in a narrow pipe, the drag force is proportional to the flow speed and to the pipe length l, i.e. $k = \kappa l$. For ordinary pipes, the drag force (and hence, the coefficient κ) depends also on the diameter of the pipe. However, let us assume that κ is constant (this would correspond to the case when we fill the pipe with a granular material, e.g. coarse sand). Pursuing the analogy, the resistance $R = V/I$ corresponds to the ratio of the height difference h, and water flux. According to the discussions above, this is proportional to the pipe length l, and inversely proportional to the cross-sectional area S of the pipe (because for a fixed v, the flux is proportional to S). Hence, we arrive at the following fact.

fact 4: The electrical resistance of a wire (of a length l and cross-sectional area S)

$$R = \rho l/S,$$

where ρ is called the electrical resistivity of the wire material ($\sigma = 1/\rho$ is called the conductivity).

The proportionality law between V and I fails actually quite often: for instance, in the case of light bulbs, the dependence between the voltage and current is nonlinear. Even then, the ratio V/I is referred to as the resistance. In the case of a non-linear $V - I$ dependence, the resistance just depends on the voltage; the derivative $\frac{\mathrm{d}V}{\mathrm{d}t}$ is referred to as the differential resistance. If a circuit element is referred to as a resistor, its resistance is assumed to be constant.

Finally, the Joule's law:

fact 5: The power dissipated on a circuit element

$$P = IV,$$

where V is the voltage on its leads, and I — the current through it. Alternatively, bearing in mind that

$$I = Q/t,$$

where Q is a charge flowing through the element and t is a time interval, we can say that the current performs work

$$A = QV.$$

Using the analogy of the rivers (channels), the power of a waterfall's power plant is given by the gravitational potential energy released per unit time, which is proportional to the product of the waterfall's height and the water flow rate.

Using mathematical induction, it is not too difficult to show the following fact.

fact 6: If all the resistors and battery voltages (the electromotive forces) are known, and currents of the wires are considered as unknown variables then the Kirchoff's laws and the Ohm's law form a closed set of linear equations which can be solved to find all the currents and voltages in the circuit (i.e. the solution is unique).

This fact itself can be sometimes useful: if you are able to "guess" the solution, it will suffice to show that all the Kirchoff's equations are satisfied (there is no need to derive the answer systematically).

Typically, the number of unknown variables in the Kirchoff's equations (and hence, the number of equations) is large, and solving can be tedious. In order to make calculations easier, several tricks and techniques can be applied.

idea 1: If a circuit can be presented as a combination of series- and parallel connections (see below for an algorithm), the system of Kirchoff's equations becomes decoupled, and there is no need to write the full system of equations. Instead, the following rules can be applied. (A) For parallel connections, the net conductance (inverse resistance) is the sum of conductances and the net current splits in proportions proportional to the conductances:

$$\frac{1}{R_{\text{par}}} = \sum_i \frac{1}{R_i}; \quad I_i = \frac{R_{\text{par}} I}{R_i}.$$

(B) For series connections, the net resistance is the sum of resistances and the voltage is distributed proportionally to the resistances:

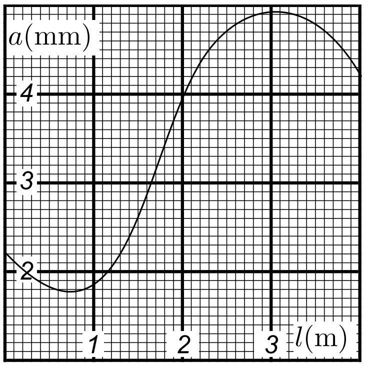
$$R_{\text{ser}} = \sum_i R_i; \quad V_i = \frac{R_i V}{R_{\text{ser}}}.$$

Algorithmically, the procedure of applying the idea 1 can be formulated as follows. If two or more resistors are connected between the same pair of nodes A and B, substitute these with an equivalent resistance according to the formula for R_{par}; if two or more resistors form a branch-less chain connecting nodes A and B, substitute this chain with an equivalent resistance according to the formula for R_{ser}; remove all the "dangling ends" (parts of the circuit which are connected to the rest of the circuit only via a single wire); repeat the process iteratively. The process will stop if (A) only one equivalent resistance remains, or (B) if a bridge is formed (i.e. for a set of four nodes, five or more pairs are connected via resistors).

These rules and formulae can be easily derived using the Kirchoff's laws. Indeed, all the resistors connected in parallel between A and B have the same lead voltage V_{AB}; hence, the currents $I_i = V_{AB}/R_i$ are proportional to the conductances. This gives rise to total current between A and B $I = \sum_i I_i = \sum_i V_{AB}/R_i = V_{AB}\sum_i 1/R_i$, which leads us to the above given formula for $R_{\text{par}} \equiv V_{AB}/I$. All the resistors connected in series between A and B have the same current I_{AB} passing through, so that the voltage on each of them $V_i = I_{AB}R_i$, i.e. the voltage is proportional to the resistance. The total voltage is $V = \sum_i V_i = I_{AB}\sum_i R_i$, hence $R_{\text{ser}} \equiv V_{AB}/I = \sum_i R_i$. Finally, regarding the removal of the "dangling ends": due to the Kirchoff's laws, the sum of currents entering a subset of a circuit needs to be zero; if there is only one wire connecting a circuit's subset to the rest of the circuit, its current needs to be zero, hence it does not affect the current distribution and its presence can be ignored.

The following problem illustrates the idea 1 in its simplest form (for a series connection), together with the fact 4.

pr 1. A uniform wire of cross-sectional area $A_0 = 1\,\text{mm}^2$ had a millimetre scale marked on it: an array of streaks with inter-streak distance $a_0 = 1\,\text{mm}$ covered the entire length of the wire. The wire was stretched in a non-uniform way, so that the inter-streak distance a is now a function of the distance l from one end of the wire (as measured after the stretching), see figure. The new length of the wire is $L = 4\,\text{m}$. Using the graph, determine the electrical resistance R of the stretched wire assuming that the resistivity of the wire material is $\rho = 1.0 \times 10^{-6}\,\Omega\,\text{m}$. During the stretching, the density of the wire material remains constant.



However, for this problem we need one more idea, which is very universal, not limited to electricity.

idea 2: Many physical quantities can be expressed as integrals of other quantities — these can be found as surface areas under graphs. In order to figure out, which surface area is needed, the following technique can be used. Divide the parameter range (for the problem above, the parameter l) into small intervals; if each interval makes an additive contribution to the given physical quantity (here, the resistance R), express this contribution in terms of the interval width and other relevant parameters; design such graph axes that this contribution is proportional to the surface area of a thin rectangular region in the graph. Then, once we sum up the contributions of all the intervals and tend the interval widths to zero, the physical quantity of our interest will be expressed in terms of a graph area.

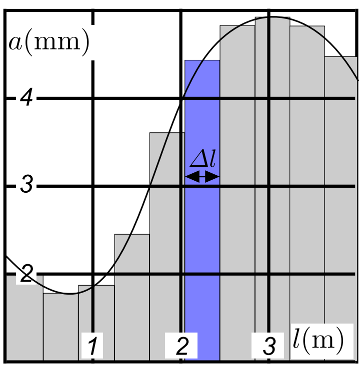
For the problem above, each wire segment of length Δl will contribute a resistance $\Delta R = \rho\Delta l/A$ to the overall resistance R; these wire pieces are connected in series, so the resistances can be just added up. The wire volume remains constant, $Aa = A_0a_0$, hence $A = A_0a_0/a$ so that

$$\Delta R = \frac{\rho}{A_0a_0}a \cdot \Delta l.$$

Note that $\frac{\rho}{A_0a_0}$ is a constant (does no depend on l), and $a \cdot \Delta l$ is the surface area of the blue rectangle in the $a - l$, see fig. below. The sum of all these rectangles (the grey and blue region in figure) approximates the surface area of the region between the $a(l)$-curve and the l-axis, and at the limit $\Delta l \to 0$ becomes equal to that area. Such infinitesimally small increments are called differentials and are denoted by the prefix d (substitutes the prefix Δ which we used for finite increments); the sum over all the infinitesimally small intervals is denoted by the integral sign $\int$. So, we can say that $R \approx \frac{\rho}{A_0a_0}\sum a \cdot \Delta l$ (where the sign $\sum$ denotes summing over all the intervals), and at the limit of infinitesimal increments dl we obtain equality

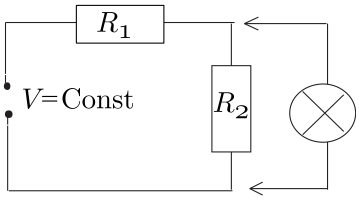
$$R = \frac{\rho}{A_0a_0}\int_0^{4\,\mathrm{m}} a \cdot dl,$$

where $\int_0^{4\,\mathrm{m}} a \cdot dl$ is the surface area under the $a(l)$ graph.



The next problem serves as another simple example of the idea 1.

pr 2. In the figure, $R_1/R_2 = 4$. If we add a lamp as shown if figure, current through R_1 will increase by $\Delta I = 0.1\,\mathrm{A}$. Find the current through the lamp.



It can be solved in a long way, and in a short way. For the long solution, another very generic idea is used.

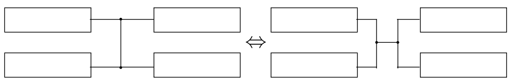
idea 3: If it seems that there are too few data provided in the problem text, just assume the “missing” data to be known (here, for instance, the lead voltage U and the resistance R_1); if everything goes well, the “missing” data will cancel out from the answer.

For the short solution, a useful modification of the Kirchoff’s laws can be applied.

idea 4: Kirchoff’s laws are not valid only for the currents and voltages, but also for voltage increments $\Delta V_i = V_i(\text{after}) - V_i(\text{before})$ and current increments $\Delta I_i = I_i(\text{after}) - I_i(\text{before})$.

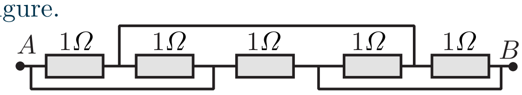
Sometimes the circuit is drawn so that it is not very easy to understand, does it break down into parallel- and series connections or not. In that case, the following idea is to be used.

idea 5: Redraw a circuit so that its structure becomes as clear and simple as possible: contract plain wires (which connect a pair of leads) into a single point and if possible, emphasize the structure of parallel- and series connections. Bear in mind that if several leads are all connected together with a plain wire, the wiring can be arbitrarily rearranged (as long as the relevant leads remain connected), for instance as shown in figure below. Indeed, one can say that the effect of a wire is equalling the voltages on two leads, and in the case of several leads, it doesn’t matter in which order the lead voltages are made equal.



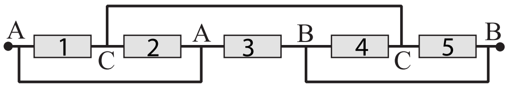
This idea can be illustrated with a task from the 27th IPhO, see below.

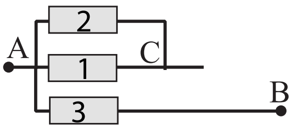
pr 3. Determine the resistance between the leads of the circuit in figure.



[IPhO-1996] For complex circuits, it is easy to make mistakes while simplifying the circuit; typically, this happens when the remote nodes are connected with wires. To avoid mistakes, the following technique can be applied. Label all the resistors, e.g. with letters; if there is more than one battery, label the batteries, as well. Label also all the nodes, so that all the nodes connected with a plain wire bear the same label, and those which have no direct wire connection have different label. Then, start redrawing the circuit by marking (on a sheet of paper) one node and drawing all those resistors which are connected to it. Next, select another lead of one of the drawn resistors or batteries, mark the respective nodes and draw the resistors which are attached to that node; repeat the process until the entire circuit is redrawn.

As an example, let us consider the last problem. We mark the nodes and resistors as shown in figure. Note that due to the wire connections, the node symbols appear in two different places.





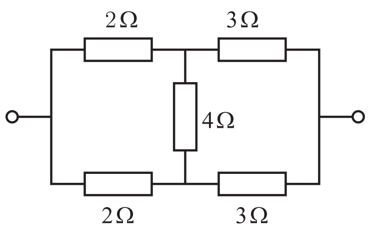
We start with drawing the node 'A', see the figure. Since the node 'A' is directly connected to the resistors '1', '2' and '3', we draw these resistors attached to the node 'A' as shown in figure. The other ends of the resistors '1' and '2' are fixed to the node 'C', hence we can connect the respective wires and designate the connection point by 'C'. Further, the other end of the resistor '3' is connected to the node 'B', so we draw a wire and mark its end with 'B'. Now, by noticing that the resistors '4' and '5' connect the nodes 'B' and 'C', it is easy to complete the circuit.

In the case of non-trivial circuit-redrawing tasks, it is highly recommended to use this technique of denoting resistors and nodes with letters and numbers (you don't want to make a mistake in redrawing!).

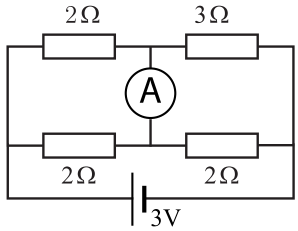
idea 6: If a bridge connection involves only an ideal ammeter (of zero resistance) or an ideal voltmeter (of infinite resistance), the bridge connection is only seemingly there, and can be essentially removed (for voltmeter) or short-circuited (for ammeter). Similarly, it can be removed if there is no current in the bridge connection due to symmetry. Once the simplified circuit is solved, it may be necessary to return to the original (non-simplified) circuit: in the case of an ammeter in a bridge connection, its current can be found from the Kirchoff's current law (written for the currents entering the node to which the ammeter is attached to); in the case of a voltmeter, its voltage can be found as the voltage difference between the nodes to which it is attached using the Kirchoff's voltage law and the voltages of the relevant resistors.

In order to illustrate this idea, let us consider the following problems.

pr 4. Determine the resistance between the leads of the circuit in figure.



pr 5. Determine the reading of the ammeter in figure.



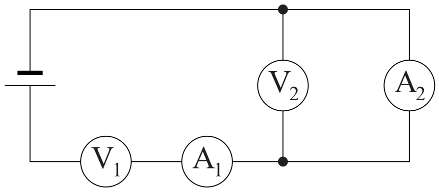
idea 7: If there are non-ideal ammeters, voltmeters, batteries or current sources included into a circuit then the following rules can be applied: (a) non-ideal battery of internal resistance r can be represented as a series connection of an ideal battery (of zero internal resistance) and a resistance r; (b) non-ideal current source of internal resistance r can be represented as a parallel connection of an ideal current source (of infinite internal resistance) and a resistance r; (c) non-ideal voltmeter can be represented as a parallel connection of an ideal voltmeter (of infinite resistance) and a resistance R; (d) non-ideal ammeter can be represented as a series connection of an ideal ammeter (of zero resistance) and a resistance R. NB! A non-ideal ammeter is not a faulty ammeter: it still shows the true current through itself; similarly, a non-ideal voltmeter shows the true voltage on its leads.

Regarding the typical values of the internal resistances of real ammeters and voltmeters, the following guideline can be used. The most common **digital voltmeters** have internal resistance of $10\,\mathrm{M\Omega}$, but cheaper ones can have also $R = 1\,\mathrm{M\Omega}$, and the expensive ones can reach a gigaohm range; typically, the internal resistance is independent of the measuring range. For **analogue voltmeters**, the resistance does depend on the selected measuring range $V_{\max}$, and can be determined by knowing the so called *full-scale deflection current* (FSDC). Essentially, an analogue voltmeter is a galvanometer (device which has a needle which deflects proportionally to the current through it), connected in series with such a resistance that with the maximal voltage $V_{\max}$ applied, the current will be equal to the FSDC. So, if $I_{\mathrm{FSDC}} = 100\,\mu A$, and the 10-volt range is selected then the resistance $r = 10\,\mathrm{V}/100\,\mu A = 100\,\mathrm{k\Omega}$. Typical values of the FSDC are in the range from $25\,\mu\mathrm{A}$ to $1\,\mathrm{mA}$.

Digital ammeters measure internally voltage on a small resistor (*shunt*) and translate the result into corresponding amperage; depending on the selected range of currents, different shunt is used; the voltage drop on the shunt is called *the burden voltage*, and the maximal burden voltage (MBV) V_{MBV} can be used to determine the resistance; for instance, for the 20-mA range and $V_{\mathrm{MBV}} = 300\,\mathrm{mV}$, the resistance is $300\,\mathrm{mV}/20\,\mathrm{mA} = 15\,\Omega$. Typical values for V_{MBV} range from 100 mV to 1 V. An **analogue ammeter** is essentially a galvanometer connected in parallel with a small resistor (shunt); the shunt controls which fraction of the net current goes through the galvanometer and ensures that the voltage on the galvanometer does not exceed the full-scale deflection voltage (FSDV). The shunt resistance can be determined in the same way as in the case of a digital ammeter: here, FSDV plays the role of the MBV.

In the case of theoretical Olympiad problems, voltmeters and ammeters are usually assumed to be ideal, unless otherwise noted. However, there is an exception to this rule: if the problem conditions contradict the assumption of ideality, you need to abandon it. Please note that in the case of theoretical problems, it is not wise to make assumptions regarding the values of the internal resistances of non-ideal ammeters and voltmeters: quite often, the authors of the problems do not check how real the numerical values of the resistances are.

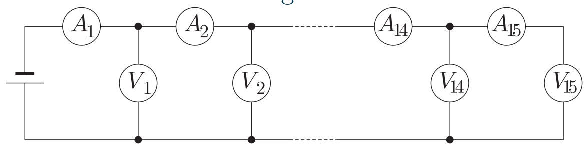
pr 6. Two identical voltmeters and two ammeters are connected to a battery as shown in figure. The readings of the devices is as follows: ammeter A_1 — $I_1 = 200\,\mu\mathrm{A}$, voltmeter $V_1 = 100\,\mathrm{V}$, and voltmeter $V_2 = 2\,\mathrm{V}$. What is the reading of ammeter A_2? Estimate, how realistic are those internal resistances which can be determined from these data; if there is something strange, is it possible to "fix" the problem by changing the circuit so that the solution would remain intact?



idea 8: Sometimes it is convenient to consider the Kirchoff's current law for a whole region and not just for a single circuit node: the sum of currents entering the region equals to the sum of outgoing currents.

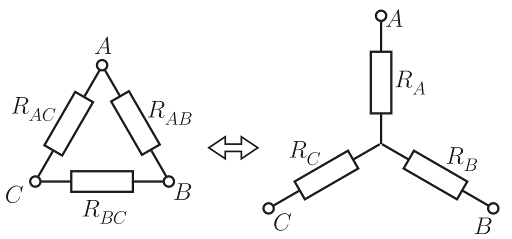
This idea can be illustrated with the following problem.

pr 7. [EstPhO-2003] 15 identical voltmeters and 15 non-identical ammeters are connected to a battery as shown in figure. The reading of the first voltmeter is $V = 9\,\mathrm{V}$, the readings of the first two ammeters are $I_1 = 2.9\,\mathrm{mA}$ and $I_2 = 2.6\,\mathrm{mA}$. What is the sum of the readings of all the other voltmeters?



In some cases, the bridge connection is real and cannot be removed. In the case of Olympiad problems, this is happens very seldom, because in that case, the difficulties are actually only mathematical: it is needed to solve the linear system of Kirchoff equations. There are several methods which simplify this mathematical task which are presented in what follows.

idea 9: Any circuit which consists only of resistors and has three ports is equivalent to a Δ- or a Y-connection of three appropriately chosen resistors[7] In particular, a Y-connection can be substituted by a Δ-connection and vice versa[8].



Note that Δ-connection is also called "triangular", and Y-connection — a "star". So, the idea is to substitute either a triangular connection with a star connection or vice versa so that the resulting circuit is simpler to analyse than the original one. While doing so, all the three lead potentials need to remain unchanged. Let us consider the simplest case when all the three resistors are equal: for a Δ-connection R, and for a Y-connection — R. Then, the inter-lead resistance of the Y-connection is $2R$ (two resistors in series), and for the Δ-connection — $\frac{2}{3}r$ ($2r$ in parallel with r). Therefore, there is matching between these circuits if $2R = \frac{2}{3}r$, hence $r = 3R$: the Δ-connection needs to have thrice as large resistances as in the case of a Y-connection. This rule — if forgotten — can easily derived whenever needed.

In the generic case of non-equal resistances, the $Y-\Delta$-substitution formulae are derived by solving the system of three equations stating pair-wise equality of the inter-lead resistances r_{AB}, r_{BC}, and r_{CA}; the result is as follows: for a Δ-to-Y-substitution

$$R_A = \frac{R_{AB}R_{AC}}{R_{AB}+R_{AC}+R_{BC}}, \tag{1}$$

and analogously for R_B and R_C (the indices are to be substituted cyclically); for a Y-to-Δ-substitution,

$$\frac{1}{R_{BC}} = \frac{\frac{1}{R_B}\cdot\frac{1}{R_C}}{\frac{1}{R_A}+\frac{1}{R_B}+\frac{1}{R_C}}, \tag{2}$$

and analogously for R_{BC} and R_{CA}.

It would consume quite a lot of time to derive these formulae during an Olympiad, so it is better to remember them. Remembering is actually not that difficult; first, let us talk about Δ-to-y-substitution which is typically more useful than the reverse one (there are exceptions) as it removes a loop from the circuit — loops can create bridge connections and are difficult to analyse. Even if that is not the case, Δ-to-y-substitution tends to reduce the number of parallel connections (the reverse substitution tends to increase it), leading to simpler calculations as typically, resistances and not conductances are given. The denominator of the formula is very simple - just the sum of all the resistances. The nominator is also simple, a product of two resistances, we just need to be able to figure out which of the three resistances is to be excluded from the product. This, however, can be easily figured out from the symmetry considerations: so, for a resistor attached to the node B in Y-connection, we exclude the resistor at the opposing side AC of the Δ-connection.

If we really need a y-to-Δ-substitution, the formula can be also easily deduced from the structure of the Δ-to-y-substitution: just the resistances need to be changed to conductances.

Note that the idea 9 cannot be generalized to circuits of resistors with n ports with $n > 3$.[9] An exception is the case when all the pair resistances are equal (to R), in which case the circuit is equivalent to a star connection of n resistors, each of resistance $R/2$ (though it is still not equivalent to a n-gon-connection of equal resistances, because for a polygon, close node pairs have smaller resistance than remote node pairs).

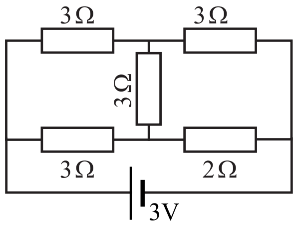
[7] The proof is provided in Appendix 1 on pg. 14.

[8] A.E. Kennelly, 1899.

[9] Indeed, there are $\frac{1}{2}n(n-1)$ different lead pairs, which can all have different resistances; for a generic case, the respective $\frac{1}{2}n(n-1)$ equivalence equations cannot be solved with respect to the n resistances of a star (or a polygon) connection as long as $\frac{1}{2}n(n-1) > n$, i.e. $n > 3$.

As an illustration, let us consider the following problem.

pr 8. Determine the current through the battery.

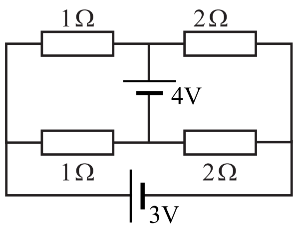


idea 10: Any circuit which consists only of resistors and batteries and has two ports A and B is equivalent to a series connection of a battery and a resistance (the Thvenin's theorem)[10]. The electromotive force $\mathcal{E}$ of the battery can be found as the voltage difference between the leads A and B when there is no load connected externally between these leads (this is because the original and the substitution circuits must behave identically when there is no load).

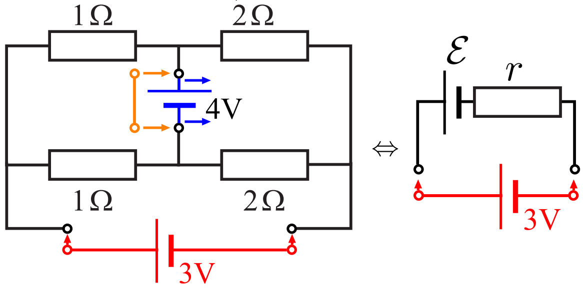
The resistance r (the internal resistance of the battery) can be found as $\mathcal{E}/I_0$, where I_0 is the current which would flow in a wire short-circuiting the two leads (this is because the original and the substitution circuits must behave identically when the leads are connected by a wire). Alternatively, r can be found as the resistance between the leads A and B when there is no external load, and all the ideal internal electromotive forces are substituted by wires (this is because the original and the substitution circuits must have identical increase of the lead voltage when there is a certain increase of the lead current, and an ideal battery and a piece of wire have identically a zero voltage response to an increase of the current).

As an illustration, let us consider the following problems.

pr 9. Determine the current through the batteries.



In order to make the application of the idea 10 more transparent, let us solve the first part of the last problem, and find the current through the 3-V-battery. In figure below, the black and blue part of the circuit will be substituted by a battery of electromotive force $\mathcal{E}$ and internal resistance r (see figure, section on right). To begin with, we assume that there is no load, i.e. the part drawn in red is missing. Then, the blue battery creates currents $4\,\mathrm{V}/2\,\Omega = 2\,\mathrm{A}$ and $4\,\mathrm{V}/4\,\Omega = 1\,\mathrm{A}$ in the left and right loops, respectively. Consequently, the voltage drops on the resistances at the bottom of the figure ($1\,\Omega$ and $1\,\Omega$) are equal to $2\,\mathrm{A}\cdot 1\,\Omega = 2\,\mathrm{V}$ and $1\,\mathrm{A}\cdot 2\,\Omega = 2\,\mathrm{V}$, respectively. Hence, the lead voltage is $2\,\mathrm{V} - 2\,\mathrm{V} = 0\,\mathrm{V}$, i.e. $\mathcal{E} = 0$. Next, we calculate the internal resistance r of the equivalent battery. To this end, we substitute the blue battery with the orange wire (see figure) and calculate the resistance between the leads: the parallel connections of 1-ohm resistors and the parallel connections of 2-ohm resistors are connected in series, so that $r = (\frac{1}{2}+1)\Omega = 1.5\,\Omega$. Finally, we return the red part of the circuit to its place for the equivalent circuit at right (keeping in mind that $\mathcal{E} = 0$ and $r = 1.5\,\Omega$): the current through the 3-V-battery is $I = 3\,\mathrm{V}/1.5\,\Omega = 2\,\mathrm{A}$.

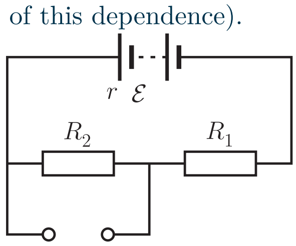


The following fact can be quite easily derived, but knowing it will can save some time during an Olympiad.

fact 7: For drawing the maximal power from a battery, the load's resistance needs to be equal to that of the internal resistance of the battery.

Indeed, the load current $I = \mathcal{E}/(R+r)$, where R is the resistance of the load. Hence, the power dissipation at the load can be found as $P = RI^2 = \mathcal{E}^2R/(R+r)^2$. Let us notice that instead of P, it would be easier to analyse $1/P$, because then the expression will break down into three additive terms: $\frac{1}{P} = \mathcal{E}^{-2}r(\frac{R}{r}+\frac{r}{R}+2)$. If P is maximal then $\frac{1}{P}$ is minimal; we need to minimize this expression over the values of R. Upon taking derivative with respect to R we obtain $\frac{d}{dR}\frac{1}{P} = \mathcal{E}^{-2}(1-r/R^2) = 0$, hence $R = r$. (Alternatively, it would have been possible to apply the fact that the sum of a number x and its reciprocal $\frac{1}{x}$ has minimum for $x = 1$, hence $\frac{R}{r} = 1$.) So, $P_{\max} = \mathcal{E}^2/4r$.

pr 10. Determine the maximal power which can be dissipated on a load connected to the leads of the circuit in figure (the power depends on the resistance of the load, you need to find the maximum of this dependence).



idea 11: Sometimes it is convenient to deal with constant current sources — instead of batteries (and sometimes, a current source is already present). A battery with electromotive force $\mathcal{E}$ and internal resistance r is equivalent to a constant current source with $I = \mathcal{E}/r$ which is connected parallel to the shunt resistance r[11].

Constant current source is a device which generates a constant current I regardless of which load is connected to the output leads — as long as the load resistance is non-infinite. The

[10] Formulated by H. Helmholtz in 1853 and L. Th venin in 1883; for a proof, note that the behaviour of a two-lead circuit is defined by the relationship between the lead voltage V and lead current I; owing to the linearity of Kirchoff's and Ohm's laws, this relationship is always linear, $V = a - Ib$. This can be always matched with a battery of electromotive $\mathcal{E}$ and internal resistance r, for which $V = \mathcal{E} - Ir$.

[11] If we apply this equvalence to the Th venin's theorem, we obtain what is called the *Norton's theorem* [E.L. Norton (1926), H.F. Mayer (1926)].

validity of this theorem can be easily verified: it suffices to check that for the same lead voltages, the lead currents are also equal. Suppose that a battery (of electromotive force $\mathcal{E}$ and internal resistance r) has lead voltage V; then, the voltage on its internal resistance is $\mathcal{E} - V$ and hence, the lead current $I_{\text{battery}} = (\mathcal{E} - V)/r$. If the same voltage is applied to a constant current source (of constant current $I = \mathcal{E}/r$), the shunt current will be V/r, i.e. the total current will be $I_{\text{c-source}} = I - V/r = \mathcal{E}/r - V/r = (\mathcal{E} - V)/r$. Indeed, $I_{\text{battery}} = I_{\text{c-source}}$ for any lead voltage V, hence this battery and this current source behave identically.

The next problem illustrates the idea 11 (although it can be also solved using the idea 10).

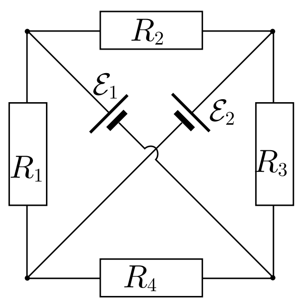
pr 11. n batteries with electromotive forces $\mathcal{E}_i$ and internal resistances r_i (with $i = 1, 2, \dots, n$) are connected in parallel. What is the effective electromotive force and the internal resistance of such a system of batteries?

idea 12: The Kirchoff's equations and the Ohm's law are linear (each term includes only a first power of a current or a voltage), hence the superposition principle is valid. More specifically, suppose we have a circuit which includes only resistors, n ideal batteries and m ideal current sources. Then the current in the j-th wire can be found as

$$I_j = \sum_{k=1}^{n+m} I_j(k),$$

where $I_j(k)$ is the current in that wire when only the k-th battery (or current source) is included into the circuit (all the other batteries are short-circuited and all the other current sources are removed by cutting off a connection wire).

pr 12. [EstPhO-2012] In the figure below, the batteries are ideal, $R_1 = R_2 = R_3 = R_4 = R$ and $\mathcal{E}_1 = \mathcal{E}_2 = \mathcal{E}$. Find the currents in the resistors (i.e. I_1, I_2, I_3 and I_4, expressed via R and $\mathcal{E}$).

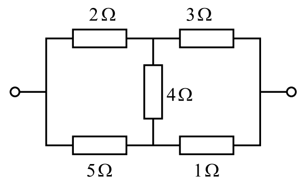


(Note that this problem can be also solved using the idea 19.)

idea 13: The number of unknowns and the number of linear equations can be reduced by using **the method of loop currents**, in which case the first set of Kirchoff's equations is automatically satisfied. The first step is selecting a full set of linearly independent loops $l_1, l_2 \dots l_n$ (the concept of linear dependences is explained below); the second step is assigning to the loops respective currents $I_1, I_2, \dots I_n$, and expressing the currents in resistors via these loop currents. The final steps is expressing the second set of Kirchoff's equations in terms of resistors' currents using the Ohm's law, and solving this set of equations with respect to the loop currents.

Let us illustrate the method and the concept of linearly independent loops using the following problem.

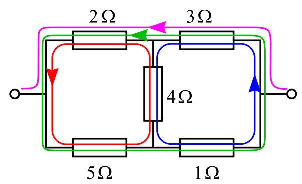
pr 13. Determine the resistance between the output leads of the circuit using the method of loop currents.



This problem can be solved using the idea 9 — and this is possibly the simplest solution. However, here we provide its solution using the idea 13. To begin with, we need one more idea.

idea 14: If the task is to find the resistance of a circuit between two leads, it is often useful to assume that either a voltage V is applied to the leads, or a current I is driven through these leads. Then we need to find the missing quantity (I or V, respectively), and calculate $R = V/I$.

And so, we assume that a current I is driven through the circuit. Let us have a look on possible shapes of loop currents on the figure below.



Let the blue loop be denoted by i_1, red — by i_2, green — by i_3, and violet — by i_4. If we take the red and blue loop currents with equal amplitude, they cancel out in the segment passing through the 4-ohm resistance, hence their sum will be equivalent to the green loop current. Therefore the green loop current is linearly dependent on red and blue loop currents: out of the three loop currents, only two can be kept as unknowns (if we were keeping all the three loop currents, the number of unknowns would be larger than the number of equations). It doesn't matter which pair of loop currents will be selected; let us opt for i_1 and i_2. However, with just the red and blue loops, we cannot obtain any current through the input leads, which means that the system of loop current is not yet closed: we need a loop passing through the input leads. Any shape of such a loop would do; let us use the one depicted by the violet curve (it can be thought to be closed via the external battery). Let us note that i_4 needs to be equal to I — to the current driven through the circuit.

Now we have a full set of loop currents, i_1, i_2, and $i_4 = I$, and we need to write down the Kirchoff's laws for the voltages. The current through the $3\,\Omega$-resistor is $i_1 + I$, so its voltage is $V_3 = 3\,\Omega(i_1 + I)$; similarly $V_4 = 4\,\Omega(i_1 - i_2)$ (the minus corresponds to the fact that the currents i_1 and i_2 are antiparallel in this resistor), and $V_1 = 1\,\Omega i_1$. Please note that the signs of these voltages have been taken corresponding to the blue loop current: positive voltage value means that when moving along the blue loop, the voltage decreases. According to the Kirchoff's laws, upon performing a full loop, the voltage drop needs to be zero:

$$0 = V_3 + V_4 + V_1 \Rightarrow 3(i_1 + I) + 4(i_1 - i_2) + i_1 = 8i_1 - 4i_2 + 3I = 0.$$

We can write down analogous equation for the red loop's voltage drop:

$$i_2(2+4+5) - 4i_1 + 2I = 0.$$

From the first Eq., $4i_1 = 2i_2 - \frac{3}{2}I$, substituting it into the second Eq. leads us to $9i_2 + \frac{7}{2}I = 0$, hence $i_2 = -\frac{7}{18}I$ and $i_1 = -\frac{1}{4}(\frac{7}{9} + \frac{3}{2})I = -\frac{41}{72}I$. Thus, $V_3 = (1 - \frac{41}{72})I \cdot 3\,\Omega = \frac{93}{72}\Omega \cdot I$, and $V_2 = \frac{22}{18}\Omega \cdot I$; the total voltage on the circuit is $V = V_2 + V_3 = \frac{181}{72}\Omega \cdot I$, which means that the resistance $R = V/I = \frac{181}{72}\Omega$.

This has been quite a lot of algebraic work, and we would like to be sure that we didn't do any mistakes. The absence of mistakes can be easily checked by calculating $V = V_1 + V_5$: we need to get the same result! Let us note that $V_1 = -i_1 \cdot 1\,\Omega = \frac{41}{72}\Omega \cdot I$ and $V_5 = -i_2 \cdot 5\,\Omega = \frac{35}{18}\Omega \cdot I$; therefore, $R = (\frac{41}{72} + \frac{35}{18})\Omega = \frac{181}{72}\Omega$.

idea 15: The number of unknowns and the number of linear equations can be reduced by using **the method of potentials**, in which case the second set of Kirchoff's equations is automatically satisfied. The first step is assigning to each node (connection point of wires) a potential φ_n (where the index n refers to the n-th node). The second step is expressing the first set of Kirchoff's equations in terms of the potentials using the Ohm's law, and solving the obtained system of equations.

pr 14. Solve the previous problem using the method of potentials.

Similarly to what we did before, we assume that the circuit leads are attached to a battery. The reference level for potential can be chosen arbitrarily, and thus it is convenient to equate the potential of one output lead to zero (let it be the left one); then the second lead's potential equals to the battery voltage V. There are two more nodes on the circuit, let the respective potentials be φ_1 (the upper one), and φ_2. The current to the upper node from the right wire $I_3 = (V - \varphi_1)/3\Omega$; the current from the upper node to the left wire $I_2 = \varphi_1/2\Omega$; the current from the upper node downwards $I_4 = (\varphi_1 - \varphi_2)/4\Omega$. According to the Kirchoff's law for currents, $I_3 = I_2 + I_4$, hence

$$(V - \varphi_1)/3\Omega = \varphi_1/2\Omega + (\varphi_1 - \varphi_2)/4\Omega \Rightarrow$$
$$13\varphi_1 - 3\varphi_2 = 4V.$$

Similarly, for the lower node, $I_4 + I_1 = I_5$, where $I_1 = (V - \varphi_2)/1\Omega$ and $I_5 = \varphi_2/5\Omega$. This leads us to

$$(\varphi_1 - \varphi_2)/4\Omega + (V - \varphi_2)/1\Omega = \varphi_2/5\Omega \Rightarrow$$
$$-5\varphi_1 + 29\varphi_2 = 20V.$$

Solving this linear system of equations results in $\varphi_1 = \frac{88}{181}V$ and $\varphi_1 = \frac{140}{181}V$; total current can be calculated using the Kirchoff's law for the leftmost node, $I = I_2 + I_5 = \frac{44+28}{181}V/\Omega$, hence $R = V/I = \frac{181}{72}\Omega$. The control of this result can be done by calculating the total current on the basis of the rightmost node.

This example shows that the difficulty level of the both methods (c.f. ideas 13 and 15) is approximately the same, so the choice is typically based on personal preferences. In the case of loop currents, selecting a good set of linearly independent loops may seem as an additional step in the solution, but in the case of *planar circuits* (i.e. circuits which can be drawn on a paper so that the wires don't intersect and meet only at the nodes), this step is not needed if we use a slight modification of the loop current method which will be referred to as the *streamfunction method*.

The concept of streamfunction $\psi(x,y)$ can be used for two-dimensional incompressible flows in which case the streamlines follow the lines of constant value $\psi(x,y) = \text{const}$, and the *flow flux*[12] between two isolines equals to the difference between the respective values of ψ. In the case of a two-dimensional fluid flow, the flow flux is the surface area which flows through a cross-section within a unit time; in the case of electrical current, the flow flux is the total electrical current which flows through a cross-section. In the case of planar circuits, current flows only along the wires and hence, the streamfunction is constant between the wires, and jumps at the position of wires[13].

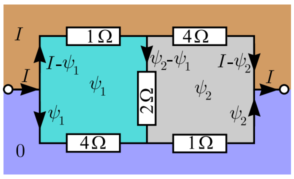
idea 16: For planar circuits, instead of the method of loop currents, the method of streamfunction can be used. Each face (the empty area between wires) of the circuit is assigned a streamfunction value: i-th face is assigned a value ψ_i which is to be found using the Kirchoff's voltage laws. The current in a wire separating i-th and j-th face is found as $I_{ij} = \psi_i - \psi_j$; the sign of I_{ij} here is chosen so that if we move along the direction of I_{ij}, the i-th face remains to our left hand.

Now we can also draw an important conclusion regarding the total number of linearly independent loops for planar circuits:

fact 8: The number of degrees of freedom (and hence, the maximal number of linearly independent current loops) for the current distribution in a planar circuit equals to the number of faces in the corresponding graph (excluding the infinite face).

Indeed, the current distribution can be fully described by the streamfunction values at the faces, and the infinite face can be postulated to have $\psi = 0$, so the number of degrees of freedom equals to the number of finite faces of the graph.

In order to illustrate the idea 16, let us consider, again, the problem 13; the unknown values of streamfunction (ψ_1 and ψ_2) are marked together with the corresponding currents in the figure below (as compared to problem 13, the resistance values are changed). The set of Kirchoff's voltage laws will be very similar to what we obtained with the method of loop currents, so we skip that part of the solution.



idea 17: Due to the symmetry of the Kirchoff's voltage law and current law, there is a duality between electrical currents and voltages[14] which means that we can interchange voltages and currents, and we'll obtain, as a result, a very similar problem. It works out most conveniently in the case of planar cir-

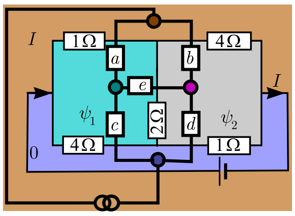
[12] For fluid flow, the fow flux is the amount of fluid carried through a cross-section per unit time; in the case of electric current, it is just the total current.

[13] This statement is valid for non-planar circuits, as well, but then there will be faces for which the streamfunction value is not a free parameter (is defined by the streamfunction values of the neighbouring faces) so that the application of the idea 16 would be less straightforward

[14] A. Russell 1904.

cuits in which case voltage values transfer into streamfunction values, and vice versa; the circuit itself is transferred into its *dual circuit* (see below). Most often, we transfer one circuit problem just into another circuit problem, but in the case of *self-dual circuits* (when the circuit is identical to its dual circuit), the symmetry may prove to be very useful[15].

Let us apply the concept of duality to the bridge-connection drawn above; the dual circuit is obtained by putting one node inside each face of the original circuit, and connecting the new nodes with wires so that each old wire is crossed by exactly one new wire, see below.



For our original circuit, we had a battery which kept the voltage between the two ports equal to $\mathcal{E}$, and in our new circuit, we have a current source which keeps the streamfunction difference between the top and bottom nodes equal to I. When using the node potential method with our old circuit, we had each node ascribed a potential; now we have each node ascribed a streamfunction value. For the old circuit, the unknown potentials φ_i were found from the Kirchoff's current laws written for each node; for j-th node

$$\sum_i (\varphi_i - \varphi_j)/R_{ij} = 0,$$

where R_{ij} is the resistance between the i-th and j-th nodes, and the sum is taken over all those nodes which are directly connected to the j-th node. For the new circuit, the unknown streamfunction values ψ_i are to be found from the Kirchoff's voltage laws written for each new node; for j-th node

$$\sum_i (\psi_i - \psi_j)R_{ij} = 0,$$

where R_{ij} is the resistance on that wire (of the old circuit) which is intersected by a new circuit segment connecting the i-th and j-th nodes, and the sum is taken over all those nodes which are directly connected to the j-th node. We would obtain exactly the same set of equations if we were considering the new circuit as a usual resistor network with resistances being equal to the conductances of the old circuit (so that in the figure above, the "resistance" of the "resistor" a is $1\,\Omega^{-1}$, the "resistance" of the "resistor" b is $0.25\,\Omega^{-1}$, etc). This procedure assumes also that the new "voltage" applied between the top and bottom nodes of the new circuit is I, and the new total "current" (the sum of "currents" in "resistors" a and b) equals to the voltage difference $\mathcal{E}$ between the two ports of the old circuit. Therefore, the "resistance" $R^\star$ of our dual circuit is expressed as

$$R^\star = \frac{I}{\mathcal{E}} \equiv \frac{1}{R},$$

where R is the total resistance of the old circuit (everything is OK with the dimensionalities as both $R^\star$ and the component-resistors of our dual circuit are measured in Ω^{-1}).

To sum up, the procedure of using dual circuits to calculate the resistance of a given circuit is as follows: build a dual circuit so that on each new circuit segment, there is a resistor the "resistance" of which is equal to the conductance of the corresponding old circuit segment (if the old circuit segment had a battery, use now a current source). Calculate the "resistance" of this dual circuit and take reciprocal to obtain the resistance of the original circuit.

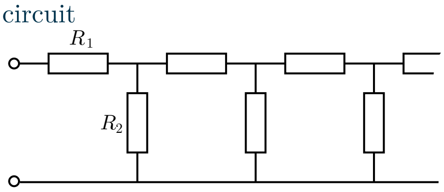
Note that the bridge connection considered above is *topologically self-dual*, because its dual circuit has the same structure — is made of five "resistors" forming a bridge. Furthermore, the numerical values of the resistors are such that if we multiply all the "resistances" by the same factor ($4\,\Omega^2$), the dual circuit becomes identical to the original circuit; this is the property which makes it a *self-dual circuit*.

pr 15. Find the resistance of the bridge connection illustrating the idea 16; use the idea 17 and the self-duality of this bridge connection.

The simple bridge connection considered above is self-dual, because its dual circuit is also a bridge connection of the same type.

idea 18: Infinite periodic chains of electronic components (resistors, capacitors etc) can be studied by making use of the self-similarity of the chain: removal of the first period does not change its properties.

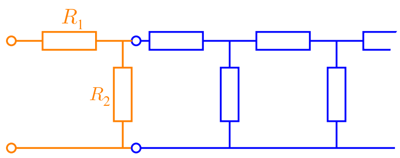
pr 16. [IPhO-1967[16]] Determine the resistance of the infinite periodic circuit



According to the idea 18, we "cut off" the first period of the infinite chain (painted in orange in the figure below); the remaining part (blue) is equivalent to the original circuit of (yet unknown) resistance R. Because of that, we can write equality

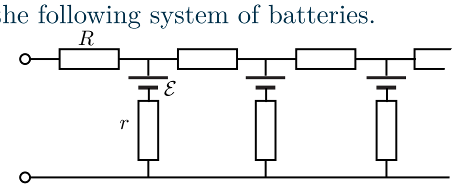
$$R = R_1 + \frac{RR_2}{R + R_2},$$

which can be solved with respect to R.



This idea can be combined with other ideas — for the next problem, together with the idea 10.

pr 17. Determine the electromotive force and internal resistance of the following system of batteries.



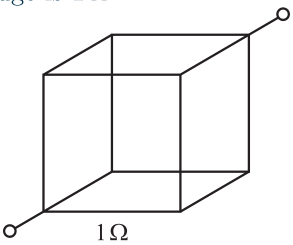
idea 19: As soon as you detect a symmetry in a problem, try exploit it.

[15] L.A. Zadeh 1951

[16] At the IPhO-1967, all resistors were equal to r

The next problem can be solved exploiting its symmetry, in conjunction with the idea 14.

pr 18. Determine the resistance between opposing corners of a cube, the edges of which are made of wire, see figure; the resistance of one edge is $1\,\Omega$



Sometimes it is convenient to use this idea in conjunction with specific algorithms how to reduce a circuit to a combination of parallel and series connections.

idea 20: Node-merging method: if two nodes have equal potential (e.g. due to symmetry), they can be short-circuited.

idea 21: Edge-splitting method: a resistor between nodes A and B can be represented as a parallel connection of two resistors, and the node A can be split into two nodes, if the potentials of the new nodes A' and A'' will be equal.

These ideas are illustrated with the following problem.

pr 19. An hexagon $ABCDEF$ with six "spokes" (connecting its centre O with the vertices) is made of 12 pieces of wire, each having a electrical resistance R. Find the resistance between the vertices A and O using methods 20 and 21.

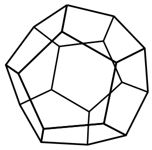
idea 22: Non-symmetric problems can be sometimes converted into symmetric ones using superposition principle.

pr 20. Determine the resistance between two neighbouring vertices A and B of an infinite square lattice assuming that the edges of the lattice are made of wire, and the resistance of each edge is R.

This problem does not possess enough symmetry to be solved immediately: if we drive a current I into the vertex A and draw it out from the vertex B, the geometry of the problem would possesses only a mirror symmetry, which is not sufficient for concluding how the current I is distributed between the four wires connected to the input vertex. However, it is possible to construct a rotationally symmetric problem: suppose that the current I is driven into a vertex A and taken out symmetrically at infinitely remote edges of the mesh. Then it is clear that the current I is distributed equally between the four outgoing wires: the current in each of them is $I/4$. Similarly, we can drive the current in a rotationally symmetric way at infinity, and draw it out from the vertex B. The superposition of these two symmetric configurations provides exactly what we need: the current is driven into A and drawn out from B; at the infinitely remote edges, the current cancels out. In the wire connecting A and B, the both superposition components have the same direction and are equal to $I/4$, hence the net current is $I/2$, which corresponds to the voltage $V = RI/2$. Therefore, the resistance $r = V/I = R/2$.

It appears that such a symmetrization technique can be also applied to finite lattices, see the next problem.

pr 21. Determine the resistance between two neighbouring vertices of a dodecahedron (see figure), the edges of which are made of wire; the resistance of each edge is R.



idea 23: Sometimes the problem symmetrization can be achieved by introducing fictitious negative resistances: there is no problem with applying Kirchoff's laws to negative resistances.[17] In particular, R and $-R$ in parallel correspond to an infinite resistance, and in series — to a zero resistance.

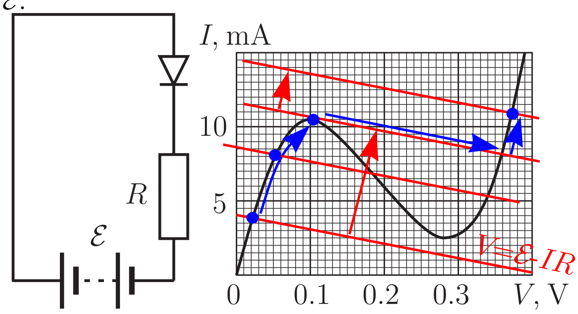
pr 22. Determine the resistance between two neighbouring vertices A and B of a dodecahedron, the edges of which are made of wire; the resistance of each edge is R, except for the edge connecting the vertices A and B, which is cut off.

idea 24: If there are nonlinear resistors included into a circuit which are characterized with a nonlinear current-voltage dependence $I(V)$ then the current through the nonlinear element can be found graphically: I–V-dependence can be also expressed using the Kirchoff's laws, in simpler cases this will be a linear law $V = U_0 - Ir$. Then, the solution will be the intersection point of the two curves, $U_0 - Ir$ and $I(V)$.

Solutions (intersection points) in the negative differential resistance range (where $R_{\rm diff} \equiv \frac{dV}{dI} < 0$) can be unstable; stability analysis requires knowledge about inductors, and so we postpone it accordingly.

fact 9: If there is more than one stable solution then the question of which solution is actually realized is resolved based on the history (e.g. if the voltage applied to the circuit has been increased or decreased) because internally, nonlinear elements obey inertia (for instance, the density of charge carriers can change fast, but not instantaneously) and it will not jump from one equilibrium state to another without a good reason (such as a loss of stability or disappearance of the current solution branch).

Let us illustrate the idea 24 on the basis of a tunnel diode connected via a resistor to a battery of variable electromotive force $\mathcal{E}$.

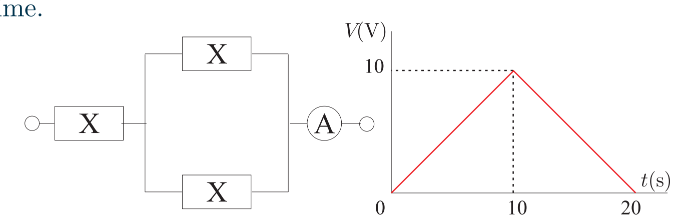


[17] Care should be taken only with oscillatory circuits which include also inductors and capacitors: positive resistance corresponds to a dissipation (decay of oscillations), negative resistance can cause instability (growth of oscillations).

If $\mathcal{E}$ is small, there is only one intersection point (the leftmost blue dot in figure); if $\mathcal{E}$ is increased, the intersection point moves up, and even though at a certain moment, there are more than one intersection points, the real current and voltage correspond to the leftmost intersection point as a continuous evolution of the original solution. When $\mathcal{E}$ is further increased, at a certain moment, this solution disappears and the solution is forced to jump rightwards as shown in figure by blue almost horizontal arrow. Now, if $\mathcal{E}$ starts decreasing, the intersection point depicting the solution moves continuously down and during the period when there are three intersection points, the rightmost one will correspond to the real solution. If $\mathcal{E}$ is further decreased, that intersection point disappears, and the solution jumps back to the only remaining intersection point.

The phenomenon when the system state depends on its history is called *hysteresis*. Hysteresis will typically appear if the system can have more than one internal states; a simple example is provided by the following problem.

pr 23. [EstOPhC-2009] Element X in the circuit below has a resistance R_X which depends on the voltage V_X on it: for $V_X \le 1\,\mathrm{V}$, $R_X = 1\,\Omega$, and for $V_X > 1\,\mathrm{V}$, $R_X = 2\,\Omega$. Three such elements are connected with an ideal ammeter as shown below; the voltage on the leads of the circuit varies in time as shown in the graph. Plot the reading of the ammeter as a function of time.



This problem is otherwise quite simple, but for certain voltages, the state of circuit's components will depend on the history. Here a typical mistake is solving the problem correctly for the first 10 seconds, and then assuming a mirror-symmetrical graph for the current. How to avoid such mistakes? The first and the best way is to always avoid rushed extrapolations (in the given case — mirror-extrapolation of the first 10 seconds to the next 10 seconds). Another way to figure out that things are not as simple as they seem is formulated as an idea.

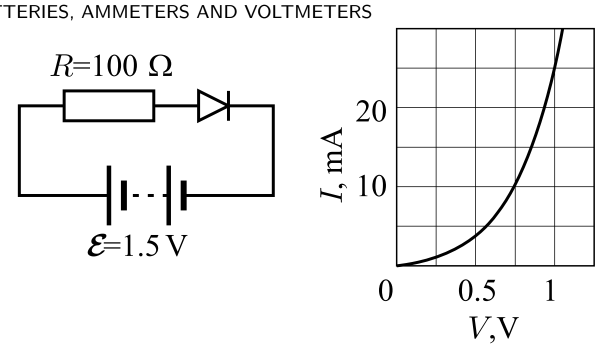
idea 25: Try to think, what was the reasoning of the author of the problem. In particular, if an Olympiad problem has seemingly similar questions, there is typically some essential difference. (As an exception, this is not a physical idea.)

In the given case, would it have been interesting to ask about the next 10 seconds if you can obtain the result by a simple mirror-extrapolation?

Returning to the idea 24, a simple illustration is provided by the next problem.

pr 24. Find the current in the circuit given below; the $I(V)$ dependence of the diode is shown in graph.

[18] Only a part of the full problem

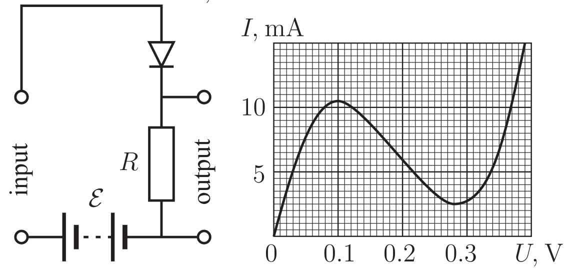


idea 26: In the case of a small variation of the voltage $\tilde{V} \equiv V - V_0$ on a nonlinear element, and a small current variation $\tilde{I} \equiv I - I_0$ through it, one can linearize the $V-I$ curve as $\tilde{V} = R_{\mathrm{diff}}\tilde{I}$, where $R_{\mathrm{diff}} = \frac{\mathrm{d}V}{\mathrm{d}I}$ is referred to as the differential resistance. Here, V_0 and I_0 are the unperturbed (equilibrium) values of the voltage and current. Then, the total voltage on the nonlinear element $V = V_0 + R_{\mathrm{diff}}\tilde{I}$. Now, if we write down the Kirhoff's voltage law in terms of the current variation $\tilde{I}$, in addition to the "Ohm's law" for the voltage variation $R_{\mathrm{diff}}\tilde{I}$, we have additional constant term V_0 which can be inerpreted as an effective electromotive force. On any linear resistor R, the voltage is also a sum of a constant term $I_0 R$ and the variation term $R\tilde{I}$. All the unperturbed constant terms together must cancel out from the Kirchoff's voltage law because V_0 and I_0 were assumed to be valid solutions of the Kirchoff's laws. Indeed, if we put all the perturbations equal to zero then $\tilde{I} = 0$ and $\tilde{V} = 0$ should provide a solution to the Kirchoff's laws, hence all the constant must cancel out.

To sum up, instead of studying voltages and currents, we study the perturbations $\tilde{V}$ and $\tilde{I}$ of these quantities; the effective circuit describing the perturbed values is obtained by removing all the unperturbed voltage and current sources (such as batteries of constant electromotive force), and by substituting nonlinear elements with their differential resistances. NB! The differential resistances of a nonlinear element depends on the current; we need to use its unperturbed value.

The usefulness of this idea is demonstrated by the following problem.

pr 25. [EstFin-2003[18]] In the figure below, the circuit of a simple tunnel-diode-based amplifier is given. Find the amplification factor for small-amplitude input signals using the following values: $R = 10\,\Omega$, $\mathcal{E} = 0.25\,\mathrm{V}$.



idea 27: It is possible to obtain upper and lower limits for the resistance of a circuit using the following theorems.

(I) For an arbitrary circuit which consists of resistors and has two leads, A and B, if a current I is driven into the lead A

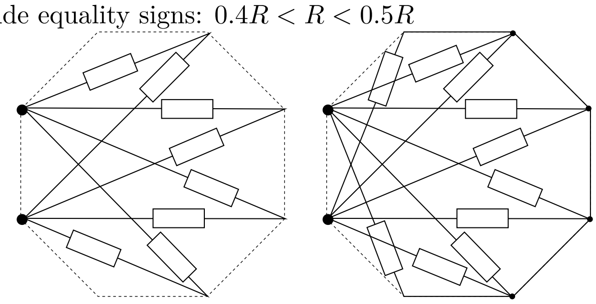
and out from the lead B, the current distributes between the resistors of the circuit so as to minimize the overall power dissipation. In other words, the power dissipation of the actual current distribution is always smaller as compared to any fictitious current distribution satisfying only the Kirchoff's law of currents. [19]

(II) For the same circuit, if there is a voltage drop V between the leads A and B, the voltage distributes between the nodes of the circuit so as to minimize the overall power dissipation. In other words, the power dissipation of the actual voltage distribution is always smaller as compared to any fictitious voltage distribution violating the Kirchoff's law of currents. [20]

Particular conclusions of these theorems are: cutting off a wire will increase the resistance, and short-circuiting a wire will decrease the resistance. Indeed, if we cut a wire, we disable the respective current and this leads to what can be considered as a fictitious current distribution, which has an increased overall power dissipation I^2R, and hence, an increased net resistance R. Similarly, short-circuiting makes it possible for the current to jump between the nodes — something which was impossible originally and violates the Kirchoff's laws of currents for the original circuit. Hence, the power dissipation in the modified circuit V^2/R is increased, and consequently, the resistance R is decreased.

pr 26. There is an octagon all diagonals of which are resistors of equal resistance R; the sides of the octagon are made of an insulating material. Find lower and upper bounds for the resistance between two neighbouring nodes of such an octagon.

The solution here is as follows. First, we cut off several resistances, and leave only those which are shown in the left figure below. The resistance of the left circuit is $\frac{2R}{4} = \frac{R}{2}$. Further, we short-circuit six nodes as shown in the right figure; the resistance is $2\frac{R}{5} = 0.4R$. So, we can conclude that $0.4R \leq r \leq 0.5R$. Since the wires we cut off did have current, and the nodes which we connected with wires did have a voltage difference, the new current- and voltage distributions are sub-optimal and we can exclude equality signs: $0.4R < R < 0.5R$

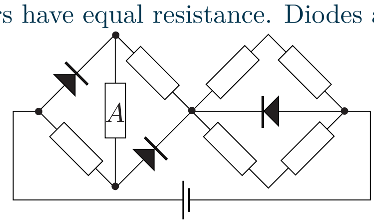


pr 27. Improve the upper bound $r < 0.5R$ for the previous problem (do not "cut off" as many wires as we did before), as well as the lower bound (short-circuit a lesser number of nodes).

Finally, let us consider circuits including ideal diodes.

idea 28: If there are ideal diodes included into the circuit (which have zero resistance for forward current, and infinite resistance for reverse current), you need to consider separately two cases: (a) assume that there is a forward current and the diode is open, hence it can be substituted by a wire; (b) assume that there is a reverse current, and hence, it can be "cut off". Depending on the problem, it may be apparent, which option is to be used, or you may need to use the calculation results to verify, which assumption was valid [21].

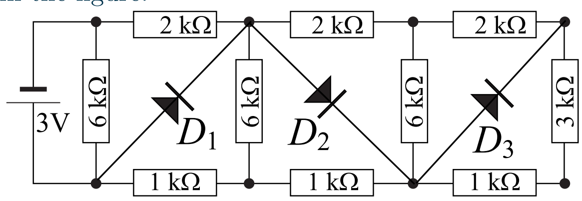
pr 28. How many times will change the power dissipation in the resistor A when the polarity of the battery is reversed? All the resistors have equal resistance. Diodes are ideal.



idea 29: **Non-ideal diodes** which are approximated with an idealized $V-I$ curve with a non-zero opening voltage V_c (there is no current for $V < V_c$, and for any forward current, $V = V_c$) ca be also handled according to the idea 28; the only difference is that for forward currents, diode needs to be replaced by a battery with emf. $\mathcal{E} = V_c$. Additionally, the power dissipation on the diode is calculated in the same way as the work done by a battery: dissipation power is V_cI, and the dissipated heat — $V_c\Delta Q$, where ΔQ is the charge passing through the diode.

Note that this idea can be made even more general: if we approximate a certain nonlinear $V-I$ curve with a curve which consists of n pieces of straight line segments (*piece-wise linear graph*) then we need to consider separately n cases; for each case, the non-linear element can be substituted by a battery the internal resistance of which is equal to the slope $\frac{\mathrm{d}V}{\mathrm{d}I} \equiv r$ of the corresponding straight line segment, and the electromotive force is equal to the V-intercept of that line. Instead of a battery, sometimes it is better to use a current source connected in parallel to the internal resistance r and supplying a current equal to the I-intercept of the graph segment; use this method in particular when $r = \infty$.

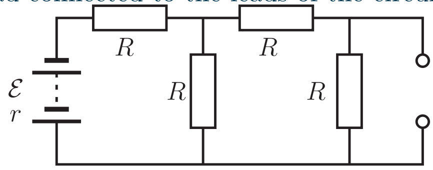
pr 29. [EstOPhC-2012] Find the power dissipation on each of the diodes in the figure below. These diodes open at the forward voltage $V_0 = 1.0\,\mathrm{V}$. It can be assumed that the diode voltage remains equal to V_0 for any forward current, and that for voltages less than V_0, there is no current through the diode. The values of the resistances and of the electromotive force are given in the figure.



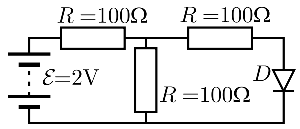
[19] Proof is provided in the appendix 3, page 15.
[20] Proof is also provided in the appendix 3.
[21] This is similar to the problems with dry friction between solid bodies when you consider separately the cases when (a) the bodies splip and there is a friction force defined by the kinetic coefficient of friction, and (b) the bodies don't slip.

Problems involving ideas 1–28

pr 30. Determine the maximal power which can be dissipated on a load connected to the leads of the circuit.

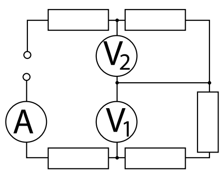


pr 31. Find the current through the diode in the circuit given below; for the diode, use the $I(V)$ dependence from the problem 24.



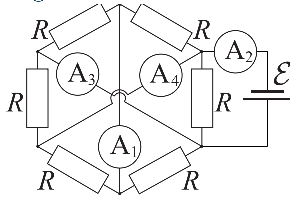
pr 32. For an overcurrent protection, there are two fuses connected in parallel: fuse A has resistance $R_A = 1\,\Omega$ and maximal current (by which it melts) $I_{A\max} = 1\,\mathrm{A}$; fuse B has resistance $R_B = 2\,\Omega$ and maximal current (by which it melts) $I_{B\max} = 1.2\,\mathrm{A}$. What is the maximal total current for such a system of fuses? What is the total current when the fuse B is substituted with a fuse C which has $R_C = 2\,\Omega$ and $I_{C\max} = 1.7\,\mathrm{A}$?

pr 33. The two voltmeters in the circuit below are identical; their readings are $V_1 = 30\,\mathrm{V}$ and $V_2 = 20\,\mathrm{V}$. The reading of the ammeter is $I = 750\,\mu\mathrm{A}$. All the five resistors have equal resistance R; find the numerical value of R.

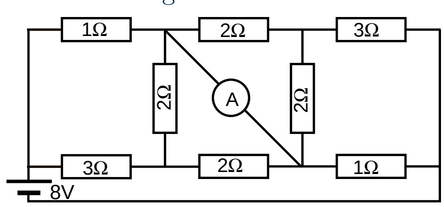


pr 34. Assuming that the resistance of a light bulb's wire is proportional to its temperature T and its heat radiation power is proportional to T^4, find the power law exponent of its V–I dependence. Neglect the heat conductivity and assume that T is much higher than the room temperature.

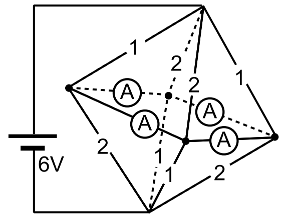
pr 35. [EstPhO-1999] All the resistors have equal resistance $R = 1\,\Omega$ Ammeters and the battery are ideal, $\mathcal{E} = 1\,\mathrm{V}$. Determine the readings of all the ammeters.



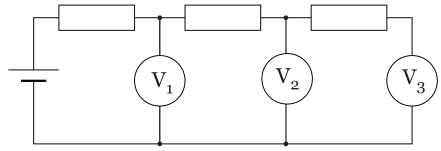
pr 36. Find the reading of the ammeter in the circuit below.



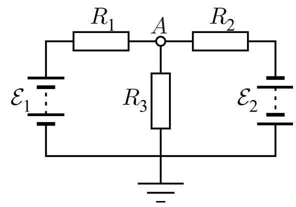
pr 37. The drawing below depicts octahedron made from wire; the number near to each edge shows the resistance of the corresponding wire in ohms. The resistance of the wires connecting the ammeters are negligibly small. Find the readings of the ammeters.



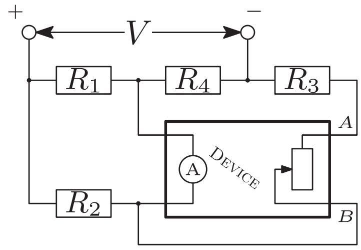
pr 38. In the figure, all three voltmeters are identical, and all three resistors are identical. The first voltmeter shows $V_1 = 10\,\mathrm{V}$, the third — $V_3 = 8\,\mathrm{V}$. What does show the second?



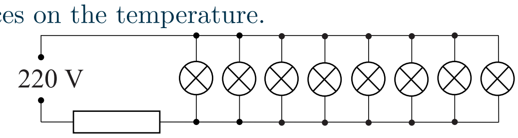
pr 39. Determine the potential of the lead A. (Note that the ground potential is always assumed to be 0.)



pr 40. In the circuit below, the "device" takes the reading of the ammeter and adjusts the resistance of the rheostat so that the ammeter reading becomes zero. Find the voltage on the resistance R_3. It is known that $V = 5\,\mathrm{V}$, $R_1 = 10\,\Omega$, $R_2 = 1\,\mathrm{k}\Omega$, $R_3 = 100\,\mathrm{k}\Omega$, $R_4 = 4.99\,\mathrm{k}\Omega$.

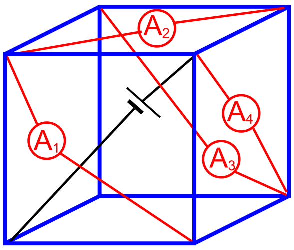


pr 41. Eight identical lamps of nominal voltage $V = 4\,\mathrm{V}$ and nominal current $I = 0.25\,\mathrm{A}$ are connected to a battery via a resistor as shown in figure. The resistor is such that the lamps will operate at the nominal regime (with nominal voltage and current). One of the lamp burns out (the lamp is essentially removed). How many times does change the overall power which is dissipated by the lamps? (The power dissipation on the resistor is NOT included.) Neglect the dependence of the lamp resistances on the temperature.

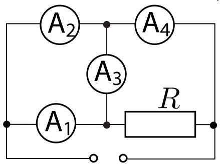


pr 42. The figure below depicts a cube, the edges of which (blue lines) are made of a resistive wire, so that the resistance of each edge is $R = 1\,\mathrm{k}\Omega$. The ammeters are connected with

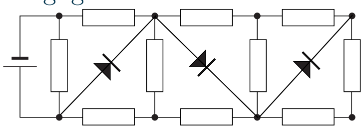
copper wires of negligible resistance to the vertices of the cube. The battery voltage is $\mathcal{E} = 9\,\text{V}$; the wires make electrical contact only at the vertices of the cube. Find the readings of the ammeters.



pr 43. Four ammeters with identical internal resistances r and a resistor of resistance R are connected to a current source as shown in figure. It is known that the reading of the ammeter A_1 is $I_1 = 3\,\text{A}$ and the reading of the ammeter A_2 is $I_2 = 5\,\text{A}$. Determine the ratio of the resistances R/r.

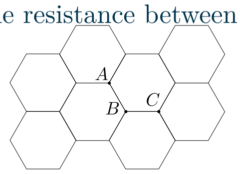


pr 44. How many times does change the current through the battery if the polarity of the battery is reversed? All the resistors are identical, diodes are ideal and internal resistance of the battery is negligible.

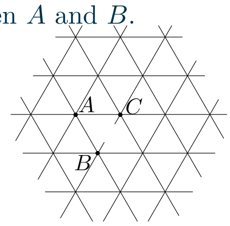


pr 45. Determine the resistance between two neighbouring nodes A and B of an infinite cubic lattice assuming that the edges of the lattice are made of wire, and the resistance of each edge is R.

pr 46. There is an infinite honeycomb lattice; the edges of the lattice are made of wire, and the resistance of each edge is R. Let us denote two neighbouring vertices of a vertex B by A and C. Determine the resistance between A and C.



pr 47. There is an infinite triangular lattice; the edges of the lattice are made of wire, and the resistance of each edge is R. Let us denote the corners of a triangular lattice face by A, B, and C. The wire connecting B and C is cut off. Determine the resistance between A and B.

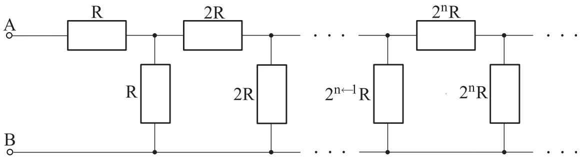


pr 48. There is a n-gon all sides and all diagonals of which are resistors of equal resistance R. What is the resistance between two neighbouring nodes of the n-gon?

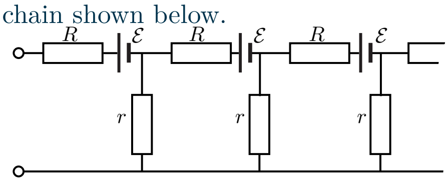
pr 49. There is a decagon all sides and all diagonals of which are resistors of equal resistance R; let A and C denote the two neighbours of a vertex B, and let D be a vertex which is not neighbouring any of the three mentioned vertices. The wires corresponding to the sides AB and BC are cut off. Determine the resistance between A and D.

pr 50. There is an octagon all diagonals of which are resistors of equal resistance R; the sides of the octagon are made of an insulating material. Find lower and upper bounds for the resistance between two opposing nodes of such an octagon without exactly calculating its value. Verify the result by calculating this resistance also exactly.

pr 51. Find the resistance between the terminals A and B for the infinite chain shown below. The resistances are as shown and increase by a factor of two for each consecutive link.



pr 52. Find the voltage between the terminals A and B for the infinite chain shown below.



pr 53. Which unequalities must be valid for the resistance between two neighbouring vertices A and B of an infinite square lattice, if the edges of the lattice were made of wire so that the resistance of each edge was R, but some parts of the lattice have been damaged: some wires have been broken and some of the broken wires have been replaced by copper wires of negligible resistance. However, within the distance of two edge lengths from the wire AB, the lattice is completely intact (this includes 13 wires parallel to AB, and 12 wires perpendicular to it).

pr 54. A wheel circuit is a circuit which can be drawn as a regular n-gon so that the rim of the "wheel" is formed by n resitors of resistance R connecting neighboring vertices of the n-gon, and the "spokes" of the "wheel" are formed by n resistors of resistance r connecting the centre of the "wheel" with each of the vertices. Let R_1 be the resistance between two neighbouring vertices of such "wheel", and R_2 — the resistance between one of the vertices and the centre. Express R_1 in terms of R_2, R and r (without using n).

appendix 1: Proof of the $Y - \Delta$ circuit theorem

Two circuits, one with ports A, B, C, and the other with ports a, b, c are equivalent if their response to external forcing is identical. This means that if we drive a current I into the lead A (or into a) and drive it out from B (or from b), the lead voltages must be pair-wise equal: $V_{AB} = V_{ab}$, and $V_{AC} = V_{ac}$. Due to the linear nature of the Kirchoff's and Ohm's laws, we know that all these voltages are proportional to the current: $V_{AB} = IR_{AB}$, $V_{ab} = IR_{ab}$, $V_{AC} = IR_{ACB}$, and $V_{ac} = IR_{acb}$, where R_{AB} is the resistance between A and B, R_{AC} is the $A - C$ resistance. Proportionality coefficients R_{ijk} ($i, j, k \in \{A, B, C, a, b, c\}$) relate the $i - k$-voltage to the $i - j$ current. Equivalence of the circuits means that

$$R_{AB} = R_{ab}, R_{AC} = R_{ac}, R_{BC} = R_{bc}, \qquad (3)$$

$$R_{ACB} = R_{abc}, R_{ABC} = R_{abc}, R_{BCA} = R_{bca}, \text{etc.} \qquad (4)$$

These 9 equations represent necessary and sufficient conditions for the equivalence between an arbitrary 3-lead circuit (with leads A, B, C) and a Y-connection (or a Δ-connection) of three resistors. Nine equations seems to be too many for determining the values of the three resistances. Luckily, it appears that if the first three equations (3) are satisfied, all the rest are satisfied automatically. It is (relatively) easy to verify via direct arithmetical calculations that for any triplet of resistances R_{AB}, R_{AC}, and R_{BC}, these three equations can always be solved with respect to the three resistances of the Y- or Δ-connection, and as long as the triangle inequalities of the form $R_{AB} \leq R_{AC} + R_{BC}$ are satisfied, the three resistances (of Y or Δ-connection) are non-negative. Indeed, for Y-connection, $R_{ab} = r_a + r_b$, $R_{ac} = r_a + r_c$, and $R_{bc} = r_b + r_c$; if we put these expressions into Eqns. (3), we obtain $r_a = \frac{1}{2}(R_{AB} + R_{AC} - R_{BC})$, and analogous expressions for r_b and r_c. The calculations for Δ-connection are analogous [alternatively, Eq. (2) can be used to find the Δ-connection-resistances from r_a, r_b and r_c]. It appears that the triangle inequalities are, indeed, satisfied for any three-lead circuit, see appendix 4 below.

What is left to do is to show that the equations (4) dealing with the three-lead-resistances follow from the equations (3). First, from the Kirchoff's voltage law we can conclude that $R_{ACB} + R_{CAB} = R_{AC}$ (and similar expressions for R_{CBA} and R_{BAC}). Indeed, if a current I is driven into A and taken out from C, we can express the voltages as $V_{AC} = IR_{AC}$, $V_{AB} = IR_{ACB}$, and $V_{BC} = IR_{CAB}$; due to the Kirchoff's voltage law, $V_{AC} = V_{AB} + V_{BC}$, hence $R_{AC} = R_{ACB} + R_{CAB}$. Second, the equality $R_{ACB} = R_{ABC}$ (and similar expressions for R_{CBA} and R_{BAC}) follows directly from the *reciprocity theorem*[22]; however, this theorem is more tricky to prove, see appendix 2.

appendix 2: Proof of the reciprocity theorem

The theorem states that if we have a four-lead system of resistors, the leads being denoted by A, B,C,and D, and we consider two cases, (i) current I is driven into A and out from B, and (ii) the same current I is driven into C and out from D, then the voltage V_{CD} induced between C and D in the first case equals to the voltage U_{AB} between A and B in the second case (the equality required for the proof of the $Y - \Delta$ circuit theorem corresponds to the particular case when D coincides with A). Let us denote the potential of the j-th node of the circuit in the first case by φ_j ($j = 1, \ldots n$), and in the second case by ψ_j; the first four nodes ($j = 1, 2, 3, 4$) are the four input leads A,B,C, and D. Due to Ohm's law, for any pair of nodes (i, j) directly connected by a wire (over a resistor), there is equality

$$(\varphi_j - \varphi_i)/I_{ji} = (\psi_j - \psi_i)/J_{ji},$$

where I_{ji} and J_{ji} are the wire's currents in the first and the second case, respectively. This can be rewritten as

$$(\varphi_j - \varphi_i)J_{ji} = (\psi_j - \psi_i)I_{ji};$$

Summing this equality over all the node pairs we obtain

$$\sum_j \varphi_j \sum_i J_{ji} - \sum_i \varphi_i \sum_j J_{ji} = \sum_j \psi_j \sum_i I_{ji} - \sum_i \psi_i \sum_j I_{ji}.$$

Note that due to the Kirchoff's current law, for any $j \neq 3, 4$, $\sum_i J_{ji} = 0$; similarly, for $j \neq 1, 2$, $\sum_i I_{ji} = 0$; for $i \neq 3, 4$, $\sum_j J_{ji} = 0$; for $i \neq 1, 2$ $\sum_j I_{ji} = 0$. Further, $\sum_i J_{4i} = -\sum_i J_{3i} = \sum_i I_{2i} = -\sum_i I_{1i} = -\sum_j J_{j4} = \sum_j J_{j4} = -\sum_j I_{j2} = \sum_j I_{1i} = I$. Therefore, the above equality simplifies into

$$2(\varphi_4 - \varphi_3)I = 2(\psi_2 - \psi_1)I.$$

Finally, as $\varphi_4 - \varphi_3 = V$ and $\psi_2 - \psi_1 = U$, we arrive at $V = U$, QED.

There is one quite difficult problem which can be solved in a somewhat similar manner to how we proved the reciprocity theorem. We might also try to formulate a respective hint.

idea 30: Sometimes it is possible to combine the equations of a long system of equations so that almost everything cancels out, leaving only few non-zero terms.

pr 55. m identical resistors of resitance R are connected in an arbitrary way; though, none of the resistors is short-circuited (there is no direct wire connection between the two leads of a resistor), and all the resistors are connected together (the resistance between any pair of nodes is finite). Overall, this resistor network has n nodes For each resistor, the resistance between the adjacent nodes (to which it is connected) is determined, and the results are added up. Show that this sum of m resistances equals always to $(n - 1)R$.[23]

appendix 3: Proof of the dissipation minimum theorem

In order to prove the first part (when the Kirchoff's voltage law remains satisfied), consider the power dissipation

$$P = \sum_{ij} (\varphi_i - \varphi_j)^2 / R_{ij},$$

where φ_i is the potential of the i-th node (for a fictitious potential distribution), and the sum is taken over all such pairs of nodes (i, j) which are directly connected via a resistor R_{ij}. If the potential of the i-th node is changed by a small increment $\Delta\varphi_i$ (while keeping the other potentials intact), the total power dissipation is changed by

$$\Delta P = \sum_j [2\Delta\varphi_i(\varphi_i - \varphi_j) + \Delta\varphi_i^2]/R_{ij}.$$

The last term here can be neglected for very small potential increments, and we can denote $(\varphi_i - \varphi_j)/R_{ij} \equiv I_{ij}$: this is

[22] H.A. Lorentz, 1896

[23] The theorem can be generalized: the resistors are different, each term in the sum is divided by the resistance of the respective resistor, the sum equals to $n - 1$.

the current flowing from i-th to j-th node. So, at the limit of infinitesimally small increments ($\Delta\varphi_i \to \mathrm{d}\varphi_i$), we obtain

$$\mathrm{d}P = 2\mathrm{d}\varphi_i \sum_j I_{ij}.$$

If $\sum_j I_{ij} > 0$ then the power dissipation can be decreased by increasing the potential φ_i ($\mathrm{d}\varphi_i < 0$ leads to $\mathrm{d}P > 0$); similarly, if $\sum_j I_{ij} < 0$, we can take $\mathrm{d}\varphi_i > 0$. So, the dissipation minimum can take place only for $\sum_j I_{ij} = 0$, i.e. when the Kirchoff's current law is satisfied. Further, if $\sum_j I_{ij} = 0$ then $\Delta P = \sum_j \Delta\varphi_i^2/R_{ij} > 0$, i.e. we have, indeed, a minimum.

The second half of the theorem is proved in the same way as the first half: we assume that there is a fictitious current distribution which satisfies the Kirchoff's current law and hence, can be represented as a sum of loop currents $I_\mu = \sum_\nu I_{\nu\mu}$, where ν enumerates the loops and $I_{\nu\mu}$ represents the current in a μ-th wire contributed by the ν-th loop. Note that $I_{\nu\mu} = 0$ if the μ-th wire does not belong to the ν-th loop; otherwise $I_{\nu\mu} = \pm i_\nu$ — the loop current has the same magnitude everywhere, and the sign of the contribution depends on which current direction is assumed to be positive for the given wire. Let us assume that all the contributions of the ν-th loop current are positive (if not, we can re-define the positive directions of the relevant wires). Then

$$P = \sum_\mu R_\mu \left(\sum_\nu I_{\nu\mu}\right)^2,$$

and for an increment di_ν of the ν-th loop current,

$$\mathrm{d}P = \mathrm{d}i_\nu \cdot 2\sum_\mu I_\mu R_\mu;$$

for a minimum, we need to have $\sum_\mu I_\mu R_\mu = 0$, which is the Kirchoff's voltage law for the ν-th loop.

appendix 4: Proof of the triangle inequality

For a three-lead circuit, let us ground the lead A. (i) First, let us connect the lead C to a voltage source providing a potential $V_0 > 0$ while keeping the lead B disconnected externally; this gives rise to a certain current I_0 which flows from C through the circuit and through the lead A into the grounding wire; this will also induce a certain potential V_1 on the lead B; apparently $0 \le V_1 \le V_0$.

(ii) Second, let us disconnect C from the voltage source and connect B to a voltage source providing the same potential V_1 what it had previously; this gives rise to a current I_1 via A and B, and an induced potential V_2 on the lead C. Now, let us apply the minimal power dissipation theorem. For case (ii), we construct a fictitious potential distribution based on the potential distribution of case (i): all these internal circuit points which have potential φ less than V_1 will have the same potential which they had previously, and all those internal circuit points which have $\varphi \ge V_1$ will have potential V_1 (if a certain resistor extends over the threshold potential V_1, we imagine the resistor as if being made of a resistive wire and "cut" this wire into two segments at the point where $\varphi = V_1$). Such a fictitious potential distribution would be the real potential distribution of a modfied circuit for which all the circuit points with potential $\varphi = V_1$ are connected via a wire to the lead B. Indeed, there is no change in the region $\varphi \le V_1$ as compared to the case (i), which means that the Kirchoff's current law is satisfied there; in the region which had originally $\varphi > V_1$, the potential is now constant, hence there is no current, hence the Kirchoff's current law is also satisfied. All the threshold points $\varphi = V_1$ are connected by a wire which directs all the total current I_0 into the lead B so that the Kirchoff's current law remains still satisfied. For such a modified circuit, the power dissipation is $V_1 I_0$; due to the power dissipation theorem, this is larger or equal to the actual dissipation $V_1 I_1$, hence $I_1 \le I_0$.

Finally, let us introduce case (iii): we disconnect A from the ground, and connect B and C to the voltage sources which provide potentials V_1 and V_2, respectively. Analogously to what we did before, we can show that the emerging current $I_2 \le I_0$ (through B and C). Now we can write inequalities for the resistances:

$$R_{AB} = \frac{V_1}{I_1} \ge \frac{V_1}{I_0}, \quad R_{BC} = \frac{V_0 - V_1}{I_1} \ge \frac{V_0 - V_1}{I_0};$$

if we sum up these inequalities we obtain

$$R_{AB} + R_{BC} \ge \frac{V_1}{I_0} + \frac{V_0 - V_1}{I_0} = \frac{V_0}{I_1} = R_{AC},$$

QED.

Note that owing to the triangle inequalities, the resistance can be used to define the distance between two circuit points (or between two points of a continuous conducting medium); then, instead of meters, the distance will be characterized in ohms.

2 Circuits including capacitors and inductances

In order to be able to solve circuits involving capacitors and inductances, the knowledge of several facts is needed. Some facts will be provided here without proof; more insight will be given in the section "Electromagnetism'.

Let us begin with **capacitors**. A capacitor can be thought of as consisting of two parallel conducting sheets (plates) which are very close to each other, and separated by a thin dielectric (insulating) layer [24]. We mentioned in the introduction of Section 1 that typically, we can neglect charges on the wires; this is because any non-negligible charge on wires would give rise to a huge electric field, and hence, to a huge voltage. However, situation is different if we have two parallel conducting plates: if these two plates have equal and opposite charges, so that the system as a whole is electrically neutral, the huge electric field is constrained into the narrow layer between the plates, hence the voltage (the product of the layer thickness and field strength) can remain moderate. Typically [25], the voltage between the plates is proportional to the charge sitting on one of the plates. Since a capacitor is electrically neutral as a whole, the Kirchoff's current law remains valid for capacitors, as well: current flowing along a wire to one plate (increasing the charge there) equals to the current flowing from the other plate (decreasing the charge there) along another wire.

fact 10: Capacitance is defined as

$$C = q/V,$$

where q is the charge on the plates of the capacitor (one plate has $+q$, the other one $-q$) and V is the potential difference

[24] There are different types of capacitors with different shapes, but such details are not important for the time being.
[25] if the inter-plate distance and the dielectric permeability of the insulator remain constant

between the plates of the capacitor. Unless otherwise mentioned, C is independent of the applied voltage V.

fact 11: The energy of a charged capacitor is

$$W = CV^2/2.$$

Indeed, consider a charging of a capacitor. If a charge $\mathrm{d}q$ crosses a potential difference V, electrical work $dA = VI\cdot \mathrm{d}t = V\cdot \mathrm{d}q$ needs to be done. So, the total work done $A = \int_0^V V\cdot \mathrm{d}q = \int_0^V V\cdot d(CV) = C\int_0^V VdV = CV^2/2$.

fact 12: The voltage on a capacitor cannot change momentarily, because a momentary change of the charge would require an infinite current; the characteristic time of the voltage change (with which the voltage will relax towards its equilibrium value) is

$$\tau = CR,$$

where R is the net resistance of the circuit connected to the capacitor's leads.

Indeed, consider a capacitor with voltage V, the leads of which are attached to a resistance R. According to Kirchoff's laws, $R\frac{\mathrm{d}q}{\mathrm{d}t} + \frac{q}{C} = 0$, hence

$$\frac{\mathrm{d}q}{q} = -\frac{\mathrm{d}t}{CR} \Rightarrow \ln q - \ln q_0 = -\frac{t}{CR} \Rightarrow q = q_0\mathrm{e}^{-t/RC}.$$

Here, $-\ln q_0$ serves as an integration constant.

fact 13: In a simple $R-C$-circuit, charge (and voltage) on the capacitor, as well as the current decay exponentially, $\propto \mathrm{e}^{-t/\tau}$.

Now, let us consider **inductors**. In the section "Electromagnetism' we'll learn that similarly to how electrical charges give rise to an electric field, currents (moving charges) give rise to a magnetic field, which is characterized by *magnetic induction* B (also referred to as the magnetic B-field). We'll need also the concept of *magnetic flux* Φ, which can be interpreted intuitively (and loosely) as the number of magnetic field lines passing through a closed (possibly fictitious) loop; in the case of an homogeneous magnetic field perpendicular to the loop, $\Phi = BS$, where S is the surface area of the loop. The importance of the concept of magnetic flux lies in the fact that if it changes in time, an emf. is created in the loop (circuit), see below.

So, any current in a circuit gives rise to a magnetic field, which, in its turn, will cause a magnetic flux passing through that electric circuit. Typically, however, that flux is relatively small so that the emf. caused by it can be neglected. In order to create a larger flux, coils (inductors) are used. Increasing the number N of overlapping wire loops has two-fold effect: first, the current in the circuit will pass N times parallel to itself, giving rise to a N-fold increase of the magnetic field; second, the magnetic field lines pass now the circuit N times, giving rise to another factor N for the magnetic flux.

fact 14: Self-inductance of an inductor (often called just "inductance") is defined as

$$L = \Phi/I,$$

where I is the current flowing through the inductor, and Φ is the magnetic flux created by that current passing through the inductor itself [26]. Unless otherwise noted, the inductance may be assumed to be independent of current [27].

fact 15: Electromotive force created in a circuit due to changing magnetic field

$$\mathcal{E} = -\mathrm{d}\Phi/\mathrm{d}t,$$

where Φ is the magnetic flux through the circuit. If Φ is created by the self-inductance effect in an inductor, we obtain

$$\mathcal{E} = -L\mathrm{d}I/\mathrm{d}t.$$

The minus sign refers to the fact that this electromotive force tries to oppose the current change.

fact 16: The energy stored in an inductor

$$W = LI^2/2.$$

Indeed, consider the electrical work needed to create a current in an inductance: $A = \int \mathcal{E}\cdot \mathrm{d}q = \int \mathcal{E}\cdot I\mathrm{d}t = \int L\frac{\mathrm{d}I}{\mathrm{d}t}\cdot I\mathrm{d}t = L\int IdI = LI^2/2$.

fact 17: The current through an inductance cannot change momentarily, because this would cause an infinite electromotive force; the characteristic time of the current change (with which the current will relax towards its equilibrium value) is

$$\tau = L/R,$$

where R is the net resistance of the circuit connected to the inductance leads.

Indeed, consider an inductance with current I, the leads of which are attached to a resistance R. According to Kirchoff's laws, $RI + L\frac{\mathrm{d}I}{\mathrm{d}t} = 0$, hence

$$\frac{dI}{I} = -\frac{R\mathrm{d}t}{L} \Rightarrow \ln I - \ln I_0 = -\frac{Rt}{L} \Rightarrow I = I_0\mathrm{e}^{-Rt/L}.$$

Here, $-\ln I_0$ serves as an integration constant.

fact 18: In a simple $L-R$-circuit, inductor current (and voltage) decays exponentially, $\propto \mathrm{e}^{-t/\tau}$.

With this result, we are finally ready to return to the problem of stability of circuits with nonlinear elements obeying negative differential resistance, cf. idea 24 and fact 9.

idea 31: When you are asked to perform a stability analysis, keep in mind that
(a) the most standard way of doing it is by assuming that the departure from a stationary state is very small, hence the idea 26 can be applied, i.e. all the nonlinear dependences can be linearized[28];
(b) the system needs to involve inertia which, in the case of circuits, is most typically provided by inductance: every wire has a non-zero inductance (a very rough rule is that one millimetre of wire length contributes 0.5 nH to the overall inductance). Introduction of inductance may not be needed for systems with capacitors which already obey inertia of capacitor charge (and hence, of capacitor voltage).
(c) the linearized differential equation may have more than one solution; the system as a whole is stable if none of the solutions

[26] One can also speak about the inductance of simple circuit wires: although the inductance of simple wires is small, there are applications where it cannot be neglected

[27] However, in the case of inductors with ferromagnetic coils, there is an essential non-linearity: the inductance will decrease with increasing current.

[28] Alternatively, conservation laws can be studied, cf. http://www.ipho2012.ee/physicscup/physics-solvers-mosaic/1-minimum-or-maximum/

is unstable (if there is even one unstable solution, its exponential growth would lead to an eventual departure of the system from the equilibrium state).

pr 56. Consider a tunnel diode which is connected in series with a resistance R to a battery (for a typical $V-I$-curve of a tunnel diode, see problem 25). Let the parameters of the system be such that there is a stationary state at such diode voltage that diode's differential resistance $R_{\text{diff}} \equiv \frac{\mathrm{d}V}{\mathrm{d}I}$ is negative. Under which condition will this state be stable?

The answer of this problem has an interesting implication. Indeed, suppose we try to measure the V-I-curve of a tunnel diode; we would use the same circuit as in the case of this problem (with the addition of an ammeter), but according to the results this problem, the state with $R_{\text{diff}} < 0$ would be either unstable or cannot be reached due to fact 9. If that is so then how can we measure the full $V-I$-curve? It appears that for this purpose, the tunnel diode can be stabilized if a suitably selected capacitor and resistor are connected in parallel to the diode. In that case, the stability analysis becomes more complicated; then, the most efficient approach makes use few more ideas and therefore the corresponding problem is considered at the very end of the booklet (see problem 82).

idea 32: Energy conservation law can be used to calculate heat dissipation. In addition to capacitors' and inductances' energies (c.f. facts 11 and 16), the work done by electromotive force needs to be taken into account: $A = \int \mathcal{E} I \cdot \mathrm{d}t = \int \mathcal{E} \mathrm{d}q$; if $\mathcal{E}$ is constant, this simplifies into

$$A = \mathcal{E} \cdot \delta q,$$

where δq is the charge passing through the electromotive force.

idea 33: If a battery is connected in series to a capacitor, the charge passing through the battery can be found as the change of charge on a plate of the capacitor:

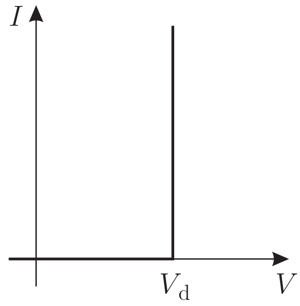
$$\delta q = C \cdot \delta V,$$

where δV is the change of the capacitor's voltage.

pr 57. A capacitor of capacitance C is charged using a battery of electromotive force $\mathcal{E}$. Find the heat dissipated during the charging process (either via a spark or in the wires or in the battery due to (internal) resistance.

This is a very simple problem which we solve here to show the procedure. During the charging process, a plate of the capacitor obtains charge q; this charge necessarily needs to come through the battery, hence the work done by the battery $A = q\mathcal{E} = C\mathcal{E}^2$. Part of this work is accumulated as the potential energy of the capacitor, $W = C\mathcal{E}^2/2$; the rest is dissipated as a heat, $Q = A - W = C\mathcal{E}^2/2$.

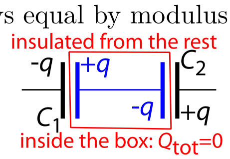
pr 58. A capacitor of capacitance C is charged so that its voltage is V_0. The capacitor is discharged on a series connection of a diode and resistor R. Assume that the following graph provides a good approximation for the V–I dependence of the diode and that the capacitor is discharged down to the voltage V_d . Find the amount of heat which is dissipated on the resistor.



pr 59. A capacitor is charged by connecting it to a series connection of a battery of electromotive force $\mathcal{E}$, inductor of inductance L, and a diode. For the V–I dependence of the diode use the graph of the previous problem; internal resistance of the battery is negligible. To which voltage the capacitor will be charged, assuming that $\mathcal{E} > V_d$?

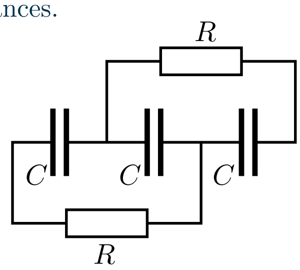
idea 34: If a circuit includes a set of the plates of capacitors which is isolated electrically from the rest of circuit by the dielectric insulating layers of the capacitors, the net charge on these plates is conserved.

For instance, consider a series connection of two capacitors which were initially charge-free. Then the set of two plates (shown in figure) forms an electrically insulated system, hence the net charge there will remain always zero, i.e. the two capacitors will bear always equal by modulus charge.



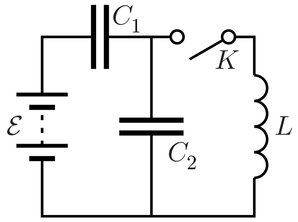
pr 60. Show that the series connection of capacitors of capacitance C_1, C_2,... C_n has net capacitance $C = (C_1^{-1} + C_2^{-1} + \ldots C_n^{-1})^{-1}$.

pr 61. Three identical charge-less capacitors of capacitance C are connected in series. The capacitors are charged by connecting a battery of electromotive force $\mathcal{E}$ to the terminal leads of this circuit. Next, the battery is disconnected, and two resistors of resistance R are connected simultaneously as shown in figure below. Find the net heat which will be dissipated on each of the resistances.



idea 35: Extremal currents and voltages can be often found from the energy conservation law by noting that (a) at the moment of an inductor's current extremum, $\frac{dI}{dt} = 0$, hence the voltage on the inductor $V = L\frac{dI}{dt} = 0$; (b) at the moment of a capacitor's voltage extremum, $\frac{dV}{dt} = C^{-1}\frac{\mathrm{d}q}{\mathrm{d}t} = 0$, hence the current through capacitor's leads $I = \frac{\mathrm{d}q}{\mathrm{d}t} = 0$.

pr 62. Consider the electrical circuit given below: initially chargeless capacitors C_1 and C_2 were connected to a battery, and at certain moment, the key K will be closed. After that moment, current and voltage will start oscillating. For these oscillations, you need to find (a) the maximal current $I_{\max}$ through the inductor; (b) the maximal voltage $V_{\max}$ on the capacitor C_1.



One way of solving this problem is using the idea 35, together with the energy conservation law. The second way is to study the voltage (and current) oscillations in the circuit. LC-circuit oscillations will be studied later in more details; here it is enough to formulate one more "fact".

fact 19: In a closed circuit consisting of a capacitor C and an inductor L, current through the inductor and voltage on the capacitor will oscillate sinusoidally with circular frequency $\omega_0 = 1/\sqrt{LC}$, e.g. $V = V_0 \sin(\omega_0 t + \varphi)$.[29]

Indeed, for such a circuit, Kirchoff's voltage law states that $q/C + L\frac{dI}{dt} = 0$; here, q is the capacitor's charge, and $I = \frac{dq}{dt}$, hence $q + LC\frac{d^2q}{dt^2} = 0$. This is a second order linear differential equation, the solution of which is given by $q = q_0 \sin(\omega_0 t + \varphi)$, where the constants q_0 and φ can be found using the initial conditions (e.g. the current and voltage values at $t = 0$), c.f. Formula sheet I-3.

idea 36: If the task is to find a temporal dependence of a voltage or current, and the circuit contains one or more batteries or constant current sources, the solution can be found as a superposition of a stationary solution (when all the voltages and currents are constant), and a solution obtained for a simplified circuit, where all the ideal batteries are substituted with wires, and the current sources are "cut off".

This idea is based on the fact that if there are neither batteries nor current sources present, circuits containing linear resistors, capacitors, and/or inductors are described by Kirchoff's laws which represent a set of *homogeneous* linear differential equations. The word *homogeneous* here means that *each term in these equations contains **exactly one** unknown function* (or its time derivative), $I_k(t)$ or $V_l(t)$ (the current in the k-th wire segment and the voltage on the l-th circuit element). On the other hand, if there are also batteries or current sources in the circuit, there would be also *terms without any of the unknown functions being involved* — these are the terms involving the corresponding electromotive forces and/or constant current values; such systems of equations are called *linear nonhomogeneous differential equations*, and if we remove from these equations the terms without unknown functions, we obtain what is called *the homogeneous part* of the equations.

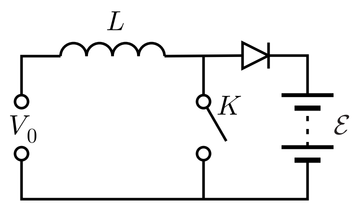
General theory of linear differential equations tells us that the generic solution of the nonhomogeneous differential equations is obtained as the sum of *(a) the generic solution of the homogeneous part of these differential equations* and *(b) one single (any) solution of the nonhomogeneous equations*. If the *nonhomogeneous part of the equations* are constant terms then one solution can be easily found as a *stationary solution* which is constant in time and for which all the time derivatives are equal to zero. In the case of our circuits, the nonhomogeneous terms are indeed constant, and usually the stationary solutions (currents in all wires and voltages on all elements) can be figured out without writing down the differential equations.

In order to find the generic solution of the homogeneous equations, we need to notice first that the homogeneous part of the equations corresponds to the case when all the electromotive forces and currents supplied by current sources are set equal to zero; this corresponds to short-circuiting all the electromotive forces and removal of all the current sources, as suggested by idea 36. Now, the task of finding the generic solution may become already easy enough: if parallel and series connections of resistors can be combined into one single resistor, and the same applies to capacitors, as well as to inductors then we'll be having either a $R-C$ circuit, $L-R$-circuit, or $L-C$-circuit. In each of these cases, we already know the solution as provided by the facts 13, 18, and 19, which is either an exponential decay $I = I_0 e^{-t/\tau}$ or a sinusoidal oscillation. In more complicated cases, it is still possible to find the generic solution without writing down the system of equations, by using AC-resonance as explained by idea 48.

pr 63. Under the assumptions of the previous problem, sketch the voltage on the capacitor C_1 as a function of time.

idea 37: If a constant voltage V is applied to the leads of an inductor, its current will start changing linearly in time: $L\frac{dI}{dt} = V \Rightarrow I = I_0 + Vt/L$.

pr 64. The circuit below makes it possible to charge a rechargeable battery of voltage $\mathcal{E} = 12\,\mathrm{V}$ with a direct voltage source of a voltage lower than $\mathcal{E}$, $V_0 = 5\,\mathrm{V}$. To that end, the key K is periodically switched on and off — the open and closed periods have equal length of $\tau = 10\,\mathrm{ms}$. Find the average charging current assuming that $L = 1\,\mathrm{H}$. The diode can be considered to be ideal; neglect the ohmic resistance of the inductor.



idea 38: For circuits containing L and R or C and R, *at time-scales much shorter then the characteristic times*

$$\tau = RC \quad \text{or} \quad \tau = L/R,$$

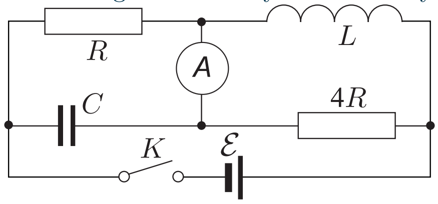
the capacitor's charge and inductor's current remain almost constant. In particular, if a capacitor was chargeless, its voltage remains almost zero, i.e. it is essentially short-circuited; if there was no current in an inductor, its current remains zero, i.e. the wire leading to the inductor can be considered as broken. If a capacitor had a charge Q corresponding to a voltage V_0, its voltage remains essentially constant, i.e. it acts as (and can be substituted by) a battery of emf. $\mathcal{E} = V_0$. Similarly, if an

[29] If there is also a small resistance R connected in series then $V = V_0 e^{-\gamma t} \sin(t\sqrt{\omega_0^2 - \gamma^2} + \varphi)$ with $\gamma = R/2L$; this will be derived after idea 48

inductor had a current I_0, it can be substituted by a respective constant current source. If we try to forcefully break the current through an inductor by switching it off, a rapid fall of current I creates a huge voltage $L\frac{dI}{dt}$ which usually leads to a spark at the switch[30]

At time-scales which are much longer than the characteristic times, the situation is reversed: inductor can be considered as a short-circuiting wire, and capacitor as an insulator. This is because all the currents and voltages tend exponentially towards the equilibrium state so that the difference from the equilibrium value $\Delta \propto e^{-t/\tau}$: the capacitor charge is almost constant, hence there is no current, and the inductor current is almost constant, hence no electromotive force.

pr 65. The key of the circuit given below has been kept open; at certain moment, it is closed. (a) What is the ammeter reading immediately after the key is closed? (b) The key is kept closed until an equilibrium state is achieved; what is the ammeter reading now? (c) Now, the key is opened, again; what is the ammeter reading immediately after the key is opened?

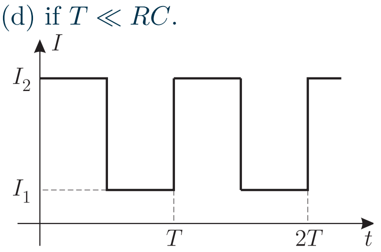


The short-time-approximation of the idea 38 can be further improved with the help of the following idea.

idea 39: If the considered time interval is much less than RC or L/R, the time dependence of the capacitors' charges and inductors' currents can be linearized: $q = q_0 + I_c t$, where I_c is an almost constant current feeding the capacitor, and $I_L = I_0 + V_L t/L$, where V_L is an almost constant voltage applied to the leads of the inductor.

pr 66. Capacitor of capacitance C and resistor of resistance R are connected in parallel, and rectangular current pulses (see figure) are applied to the leads of the system. Assuming that $I_2 = -I_1$ and that at the moment $t = 0$, the capacitor had no charge, sketch the voltage on the capacitor as a function of time (a) if $T \gg RC$, and (b) if $T \ll RC$.

Now assume that the periodic input current has been applied for a very long time (for much longer than RC), and let us no longer assume $I_2 = -I_1$. Find the average voltage and the amplitude of the voltage oscillations on the capacitor if (c) $T \gg RC$, and (d) if $T \ll RC$.



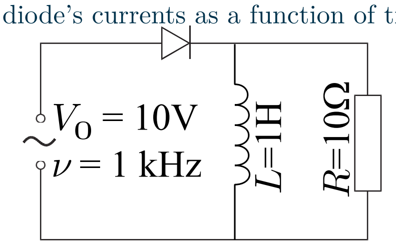
The last part of this problem requires one more idea.

idea 40: Suppose that a periodic signal is applied to a circuit containing two or more of the following elements: resistances (R), capacitors (C), inductances L, nonlinear elements such as diodes. If the system has evolved long enough (much longer than RC and L/R, so that the system response has also become periodic), the average voltage on the leads of an inductor is zero, and the average current through each capacitor is zero.

Indeed, the voltage on an inductor can be expressed via its current, $V = L\frac{dI}{dt}$, and an average non-zero voltage would imply a non-constant average current, $\frac{d\langle I\rangle}{dt} = \langle V\rangle \neq 0$, which violates the assumption that the system response has become periodic. Similarly, a non-zero average current through the wires leading to a capacitor would imply a non-constant average charge on the plates of it.

This idea is illustrated with one more problem.

pr 67. Alternating voltage $V = V_0\cos(2\pi\nu t)$ is applied to the leads of the circuit shown below. Sketch the graphs of the resistor's and diode's currents as a function of time.



Finally, there is one more idea which can be used when it is needed to find a charge passing through a resistor.

idea 41: If a circuit contains a current loop (as defined for idea 13) which contains a resistor R, an inductor L, and/or embraces an externally applied magnetic flux Φ_e, the charge passing through the resistor can be expressed in terms of the change of the magnetic flux (both external and self-induced):

$$q = (\delta\Phi_e + L\delta I)/R.$$

Indeed, this follows immediately from the Kirchoff's voltage law $\frac{d\Phi_e}{dt} + L\frac{d\Phi_i}{dt} = -RI = -R\frac{dq}{dt}$ (where $\Phi_i = LI$), which can be written as $d\Phi + L\cdot dI = -R\cdot dq$ and easily integrated.

pr 68. In order to measure magnetic induction, the following device can be used. A small coil with N loops, surface area S and inductivity L is connected to a ballistic galvanometer which is graduated to show the total charge of a current pulse [31]. The coil is placed into a magnetic field so that the axis of the coil is parallel to the magnetic field. With a fast motion, the coil is flipped around by 180° (so that axis is again parallel to the magnetic field); find the total charge of the current pulse passing through the galvanometer if the total ohmic resistance of the coil and wires is R

The next idea can be considered to be a limit case of the idea 38, but it can be formulated as a more generic conservation law.

[30] This effect can be used intentionally to create a short pulse of high voltage.

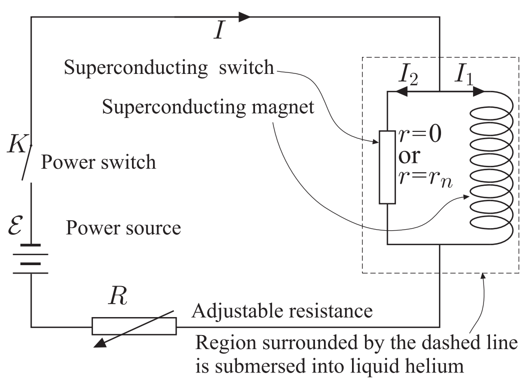
[31] The needle of the ballistic galvanometer has a large inertia, it will take some time before it will reach the equilibrium position; because of that, if a short current pulse is let through such a galvanometer (shorter than the response time of the galvanometer), the maximal declination of the needle will be proportional to the total charge of the pulse.

idea 42: If a circuit includes a current loop which is entirely in a superconducting state (i.e. with strictly zero resistance), the magnetic flux through it is conserved, $\Phi = \text{Const}$. This follows directly from the Kirchoff's voltage law for the superconducting circuit, $\frac{d\Phi}{dt} = 0$. If the flux is only due to the self-inductance, and there is only one inductor of inductance L in the circuit then $LI = \text{Const}$; if L is constant then also I is constant.

In the Section "Electromagnetism', there will be more examples for the application of this idea (involving external fields and mutual induction); here just one problem is provided.

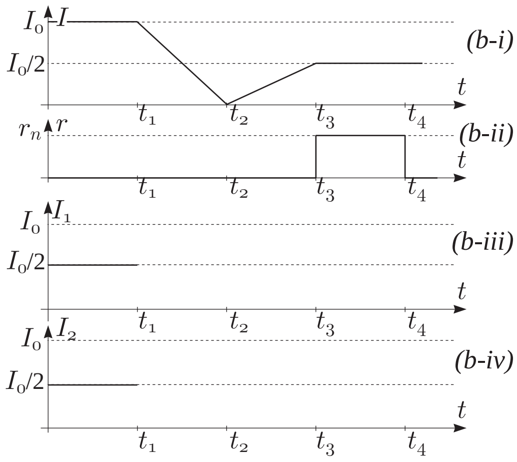
pr 69. [IPhO-1994] Superconducting magnets are widely used in laboratories. The most common form of superconducting magnets is a solenoid made of superconducting wire. The wonderful thing about a superconducting magnet is that it produces high magnetic fields without any energy dissipation due to Joule heating, since the electrical resistance of the superconducting wire becomes zero when the magnet is immersed in liquid helium at a temperature of 4.2 K. Usually, the magnet is provided with a specially designed superconducting switch, as shown in Fig. 1. The resistance r of the switch can be controlled: either $r = 0$ in the superconducting state, or in the normal state. When the persistent mode, with a current circulating through the magnet and superconducting switch indefinitely. The persistent mode allows a steady magnetic field to be maintained for long periods with the external source cut off.

The details of the superconducting switch are not given in Fig. (a). It is usually a small length of superconducting wire wrapped with a heater wire and suitably thermally insulated from the liquid helium bath. On being heated, the temperature of the superconducting wire increases and it reverts to the resistive normal state. The typical value of is a few ohms. Here we assume it to be 5 Ω. The inductance of a superconducting magnet depends on its size; assume it be 10 H for the magnet in Fig. (a). The total current I can be changed by adjusting the resistance R.

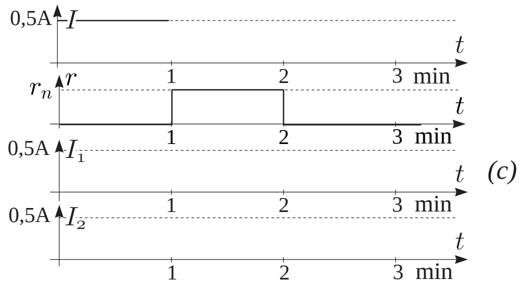


The arrows denote the positive direction of I, I_1 and I_2.

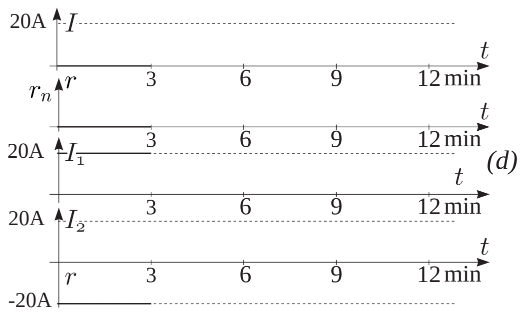
(a) If the total current I and the resistance r of the superconducting switch are controlled to vary with time in the way shown in Figs. (b)-i and (b)-ii respectively, and assuming the currents I_1 and I_2 flowing through the magnet and the switch respectively are equal at the beginning (Fig. (b)-iii and Fig. (b)-iv), how do they vary with time from t_1 to t_4? Plot your answer in Fig. (b)-iii and Fig. (b)-iv.



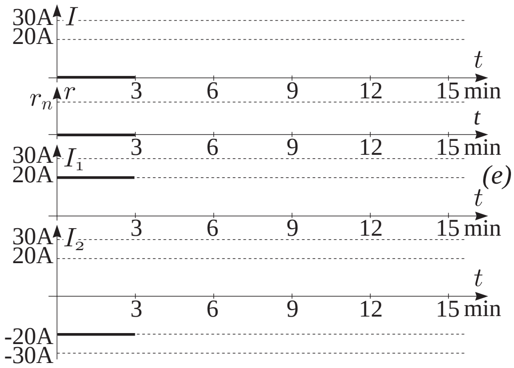
(b) Suppose the power switch K is turned on at time $t = 0$ when $r = 0$, $I_1 = 0$ and $R = 7.5\,\Omega$, and the total current I is 0.5 A. With K kept closed, the resistance r of the superconducting switch is varied in he way shown in Fig. (c)-ii. Plot the corresponding time dependences of I, I_1 and I_2 in Figs. (c)-i, (c)-iii and (c)-iv respectively.



(c) Only small currents, less than 0.5 A, are allowed to flow through the superconducting switch when it is in the normal state, with larger currents the switch will be burnt out. Suppose the superconducting magnet is operated in a persistent mode, i. e. $I = 0$, and $I_1 = i_1$(e.g. 20 A), $I_2 = -i_1$, as shown in Fig. (d), from $t = 0$ to $t = 3$ min. If the experiment is to be stopped by reducing the current through the magnet to zero, how would you do it? This has to be done in several operation steps. Plot the corresponding changes of I, r, I_1 and I_2 in Fig. (d)



(d) Suppose the magnet is operated in a persistent mode with a persistent current of 20 A [$t = 0$ to $t = 3$ min. See Fig. (e)]. How would you change it to a persistent mode with a current of 30 A? plot your answer in Fig. (e).



Previously we introduced inductance with facts 14–16. In particular, we learned that electric current I in a circuit will cause a flux of magnetic field $\Phi = LI$, where L is the self-inductance of the loop. Consider now two loops which are positioned in each other's neighbourhood. Then, similarly to what we observed for a single loop, a current I_1 *in the first loop* will cause a magnetic field in the position of the other loop, and hence, gives rise to a flux Φ_{21} through the second loop. Due to the linearity of the Maxwell's equations[32], the flux is proportional to the current, so $\Phi_{21} \propto I_1$; the coefficient of proportionality L_{21} is called *the mutual inductance.* Now we can express the total flux in the second loop as

$$\Phi_2 = L_{21}I_1 + L_2I_2,$$

where L_2 is the self-inductance of the second loop. Similarly, the flux through the first loop is expressed as

$$\Phi_1 = L_{12}I_2 + L_1I_1,$$

where L_1 is the self-inductance of the first loop.

fact 20: If there are two loops (or two inductors) then the mutual inductance L_{21} is defined as the coefficient of proportionality between the flux in the second loop, caused by the current I_1 in the first loop, and the current I_1. L_{12} is defined analogously; it appears that always, $L_{12} = L_{21}$. If there are only two magnetically coupled circuits, both are usually denoted as M, so that total flux through the first loop is expressed as

$$\Phi_1 = MI_1 + L_1I_1.$$

The total energy of the system is

$$W = \tfrac{1}{2}\left(L_1I_1^2 + 2MI_1I_2 + L_2I_2^2\right).$$

(Both equalities will be proved below.) As long as the currents I_1 and I_2 flow in independent circuits, the sign of M is not important as we can just change the sign of one of the currents. However, if I_1 and I_2 are connected to each other via Kirchoff's current law[33], we need to be careful to select the correct sign of M.

The equality $L_{12} = L_{21}$ is very useful, it is analogous to the reciprocity theorem (cf. appendix 2) and to Newton's 3rd law. Sometimes it is difficult to calculate a force exerted by a body A to a body B, but it is easy to calculate the force exerted by the body B to the body A (for instance, when A is a point charge and B — an homogeneously charged plate); similarly, it is sometimes difficult to calculate L_{12}, but it is easy to calculate L_{21} (naturally we determine M then by calculating L_{21}).

In order to prove this fact, let us first derive an expression for the total energy for the system consisting of the two current-carrying circuits similarly to how we proved the fact 16. So we have two simple circuits, each consisting of an inductor and an adjustable current source. We start with increasing the current in the first loop while keeping the current in the second loop zero. The work done by the current source in the second loop is zero because no charge will pass through the current source. If the final value of the current in the first loop is I_1 then the work done by the first current source is $\frac{1}{2}L_1I_1^2$ (due to the fact 16). Next we start increasing the current in the second loop while keeping the current in the first loop constant. Let us consider a current increment $\mathrm{d}I_2$; this will give rise to a voltage in the first loop which needs to be compensated by an electromotive force of the current source $\mathcal{E}_1 = L_{12}\frac{\mathrm{d}I_2}{\mathrm{d}t}$; similarly for the current source of the second loop $\mathcal{E}_2 = L_2\frac{\mathrm{d}I_2}{\mathrm{d}t}$. The work done by the first current source

$$A_1 = \int \mathcal{E}_1\mathrm{d}q_1 = \int L_{12}\frac{\mathrm{d}I_2}{\mathrm{d}t}\cdot\mathrm{d}q_1 = \int L_{12}\frac{\mathrm{d}q_1}{\mathrm{d}t}\mathrm{d}I_2 = L_{12}\int I_1\mathrm{d}I_2.$$

Now let us recall that in our process, I_1 is constant, so the integral is easily taken, resulting in $A_1 = L_{12}I_1I_2$. The work done by the second current source is calculated similarly, resulting in $A_2 = \mathcal{E}_2\mathrm{d}q_2 = \frac{1}{2}L_2I_2^2$. Therefore, the total amount of work which has been done is

$$W = \tfrac{1}{2}L_1I_1^2 + A_1 + A_2 = \tfrac{1}{2}\left(L_1I_1^2 + 2L_{12}I_1I_2 + L_2I_2^2\right).$$

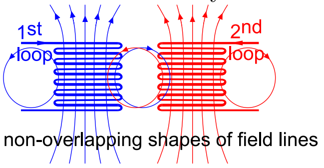
Now it is clear that if we increase the currents in the reverse order, we shall obtain a result where the indices 1 and 2 are swapped: $W = \frac{1}{2}\left(L_1I_1^2 + 2L_{21}I_1I_2 + L_2I_2^2\right)$. The work of the current sources was transferred to the energy of the magnetic field and its value can depend only on the final state of the system, hence these two expressions must provide the same result and therefore, $L_{12} = L_{21}$.

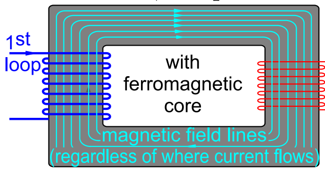
Magnetic field energy density is proportional to the squared magnetic field strength and hence, is always positive; because of that, any system of magnetically coupled current-carrying inductors must also have a positive total energy. From this condition, we can derive a useful inequality for the mutual inductance M. Indeed, for two loops, the total energy can expressed as $W = \frac{1}{2}I_1^2\left(L_1 + 2Mx + L_2x^2\right)$, where $x = I_2/I_1$. Note that x can take any values, including negative ones, but W must remain non-negative. Therefore, the roots x_1 and x_2 of the quadratic equation $L_1 + 2Mx + L_2x^2 = 0$ cannot take real values (unless the two roots are equal): otherwise, for any value of x between x_1 and x_2, W would be negative. Thus, the discriminant must be nonnegative, $M^2 \le L_1L_2$.

Equality $M^2 = L_1L_2$ would mean that the total energy of the system can be zero even if there are non-zero currents in the loops. Zero energy means zero magnetic field: the magnetic field created by a current in one loop must negate everywhere the magnetic field created by the current in the other loop. This is possible only if the shape of the field lines of the both loops are identical. To achieve this, there are two possibilities:
(a) they must have identical and overlapping in space geometrical shape (two solenoids of identical length and cross-sectional area but possibly with different winding densities, one of which is inserted tightly into the other), or
(b) the windings of the both inductors must be made around the same closed ferromagnetic core as shown in figure — such devices are called transformers. As we shall learn in electro-

[32] To be discussed in more details in the Electromagnetism booklet
[33] For instance, by problem 85

magnetism, magnetic field lines keep, if possible, inside ferromagnetic materials; if the ferromagnetic core is closed (e.g. having a toroidal shape), there is no need for the field lines caused by the current in the winding to exit the core. In that case, the shape of the field lines inside the core is defined by the shape of the core, and not by where and how the winding is made. Note that electrical transformers are usually made using a closed ferromagnetic core as shown in the figure below; for the electrical symbol of such transformer, see problem 70.



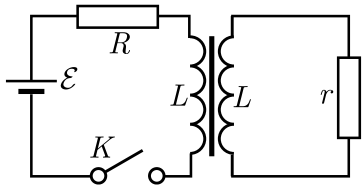


fact 21: Mutual inductance cannot be larger than the geometric average of the self-inductances; equality $M = \sqrt{L_1 L_2}$ is achieved for transformers when all the magnetic field lines created by the both coils have identical shapes.

idea 43: If there is no leakage of magnetic field lines from a transformer, i.e. if $M = \sqrt{L_1 L_2}$ then the total inductive electromotive force in both inductors is defined by the same linear combination of currents $J \equiv I_1\sqrt{L_1} + I_2\sqrt{L_2}$ (indeed, $\Phi_1 = J\sqrt{L_1}$ and $\Phi_2 = J\sqrt{L_2}$). This has two important consequences.
(i) The idea 38 states that if there are inductors, the inductor currents need to be continuous; in the case of transformers with $M = \sqrt{L_1 L_2}$, this statement has to be modified: what need to be continuous are the fluxes, and this is achieved as long as J is a contiuous function of time, i.e. current jumps satsify the condition $\Delta I_1 = -\Delta I_2\sqrt{I_2/I_1}$.
(ii) Assuming that there are no other inductors affected by the currents I_1 and I_2, if we write down all the Kirchoff's voltage laws, we can reduce the order of the system of differential equations by one if we use J (or $J/\sqrt{L_1}$ which has the dimensionality of an electric current) as one unknown function, and any other linear combination of I_1 and I_2 as the other unknown.

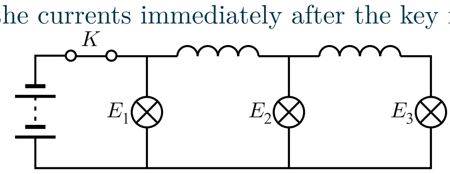
Let us elaborate on the statement (i) and consider the simplest case of only two unknown functions, I_1 and I_2. Then we would be able to get two equations given by the Kirchoff's voltage law. Inductors yield us terms containing derivatives of I_1 and I_2 so that with $M < \sqrt{L_1 L_2}$ we would have a second order system of linear differential equations (if we take derivative of both equations, and eliminate from the four equations I_2 and $\frac{\mathrm{d}I_2}{\mathrm{d}t}$, we obtain a second order differential equation for I_1). Now with $M = \sqrt{L_1 L_2}$, however, we can use J as one of the variables, and eliminate $\frac{\mathrm{d}J}{\mathrm{d}t}$ from the set of equations; we result in an expression relating I_1 and I_2 to each other, i.e. we can express I_2 in terms of I_1. If we substitute I_2 in one of the original differential equations using this expression, we obtain a first order differential equation for I_1.

pr 70. An electrical transformer is connected as shown in the circuit below. Both windings of the transformer have the same number of loops and the self-inductance of the both coils is equal to L; there is no leakage of the magnetic field lines from the core so that the mutual inductance is also equal to L.
(a) Find the current in the both loops immediately after the switch is closed.
(b) Find the currents as a function of time.

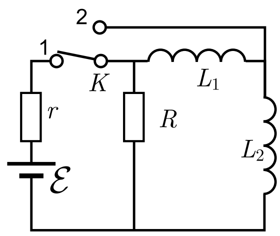


Problems involving ideas 32–43

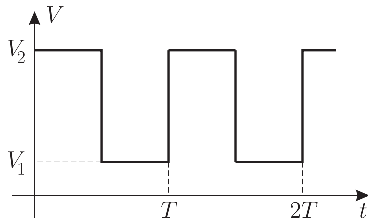
pr 71. There are three identical lamps which are connected to a battery as shown in figure; the current through each lamp is I. Find the currents immediately after the key is opened.



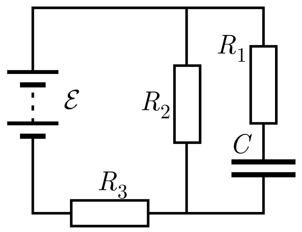
pr 72. In the circuit shown below, the key K has been kept in the position 1; after an equilibrium state has been reached, the key is thrown over to the position 2. This is done much faster than the characteristic time $(L_1 + L_2)/R$. After that, we wait for a very long time until a new equilibrium state is reached. Find the amount of heat Q which was dissipated in the resistor R after the key was switched to the position 2. Also, find the total charge q which flowed through the resistor R during the same period of time. Neglect the internal resistance of the inductors. Note that the resistance of all the wires is also negligible, but there is no superconductivity.



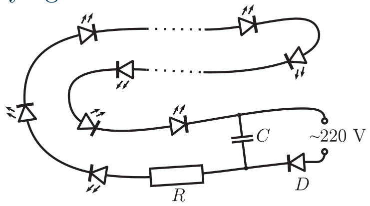
pr 73. Capacitor of capacitance C and resistor of resistance R are connected in series, and rectangular voltage pulses (see figure) are applied to the leads of the system. Find the average power which dissipates on the resistors assuming (a) that $T \gg RC$; (b) that $T \ll RC$.



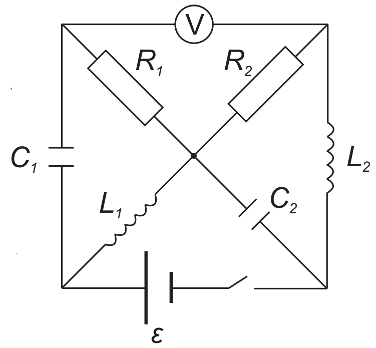
pr 74. Determine the time constant for the circuit shown in figure (i.e. for the process of charging the capacitor, time interval during which the charging rate drops e times).



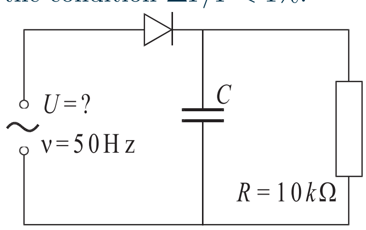
pr 75. A boy wants to build decorative lights using 50 light emitting diodes, to be fed by AC-voltage $V = V_0 \cos(2\pi\nu t)$, with $V_0 = 311\,\mathrm{V}$ and $\nu = 50\,\mathrm{Hz}$. The circuit he plans to use is given below. The voltage of his light emitting diodes can be taken equal to $3\,\mathrm{V}$ (it remains constant for a wide range of forward currents); the nominal current is $20\,\mathrm{mA}$. Find the optimal value of the resistor R (ensuring a nominal operation of the diodes), and minimal value of the capacitance C, if the current variations need to be less than 5%. The rectifying diode D can be considered to be ideal.

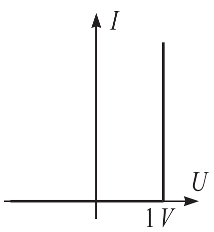


pr 76. [EstFin-2012] For the circuit shown in Figure, $R_1 = 3R$, $R_2 = R$, $C_1 = C_2 = C$, and $L_1 = L_2 = L$. The electromotive force of the battery is $\mathcal{E}$. Initially the switch is closed and the system is operating in a stationary regime.
(a) Find the reading of the voltmeter in the stationary regime.
(b) Now, the switch is opened. Find the reading of the voltmeter immediately after the opening.
(c) Find the total amount of heat which will be dissipated on each of the resistors after opening the switch, and until a new equilibrium state is achieved.



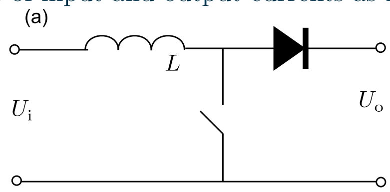
pr 77. [EstFin-2008] A voltage rectifier is made according to the circuit depicted in Figure. The load $R = 10\,\mathrm{k\Omega}$ is fed with DC, equal to $I = 2\,\mathrm{mA}$. In what follows we approximate the U-I characteristic of the diode with the curve depicted in Figure. The relative variation of the current at the load has to satisfy the condition $\Delta I/I < 1\%$.



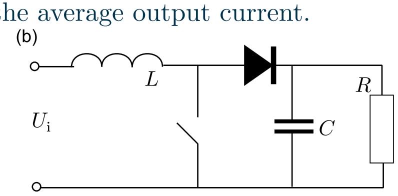


(a) Find the average power dissipation at the diode at the working regime of such a circuit.
(b) Determine the amplitude of the AC voltage (with frequency $\nu = 50\,\mathrm{Hz}$), which has to be applied at the input of the circuit.
(c) Find the required capacitance C.
(d) Find the average power dissipation at the diode during the first period (of AC input voltage) immediately following the application of AC voltage to the input of the circuit.

pr 78. [Est-Fin-2010]
(a) Consider the circuit given in Fig (a), where the diode can be assumed to be ideal (i.e. having zero resistance for forward current and infinite resistance for reverse current. The key is switched on for a time τ_c and then switched off, again. The input and output voltages are during the whole process constant and equal to U_i and U_o, respectively ($2U_i < U_o$). Plot the graphs of input and output currents as functions of time.



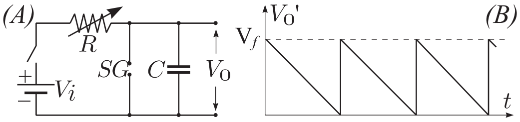
(b) Now, the key is switched on and off periodically; each time, the key is kept closed for time interval τ_c and open — also for τ_c. Find the average output current.



(c) Now, circuit (a) is substituted by circuit (b); the switch is switched on and off as in part ii. What will be the voltage on the load R, when a stationary working regime has been reached? You may assume that $\tau_c \ll RC$, i.e. the voltage variation on the load (and capacitor) is negligible during the whole period (i.e. the charge on the capacitor has no time to change significantly).

pr 79. [IPhO-2001] A sawtooth voltage waveform V_0 can be obtained across the capacitor C in Fig. (A); R is a variable resistor, V_i is an ideal battery, and SG is a spark gap consisting of two electrodes with an adjustable distance between them. When the voltage across the electrodes exceeds the firing voltage V_f, the air between the electrodes breaks down, hence the gap becomes a short circuit and remains so until the voltage across the gap becomes very small.
(a) Draw the voltage waveform V_0 versus time t, after the switch is closed.
(b) What condition must be satisfied in order to have an almost linearly varying sawtooth voltage waveform V_0?
(c) Provided that this condition is satisfied, derive a simplified expression for the period T of the waveform.
(d) What should you vary(R and/or SG) to change the period only?
(e) What should you vary (R and/or SG) to change the amplitude only?
(f) You are given an additional, adjustable DC voltage supply. Design and draw a new circuit indicating the terminals where

you would obtain the voltage waveform described in Fig. (B).



pr 80. [Est-Fin-2013] An inductance L and a capacitor C are connected in series with a switch. Initially the switch is open and the capacitor is given a charge q_0. Now the switch is closed.

(a) What are the charge q on the capacitor and the current I in the circuit as functions of time? Draw the phase diagram of the system — the evolution of the system on a $I-q$ graph — and note the curve's parameters. Note the direction of the system's evolution with arrow(s).

A *Zener diode* is a non-linear circuit element that acts as a bi-directional diode: it allows the current to flow in the positive direction when a forward voltage on it exceeds a certain threshold value, but it also allows a current to flow in the opposite direction when exposed to sufficiently large negative voltage. Normally the two voltage scales are quite different, but for our purposes we will take a Zener diode with the following volt-ampere characteristics: for forward currents, the voltage on the diode is V_d, for reverse currents, the voltage on the diode is $-V_d$, for zero current the voltage on the diode is $-V_d < V < V_d$.

Now we connect the inductance L, the capacitor C all in series with a switch and a Zener diode. The switch is initially open. The capacitor is again given the charge $q_0 > CV_d$ and the switch is then closed.

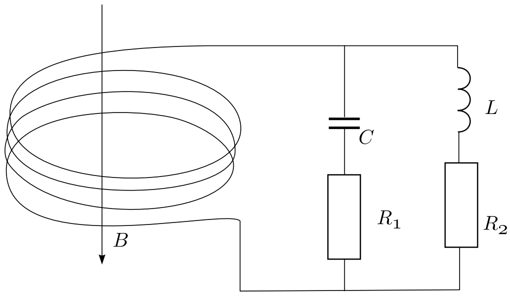
(b) Make a drawing of the phase diagram for the system. Note the direction of the system's evolution with arrow(s).

(c) Does the evolution of the system only necessarily stop for $q = 0$? Find the range of values of q on the capacitor for which the evolution of the system will necessarily come to a halt.

(d) Find the decrease Δq in the maximum positive value of the capacitor's charge q after one full oscillation. How long does it take before oscillation halts?

(e) Suggest a mechanical system which is analogous to this circuit.

pr 81. [EstFin-2009] Consider an electric circuit consisting of a coil of negligibly small inductance, consisting of $N = 10$ turns and with the surface area of a single loop $S = 10\,\mathrm{cm}^2$, resistors $R_1 = R_2 = 3\,\Omega$, capacitor $C = 0.2\,\mathrm{F}$, and an inductance $L = 1H$, connected as shown in Fig. At the moment of time $t = 0$, a magnetic field, parallel to the axis of the coil is switched on. The induction of the magnetic field starts growing linearly, starting from $B = 0$ until the maximal value $B = 1\,\mathrm{T}$ is achieved at $t = 10\,\mathrm{ms}$. Further, the induction of the magnetic field remains constant (and equal to $1\,\mathrm{T}$).



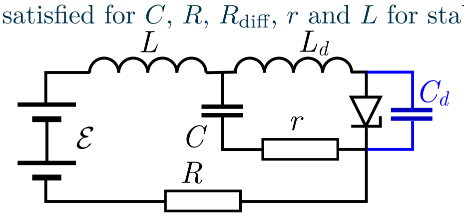
(a) Find the current through the resistors R_1 and R_2 at the moment of time $t_1 = 5\,\mathrm{ms}$.

(b) Find the current through the resistors R_1 and R_2 at the moment of time $t_2 = 15\,\mathrm{ms}$.

(c) What is the net charge passing through the resistor R_2?

pr 82. As we have learned with problem 56, if we want to measure the full $V-I$-curve of a tunnel diode, it needs to be specifically stabilized[34]. Let us study such a stabilization in more details. Let a tunnel diode be connected in series with a resistance R to a battery, and let the parameters of the system be such that at the stationary state, the diode's voltage is such that the diode's differential resistance $R_{\mathrm{diff}} < -R$.

(a) According to the results of problem 56, this stationary state is unstable; in order to stabilize it, a series connection of a capacitance C and a resistance r are connected in parallel to the diode. The wires connecting the capacitor and resistor r to the diode are so short that the corresponding inductance L_d can be assumed to be negligibly small; the inductance of the wires connecting the diode, battery, and resistor R to each other is L, (see figure; neglect the capacitance C_d). Which condition(s) need to be satisfied for C, R, R_{diff}, r and L for stabilization?



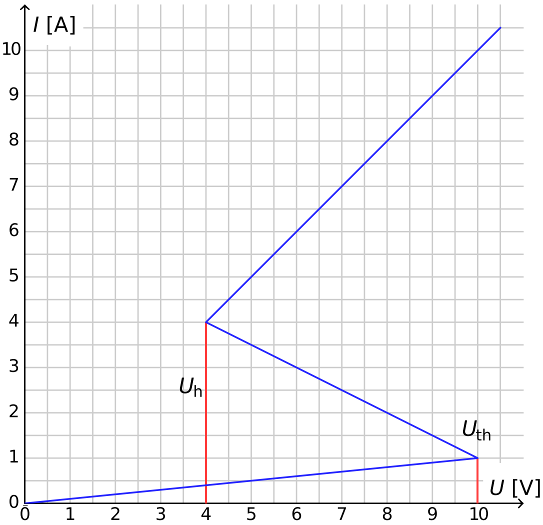
(b) In task (a) we addressed a relatively slow instability; in that case, the characteristic time of current variation cannot be much less than

$$\tau_s \equiv \min\left(\frac{L}{|R+R_{\mathrm{diff}}|}, C|r+R_{\mathrm{diff}}|\right).$$

However, with the new addition to the circuit, there is now a current loop consisting of the diode, resistance r, capacitance C, and inductance L_d in which the characteristic time scale τ_f can be much smaller than τ_s, of the order of $L_d/|r+R_{\mathrm{diff}}|$. For such fast current fluctuations, we can no longer neglect the inductance L_d (its impedance becomes comparable with $|R_{\mathrm{diff}}|$). The negative differential resistance in that loop may give rise to emergence of instabilities which develop within the time-scale $\tau_f \ll \tau_s$. For the analysis of instabilities within that loop, it is also important to notice that the tunnel diode has electrodes which perform as a capacitor of capacitance $C_d \ll C$, connected in parallel to the differential resistance R_{diff} (for very fast current fluctuations, the impedance of C_d may become comparible with $|R_{\mathrm{diff}}|$). Find the condition for the lack of fast instabilities, i.e. instabilities which would develop within a time-scale $\tau_f \ll \tau_s$; simplify your calculations by using appropriate approximations.

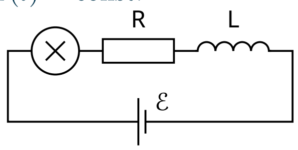
[34] Such a stabilization has been done for the black-box-experiment at IPhO-2012

pr 83. [IPhO-2016] The $I-V$ characteristics of a thyristor can be approximated by a piece-wise linear graph as shown in the figure below. Henceforth we consider such an idealized thyristor, the $I-V$ curve of which is given by figure, and refer to it as the "nonlinear element X". In the voltage range between $U_{\rm h} = 4.00$ V (the holding voltage) and $U_{\rm th} = 10.0$ V (the threshold voltage) this $I-V$ curve is multivalued.



(a) Using the graph, determine the resistance $R_{\rm on}$ of the element X on the upper branch of the $I-V$ characteristics, and $R_{\rm off}$ on the lower branch, respectively. The middle branch is described by the equation $I = I_0 - \frac{U}{R_{\rm int}}$. Find the values of the parameters I_0 and $R_{\rm int}$.

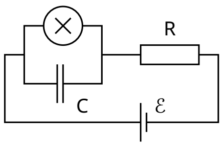
(b) The element X is connected in series with a resistor R, an inductor L and an ideal voltage source $\mathcal{E}$ (see figure below). The circuit is said to be in a stationary state if the current is constant in time, $I(t) = \text{const}$.



How many different stationary states this circuit may have for a fixed value of $\mathcal{E}$ and for $R = 3.00\,\Omega$ (consider different values of $\mathcal{E}$)? How does the answer change for $R = 1.00\,\Omega$?

(c) Let $R = 3.00\,\Omega$, $L = 1.00\,\mu$H and $\mathcal{E} = 15.0$ V in the circuit shown above. Determine the values of the current $I_{\rm st}$ and the voltage $V_{\rm st}$ on the non-linear element X in the stationary state. Is this state stable or unstable (study the effect of a small departure of the current strength from $I_{\rm st}$)?

(d) We now investigate a new circuit configuration, see figure below. This time, the non-linear element X is connected in parallel to a capacitor of capacitance $C = 1.00\,\mu$F. This block is then connected in series to a resistor of resistance $R = 3.00\,\Omega$ and an ideal constant voltage source of voltage $\mathcal{E} = 15.0$ V. It turns out that this circuit undergoes oscillations with the non-linear element X jumping from one branch of the $I-V$ characteristics to another over the course of one cycle.



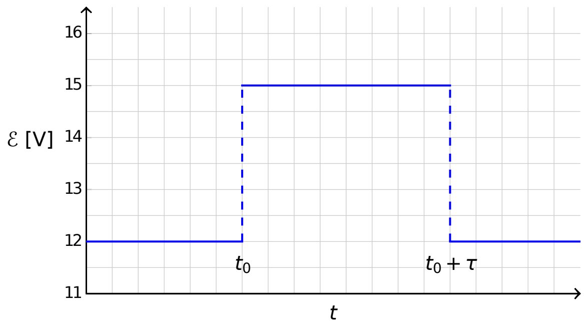
Draw the oscillation cycle on the $I-V$ graph, including its direction (clockwise or anticlockwise).

(e) Find expressions for the times t_1 and t_2 that the system spends on each branch of the $I-V$ graph during the oscillation cycle. Determine their numerical values. Find the numerical value of the oscillation period T assuming that the time needed for jumps between the branches of the $I-V$ graph is negligible.

(f) Estimate the average power P dissipated by the non-linear element over the course of one oscillation. An order of magnitude is sufficient.

(g) A neuron in a human brain has the following property: when excited by an external signal, it makes one single oscillation and then returns to its initial state. This feature is called excitability. Due to this property, pulses can propagate in the network of coupled neurons constituting the nerve systems. A semiconductor chip designed to mimic excitability and pulse propagation is called a *neuristor* (from neuron and transistor).

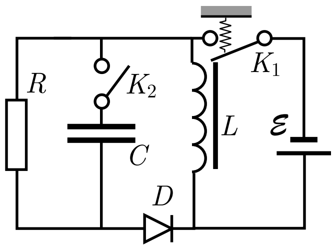
We attempt to model a simple neuristor using a circuit that includes the non-linear element X that we investigated previously. To this end, the voltage $\mathcal{E}$ in the circuit above is decreased to the value $\mathcal{E}' = 12.0$ V. The oscillations stop, and the system reaches its stationary state. Then, the voltage is rapidly increased back to the value $\mathcal{E} = 15.0$ V, and after a period of time τ (with $\tau < T$) is set again to the value $\mathcal{E}'$ (see figure below). It turns out that there is a certain critical value $\tau_{\rm crit.}$, and the system shows qualitatively different behavior for $\tau < \tau_{\rm crit}$ and for $\tau > \tau_{\rm crit}$.



Sketch the graphs of the time dependence of the current $I_X(t)$ on the non-linear element X for $\tau < \tau_{\rm crit}$ and for $\tau > \tau_{\rm crit}$.

(h) Find the expression and the numerical value of the critical time $\tau_{\rm crit}$ for which the scenario switches.

pr 84. [Est-Fin-2014]
In order to obtain high voltage supply using a battery, the following circuit is used.



An electromagnetic switch K_1 connects a battery of electromotive force $\mathcal{E}$ to an inductor of inductance L: it is closed if there is no current in the inductor (a spring keeps it closed), but if the inductor current reaches a critical value I_0, magnetic field created by the inductor pulls it open. Due to inertia, once the key is open, it takes a certain time τ_K to close again even if the current falls to zero.

For the diode D you may assume that its current is zero for any reverse voltage ($V_D < 0$), and also for any forward voltage smaller than the opening voltage V_0 (i.e. for $0 < V_D < V_0$). For any non-zero forward current, the diode voltage V_D remains equal to V_0.

You may express your answers in terms of L, $\mathcal{E}$, I_0, V_0, R, and the capacitance C (see figure).

a) At first, let the key K_2 be open. If the initial inductor current is zero, how long time τ_L will it take to open the key K_1?

b) Assuming (here and in what follows) that $L/R \ll \tau_K \ll \tau_L$, plot the inductor current as a function of time t (for $0 \leq t < 3\tau_L$).

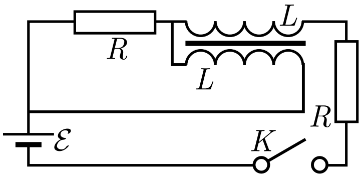
c) What is the maximal voltage $V_{\max}$ on the resistor R?

d) Assuming that $V_{\max} \gg V_0$, what is the average power dissipation on the diode?

e) Now, let the key K_2 be closed, and let us assume simplifyingly that $V_0 = 0$; also, $RC \gg \tau_L$ and $\tau_K > \pi\sqrt{LC}$. Suppose that the circuit has been operated for a very long time. Find the average voltage on the resistor.

f) Find the amplitude of voltage variations on the resistor.

pr 85. An electrical transformer is connected as shown in the circuit below. Both windings of the transformer have the same number of loops and the self-inductance of the both coils is equal to L; there is no leakage of the magnetic field lines from the core so that the mutual inductance is also equal to L.



(a) Consider the case when the coil windings are oriented so that if the both coils have current flowing from left to right then the magnetic fields in the transformer core add up destructively Find the currents in the resistors immediately after the switch is closed.
(b) Under the assumption of the task (a), find the current in the left resistor as a function of time.
(c) Now, let one of the coils have a reversed direction of winding; find the current in the right resistor as a function of time.

Alternating current

Alternating current (AC) and voltage are assumed to be sinusoidal, e.g. $I = I_0 \sin(\omega t + \varphi)$. Kirchoff's laws are linear — they involve only adding first powers of voltages and currents; hence, as long as the circuit elements are linear (i.e. their properties do not depend on the amplitude of the current or voltage), dealing with Kirchoff's laws means dealing with linear combinations of voltages and currents. However, sine and cosine are not very convenient functions for adding, in particular if different terms have different phase shift φ. Luckily, using the Euler's formula (see appendix 5), sine and cosine can be substituted with exponential function, if we switch from real numbers to complex numbers:

$$\mathrm{e}^{\mathrm{i}(\omega t+\varphi)} = \cos(\omega t + \varphi) + \mathrm{i}\sin(\omega t + \varphi).$$

So, instead of using a sine or cosine, we write $I = I_0\mathrm{e}^{i(\omega t+\varphi)}$. The exponential function is much easier to deal with, because if we add different voltages or currents with the same frequency, the term $\mathrm{e}^{\mathrm{i}\omega t}$ can be factorised, owing to the property $\mathrm{e}^{a+b} = \mathrm{e}^a \cdot \mathrm{e}^b$ (see appendix 6). There is no need to worry that physical quantities are typically measured in real numbers, and now we have suddenly a complex current (and voltage): current remains to be a real-valued quantity; when we write it in a complex form, we just keep in mind that what we actually have (in physical reality) is the real part of that complex number. So, if we write $I = I_0\mathrm{e}^{\mathrm{i}(\omega t+\varphi)}$, we assume that the physically measurable current is $I_r = \Re I_0 \mathrm{e}^{\mathrm{i}(\omega t+\varphi)} = I_0\cos(\omega t + \varphi)$ ($\Re z$ stands for "real part of z").

Now, if we accept the complex form $I = I_0\mathrm{e}^{\mathrm{i}(\omega t+\varphi)} = I_0\mathrm{e}^{\mathrm{i}\varphi}\cdot \mathrm{e}^{\mathrm{i}\omega t}$, it is convenient to combine I_0 and $\mathrm{e}^{\mathrm{i}\varphi}$ into what we call the complex amplitude of the current,

$$I_c = I_0\mathrm{e}^{\mathrm{i}\varphi}.$$

Then, all the currents and voltages are products of $\mathrm{e}^{\mathrm{i}\omega t}$ with the complex amplitude, which means that for any linear combination of currents and voltages, the time-dependent factor $\mathrm{e}^{\mathrm{i}\omega t}$ can be brought before the braces. If so, there is no need to write always that term: typically, all the calculations are done just with the complex amplitudes, the modulus of which gives us the amplitude, $|I_c| = |I_0\mathrm{e}^{\mathrm{i}\varphi}| = |I_0||\mathrm{e}^{\mathrm{i}\varphi}| = I_0$, and the argument of which gives us the phase shift, $\varphi = \arg I_0 = \arctan \Im I_c/\Re I_c$ (for more details about those properties of complex numbers which have been used here, see appendix 8).

From this brief theory we can draw the following conclusions. Operating with complex amplitudes works well as long as we have a single sinusoidal signal, and only linear circuit elements are included. Inversely, complex amplitudes **cannot be used** if (a) the signal is not sinusoidal, e.g. rectangular; (b) if there are nonlinear elements, e.g. diodes, capacitors for which capacitance depends on the charge, etc. If we have a superposition of different frequencies and these assumptions are satisfied, the different frequency signals need to be studied separately (superposition principle can be applied), and for each component-signal, the complex amplitudes can be used. An important case is the power dissipated in the circuit: this is a nonlinear function of the voltage and current, and so we need to be careful. Let I and V be the complex amplitudes of the current and voltage. Then

$$P = \left\langle \Re I\mathrm{e}^{\mathrm{i}\omega t}\cdot \Re V\mathrm{e}^{\mathrm{i}\omega t}\right\rangle = \left\langle \frac{I\mathrm{e}^{\mathrm{i}\omega t} + \overline{I}\mathrm{e}^{-\mathrm{i}\omega t}}{2}\cdot\frac{V\mathrm{e}^{\mathrm{i}\omega t} + \overline{V}\mathrm{e}^{-\mathrm{i}\omega t}}{2}\right\rangle,$$

where $\langle\ldots\rangle$ denotes averaging over time, and bar over a symbol denotes a complex conjugate ($\overline{a+b\mathrm{i}} \equiv a - b\mathrm{i}$; $\overline{\mathrm{e}^{\mathrm{i}\omega}} = \mathrm{e}^{-\mathrm{i}\omega}$). Upon opening the braces and using the fact that $\left\langle \mathrm{e}^{\mathrm{i}2\omega t}\right\rangle = \left\langle \mathrm{e}^{-\mathrm{i}2\omega t}\right\rangle = \langle\cos 2\omega t\rangle + \mathrm{i}\langle\sin 2\omega t\rangle = 0$, we obtain

$$P = \frac{I\overline{V} + V\overline{I}}{4} = |I||V|\frac{\mathrm{e}^{\mathrm{i}\varphi_1}\mathrm{e}^{-\mathrm{i}\varphi_2} + \mathrm{e}^{-\mathrm{i}\varphi_1}\mathrm{e}^{\mathrm{i}\varphi_2}}{4};$$

using the formula $\cos x = \frac{1}{2}(\mathrm{e}^{\mathrm{i}x} + \mathrm{e}^{-\mathrm{i}x})$, we end up with

$$P = |I||V|\frac{\mathrm{e}^{\mathrm{i}(\varphi_1-\varphi_2)} + \mathrm{e}^{-\mathrm{i}(\varphi_1-\varphi_2)}}{4} = \frac{1}{2}|I||V|\cos(\varphi_1 - \varphi_2).$$

Note that this can be rewritten as $P = \frac{1}{2}\Re V\overline{I}$, because $\Re V\overline{I} = |V||I|\Re\mathrm{e}^{\mathrm{i}\varphi_1}\mathrm{e}^{-\mathrm{i}\varphi_2} = |V||I|\cos(\varphi_1 - \varphi_2)$. Also, since $\Re V\overline{I} = \Re(ZI)\overline{I} = \Re Z|I|^2 = |I|^2\Re Z$, we can write

$$P = \frac{1}{2}|I|^2\Re Z.$$

In order to get rid of the factor $\frac{1}{2}$, amplitudes are often substituted by root mean square (rms) amplitudes: $\tilde{U} = U/\sqrt{2}$, $\tilde{I} = I/\sqrt{2}$ (always make clear with which amplitude you are dealing with).

And so, in the case of AC currents, it is convenient to deal with complex amplitudes. Most often, the temporal dependence in the form of $e^{i\omega t}$ is never written, and calculations involve only the complex amplitudes.

Let us recall that at the leads of an inductor, $U = L\frac{dI}{dt}$. Once we substitute here $I = I_0 e^{i\omega t}$ we obtain immediately $U = i\omega L I_0 e^{i\omega t}$. The prefactor of the exponent here is the complex amplitude $U_0 = i\omega L I_0$ of the capacitor's voltage; upon denoting

$$Z_L = i\omega L$$

we can rewrite the last equality as $U_0 = Z_L I_0$; here, Z_L is called **impedance**. So, if dealing with complex amplitudes, an inductor's voltage and current satisfy the Ohm's law in the same way as in the case of resistors with a direct current (DC) — just instead of a resistance, its impedance is to be used. Similarly, for a capacitor we have $U = q/C = \int I \cdot dt/C = \int I_0 e^{i\omega t} \cdot dt/C = I_0/i\omega C$, i.e. $U_0 = I_0 Z_C$ with

$$Z_C = \frac{1}{i\omega C}.$$

Finally, for a resistor we have still the Ohm's law $U = IR = I_0 e^{i\omega t}$, hence $U_0 = Z_R I_0$ with

$$Z_R = R.$$

Sometimes this is called the "active resistance", or "ohmic resistance", emphasizing the difference from the "reactive" and non-ohmic impedances Z_L and Z_C.

As a conclusion:

idea 44: For AC circuits, all the techniques learnt for DC currents can be used (Kirchoff's laws, method of potentials etc.), if calculations are made with the complex amplitudes, and impedances are used as resistors: for the complex amplitudes of the voltage and current, $V = IZ$, where Z is the circuit's full impedance; the phase shift between the voltage and current is given by $\varphi = \arg Z$.

The only difference is in the way how the power dissipation is to be calculated (see above).

idea 45: For AC circuits, the dissipated power

$$P = |I||V|\cos\varphi = \Re V\bar{I} = |I|^2 \Re Z.$$

NB! Here V and I are assumed to be the rms amplitudes; if we deal with the real amplitudes, the factor $\frac{1}{2}$ is to be added. Alternatively, since there is no power dissipation on the inductors and capacitors (for which $\varphi = \frac{\pi}{2}$ so that $\cos\varphi = 0$), the power can be calculated as the power dissipated in all the resistors, for each of which $P = RI_R^2$ (I_R being the resistor's current).

Note that if we deal with AC appliances and $\cos\varphi$ is small, for a given required power dissipation, the current needs to be larger than what would be in the case of larger values of $\cos\varphi$. Unnecessarily large current means unnecessarily large dissipation losses in the power lines. The appliances based on inductors (in particular those including electromagnetic motors) have intrinsically small $\cos\varphi$. Therefore, if several appliances of small $\cos\varphi$ are plugged simultaneously into a AC outlet, in order to reduce the net current in the power lines, it would be a good idea to equip some appliances with capacitors, which make the phase shift opposite without introducing any additional power dissipation: when currents of opposite (or nearly opposite) phase shift are added in the power lines, the large and opposite imaginary parts of the complex current amplitudes cancel out, giving rise to a significant reduction of the net current.

fact 22: It should be also mentioned that sometimes, the concepts of reactive and apparent powers, P_r and P_a are used, defined as

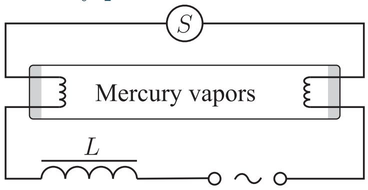
$$P_a = |V\bar{I}| \text{ and } P_r = \Im V\bar{I},$$

where $\Im z$ stands for "imaginary part of z".

However, these concepts are not very useful, and serve mainly as tools to emphasize the importance of having large $\cos\varphi$ — small reactive power.

pr 86. Consider a soldering gun of nominal power $P = 30\,\mathrm{W}$ and nominal voltage $V = 220\,\mathrm{V}$ (AC voltage with frequency $\nu = 50\,\mathrm{Hz}$). Which capacitance needs to be connected in series to the iron in order to reduce the power down to $P_1 = 20\,\mathrm{W}$?

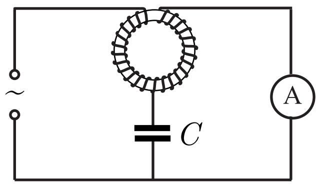
pr 87. [IPhO-1982] An alternating voltage of 50 Hz frequency is applied to the fluorescent lamp as shown in the accompanying circuit diagram. The following quantities are measured: overall voltage (main voltage) $V = 228.5\,\mathrm{V}$, electric current $I = 0.6\,\mathrm{A}$, voltage across the fluorescent lamp $U' = 84\,\mathrm{V}$, ohmic resistance of the inductor $R_d = 26.3\,\Omega$. The fluorescent lamp itself may be considered as an ohmic resistor in the calculations.
(a) What is the inductance L of the series reactor?
(b) What is the phase shift φ between voltage and current?
(c) What is the active power P_w transformed by the apparatus?
(d) Apart from limiting the current the series inductor has another important function. Name and explain this function!
Hint: The starter (denoted by a circle with "S") includes a contact which closes shortly after switching on the lamp, opens up again and stays open.
(e) In a diagram with a quantitative time scale sketch the time sequence of the luminous flux emitted by the lamp.
(f) Why has the lamp to be ignited only once although the applied alternating voltage goes through zero in regular intervals?
(g) According to the statement of the manufacturer, for a fluorescent lamp of the described type a capacitor of about $C = 4.7\,\mu\mathrm{F}$ can be switched in series with the series reactor. How does this affect the operation of the lamp and to what intent is this possibility provided for?



idea 46: With alternating currents, voltages caused by mutual inductances can be calculated in the same way as in the case of inductors: if there is a current of complex amplitude I_1 in a coil of inductance L_1 which is magnetically coupled (with mutual inductance M) to a second coil with current I_2 then

the total voltage on the first inductor is $i\omega L_1 I_1 + i\omega M I_2$. NB! Be careful with the sign of the mutual inductance, cf. fact 20.

pr 88. Around a toroidal ferromagnetic core of a very large magnetic permeability, a coil is wound; this coil has a large number of loops and its total inductivity is L. A capacitor of capacitance C is connected to the middle point of the coil's wire as shown in figure. AC voltage V_0 of circular frequency ω is applied to the input leads of the circuit; what is the reading of the ammeter (which can be considered to be ideal)?



fact 23: In AC circuits the impedance of which is dominated by capacitors and inductors, free oscillations of current and voltage can take place; the decay rate of oscillations is defined by the ohmic resistance of the circuit. The frequency of such an oscillation is called the *natural frequency*, or *eigenfrequency*; the corresponding current- and voltage oscillations are referred to as the natural modes. If the circuit contains more than one current loop, there may be more than one natural frequencies. Then, if the circuit is left isolated from external inputs, any current- and voltage dynamics in that circuit can be represented as a superposition of the natural modes.

That superposition thing means mathematically that if we have n nodes characterized by the node potentials ϕ_j, $j = 1, \ldots, n$, and m natural frequencies ω_μ, $\mu = 1, \ldots m$, and in the case of the μ-th natural frequency, the node voltages oscillate according to the law

$$\phi_j = V_{\mu j} e^{i\omega_\mu t},$$

where $V_{\mu j}$ is the complex amplitude of the potential of the j-th node in the case of the μ-th natural mode, then arbitrary motion of the system can be represented as

$$\phi_j = \sum_{\mu=1}^{m} A_\mu V_{\mu j} e^{i\omega_\mu t},$$

where A_μ is a constant — the amplitude of the μ-th natural motion. Such a decomposition into natural modes will be revisited in the section "Oscillations and waves".

idea 47: If the impedance of a circuit is dominated by inductors and capacitors[35] the number of its natural oscillation modes can be found as follows. Find the number of *linearly independent loops* N_l for the given circuit (cf. idea 13); find the number of such *linearly independent loops* $N_=$ which contain only one type of elements (all resistors, all inductors, or all capacitors); the number of natural modes is then given by $N_m = N_i - N_=$. In addition to that, each linearly independent loop containing only inductors contributes one zero-frequency mode: a constant current can circulate in each such a loop.

The proof of this idea (together with its generalization to circuits for which resistances play an important role) is given in appendix 9.

If the circuit contains resistors, the natural frequencies are typically complex numbers; then, imaginary part of the complex number gives the exponential decay of the corresponding mode: $I = e^{-i\omega_i t}\sin(\omega_r t + \varphi)$, where ω_r and ω_i stand for the real and imaginary parts of the natural frequency, respectively. If the impedance of a circuit is not dominated by inductors and capacitors, there are frequencies for which $\omega_i = 0$. For instance, one can say that the natural frequency of a simple RC-circuit is $\omega = i/RC$; indeed, with $I = I_0 e^{i\omega t} = I_0 e^{-t/RC}$ we recover the fact 12.

fact 24: Suppose at a certain circular frequency ω, impedance is very large, $\frac{1}{Z(\omega)} \approx 0$. Then, a very small current driven to the leads will give rise to a very large voltage $V = IZ$; this phenomenon is called the *voltage resonance*. Similarly, if the impedance is very small, $Z(\omega) \approx 0$, we have a *current resonance*: small input voltage will lead to a large current.

idea 48: The natural frequencies can be found as the resonance frequencies; there are two options. **First,** you can select two points A and B at the circuit, henceforth referred to as the fictitious terminals, and equate the impedance of the circuit between A and B to infinity and solve the equation with respect to the frequency: although there is no input current, there can be voltage oscillations at a resonant frequency, because with $V = IZ$, $I = 0$, and $Z = \infty$, V can take any value. **Second option:** select a point A the circuit and cut the circuit fictitiously at that point. Thus, one "half" of the point A becomes the first terminal A_1 of the new circuit, and another "half" becomes the second terminal A_2. Since in the original circuit, A_1 and A_2 coincide, they must have the same voltage: voltage between A_1 and A_2 is zero. Finally, equate the impedance between A_1 and A_2 to zero and solve it: although the voltage is zero, there can be a non-zero current $I = U/R$.

The technique described by this idea is a shortcut substituting the standard method for finding natural frequencies. The standard method involves two steps: (a) writing down the full set of linear homogeneous differential equations using the Kirchoff's laws for the circuit; (b) writing down the characteristic equations where derivatives of an unknown function are substituted by powers of an unknown variable (cf. Formula sheet pt. I-3). When we use the idea 48, we basically bypass the first step and obtain directly the characteristic equation by equating the impedance (or its reciprocal) to zero.

As an illustration, let us consider a simple circuit where a resistor of resistance R, inductor of inductance L, and a capacitor of capacitance C are connected in series. If we denote the charge of the capacitor as q, the current in the circuit is expressed as $I = \frac{dq}{dt}$ so that the Kirchoff's voltage law gives us a differential equation

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = 0;$$

the corresponding characteristic equation is $L\lambda^2 + R\lambda + C^{-1} = 0$. Meanwhile, if we "cut" one of the wires then the impedance will be $Z = i\omega L + R + \frac{1}{i\omega C} = 0$; this equation can be rewritten as $Z(\omega) \equiv -\omega^2 L + i\omega R + C^{-1} = 0$, which is identical to the

[35] Resistances connected in series to inductors are small, and resistances connected in parallel to capacitors are large.

characteristic equation with $\lambda = \mathrm{i}\omega$. This equation yields

$$\omega = \frac{\mathrm{i}R}{2L} \pm \sqrt{(LC)^{-2} - \frac{R^2}{4L^2}}.$$

When using impedances we assume that the temporal dependence of all the quantities is proportional to $\mathrm{e}^{\mathrm{i}\omega t}$. Therefore, the impedance is zero and free current oscillations are possible with

$$I = I_0 \mathrm{e}^{\mathrm{i}\omega t} = I_0 \mathrm{e}^{-\frac{R}{2L}t} \mathrm{e}^{\pm \mathrm{i}\omega_0 t},$$

where $\omega_0 \equiv \sqrt{(LC)^{-2} - \frac{R^2}{4L^2}}$. Let us recall now that with complex-number-representation, we assume that the real current value is given by the real part of a complex number; with a complex amplitude $I_0 = |I_0| \mathrm{e}^{\mathrm{i}\varphi}$, this leads us to a real-valued solution $I = |I_0| \mathrm{e}^{-\frac{R}{2L}t} \cos(\omega_0 t \pm \varphi)$.

It is useful to keep in mind that the idea 48 can be applied not only in the case of LC-circuits, but also in the case of $L-R$ circuits and $R-C$ circuits in which case the solution of the equation $Z(\omega) = 0$ will be purely imaginary, corresponding to an oscillation-less exponential decay. In such cases, it is more convenient to subsitute $\mathrm{i}\omega = -\gamma$ which corresponds to exponential dependence of currents and voltages, proportional to $\mathrm{e}^{-\gamma t}$. With such substitution, the impedance of a capacitance C becomes $-\frac{1}{\gamma C}$, and that of an inductance L — $-\gamma L$; the current resonant condition remains still the same, $Z(\gamma) = 0$.

Finally, it should emphasized that with a low probability, for symmetric circuits, one or more natural frequencies may be lost with this method. Therefore, it is a good idea to compare the number of obtained frequencies $\omega (> 0)$ with the expected number (deduced using the idea 47). Losing one or more solutions in such a way is actually not a bad thing, because losing a solution means obtaining a lower-degree equation for $x = \omega^2$ which is easier to solve. The "lost" solutions can be recovered; one option is to "cut" the circuit at a different place (or to add fictitious nodes to different places if the voltage resonance approach is used). If the resulting circuit is non-symmetric, the obtained equation for $x = \omega^2$ will have its degree (and the number of different solutions) equal to the number of degrees of freedom of the original circuit. Among the solutions, there are also the ones we already know based on the symmetric circuit. Therefore, it would be possible to reduce the degree by long dividing the corresponding polynomial with $x - x_1$, where x_1 is a known solution.

However, in many cases, there is actually no need to long divide the polynomial, because an easier method exists. Before we proceed to this method, let us analyse first why one or more solutions were lost. When the current resonance method is used, some natural frequencies will be lost if the circuit is "cut" at such a point A where one of the natural modes has (due to symmetry) always zero current: for such an oscillation mode, the circuit terminals A_1 and A_2 will have, in addition to a zero voltage, also a zero current; hence the impedance does not need to be vanishing. Similarly, when the voltage resonance method is used, a natural frequency is lost if the corresponding mode has always zero voltage between the fictitious terminals A and B.

Keeping this is mind, it becomes evident how to recover the lost solutions in the most efficient way. In the case of current resonance, we use the fact that for the lost modes (which we want to recover), the voltage between the fictitious terminals is constantly zero, hence we can short-circuit these terminals and thereby simplify the circuit. For those "lost" modes, the new circuit is no different from the old circuit, so among the solutions of the new circuit, there must be the "lost" frequency. Analogously, if the solutions were lost when the voltage resonance condition was used, the lost solutions are among the natural frequencies of the cut circuit (terminals A_1 and A_2 remain disconnected).

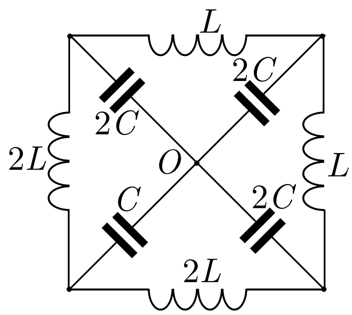
However, one must keep in mind that such a simplified circuit may have other natural frequencies (additional to the previously "lost" solutions), possibly different from the natural frequencies of the original circuit. So, the method is safe, if the simplified circuit has as many natural frequency as is the number of the "lost" solutions (otherwise we would need to study, which of the solutions of the simplified circuit are the "lost" solution of the original circuit).

Let us sum up what has been said above as

idea 49: When finding the natural frequencies of a circuit obeying certain symmetries while using the idea 48, it is useful to exploit the symmetry: select symmetric positions for the fictitious terminals (for which $Z = \infty$), or select symmetric point for "cutting" ($Z = 0$). This will lead to the loss of one or more solutions which can be found as the natural frequencies of a simplified circuit — we either short-circuit the fictitious terminals (if $Z = \infty$ was used), or we leave the "cut" wire broken (if $Z = 0$ was used). NB! count carefully the number of solutions[36]: if the number of degrees of freedom of the simplified circuit exceeds the number of "lost" solutions then some of the natural frequencies of the simplified circuit may differ from the frequencies of the original circuit.

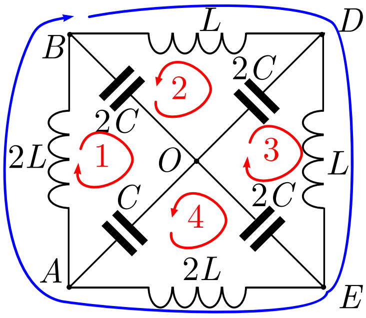
To show how to count the number of modes and to make use of the symmetry of the problem, let us consider the following problem.

pr 89. Find the natural frequencies of the circuit shown in the figure.



Using the fact 8, we can easily conclude that the maximal number of linearly independent loops is four (red arrows in figure).

[36] When counting the number of frequencies, keep in mind that an equation for finding frequencies may have repeated roots; double root needs to be counted twice, triple root trice, etc.

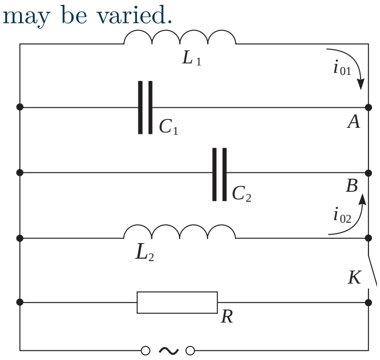


Since the circuit consists of only inductors and resistors, we could expect the number of natural frequencies to be also four. However, we need to pay attention that one can find one current loop consisting only of inductors (blue arrow). Therefore, we have 4-1=3 non-zero frequencies, and 1 zero frequency. The circuit has mirror symmetry which we are going to use: we use nodes A and D (see figure) for voltage resonance, and require $Z_{AD} = \infty$. When calculating Z_{AD}, the nodes A and D serve as the terminals; then, due to symmetry, potentials of the nodes B, O, and E are equal (because $z_{AB}/z_{BD} = z_{AO}/z_{OD} = z_{AE}/z_{ED}$, where z_{XY} denotes the impedance of the component connected directly between the nodes X and Y), hence there is no current in the wires BO and OE which can be removed without changing the impedance Z_{AD}. Then there are three parallel paths connecting A and D, out of which the paths ABD and AED can be replaced with a single equivalent inductor of inductance $\frac{3}{2}L$; the two capacitors in the path AOD can be replaced with a single capacitor of capacitance $\frac{2}{3}C$. So, we resulted in a circuit with one capacitor and one inductor, and which has clearly only one resonance frequency $\omega_1 = 1/\sqrt{LC}$. Next, we'll proceed in the same way, but now we select the nodes B and E as the terminals for voltage resonance. In a very similar way, we end up, again, with an equivalent circuit consisting of one inductor of inductance $\frac{4}{3}L$ and one capacitor of capacitance C; the corresponding natural frequency $\omega_2 = 1/\sqrt{\frac{4}{3}LC}$.

We have found two frequencies, but there are three in total. In the first step, we missed those natural modes which had zero voltage between A and D; in the second step, we missed those natural modes which had zero voltage between B and E. So, now we can be missing only those modes for which both these voltages are zero, i.e. we can connect the nodes B and E with a wire, and also connect the nodes A and D with a wire; the resulting circuit must have the missing frequency as one of its natural frequencies. Once we merge nodes B with E and D with A, we have only three nodes: B (with E) is connected to O via an effective capacitance $4C$; A (with D) is connected to O via an effective capacitance $3C$; A (with D) is connected to B (with E) via an effective inductance $\frac{1}{3}L$. This circuit has only one degree of freedom and hence, its natural frequency is the "lost" frequency of the original circuit, $\omega_3 = 1/\sqrt{\frac{4}{7}LC}$.

So, we were lucky in that the number of missing natural frequencies was equal to the number of degrees of freedom of the simplified circuit. What should have been done if that were not the case? Of course, we could have written down the impedance for a pair of non-symmetric terminals; in that case we would have ended up with a cubic equation for which we know already two solutions. However, that would have been long way. Meanwhile, we know that the missing solution is, in fact, very symmetric; would it be possible to use this fact? It appears, yes! We can make use of the voltage equalities $V_B = V_E$ and $V_D = V_E$ to conclude that in the case of the missing oscillation mode, the loop currents 1–4 (as shown in the figure) satisfy equalities $I_2 = I_3 = 2I_1 = 2I_4$. With this knowledge, we can write down Kirchoff voltage law, for instance, for the first loop: $(I_1 + I_4)/(\mathrm{i}\omega C) + (I_1 + I_2)/(2\mathrm{i}\omega C) + 2\mathrm{i}I_1\omega L$ from where we obtain easily the same result as before.

pr 90. [IPhO-1983] Let us consider the electric circuit in the figure, for which $L_1 = 10\,\mathrm{mH}$, $L_2 = 20\,\mathrm{mH}$, $C_1 = 10\,\mathrm{nF}$, $C_2 = 5\,\mathrm{nF}$, $R = 100\,\mathrm{k\Omega}$. The switch K being closed, the circuit is coupled with a source of alternating current. The current furnished by the source has constant intensity while the frequency of the current may be varied.



(a) Find the ratio of frequency $\nu_M/\Delta\nu$, where ν_m is the frequency for which the active power in circuit has the maximum value P_m, and the frequency difference $\Delta\nu = \nu_+ - \nu_-$, where ν_+ and ν_- are the frequencies for which the active power in the circuit is half of the maximum power $P = \frac{1}{2}P_m$.

The switch is opened in the moment t_0 when there is no current through the resistor. Immediately after the switch is open, the intensities of the currents in the coils L_1 and L_2 are respectively $i_{01} = 0.1\,\mathrm{A}$ and $i_{02} = 0.2\,\mathrm{A}$. (the currents flow as in the figure); at the same moment, the potential difference on the capacitor with capacity C_1 is $U_0 = 40\,\mathrm{V}$.

(b) Calculate the frequency of electromagnetic oscillation in $L_1C_1C_2L_2$ circuit;

(c) Determine the intensity of the electric current in the AB conductor;

(d) Calculate the amplitude of the oscillation of the intensity of electric current in the coil L_1.

The idea 38 is useful in the case of AC, as well; let us formulate this as another idea, which can be used to find qualitatively or asymptotically[37] the dependence of something on the frequency of the input signal, or to simplify the analysis according to the idea 90 in those cases when the circuit includes both large and small inductances and/or capacitors.

idea 50: At the limit of low frequencies, capacitors can be "cut off", and inductors — "short-circuited"; similarly, at the limit of high frequencies, inductors can be "cut off", and capacitors — "short-circuited". Systematic analysis assumes that all the appropriate limit cases are considered, e.g. for $\omega \ll 1/RC$, $|Z_C| \ll |Z_R|$ and hence, if connected in parallel, the resistor can be "cut off", and if connected in series, the capacitor can

[37] at the limit of high- or low frequencies

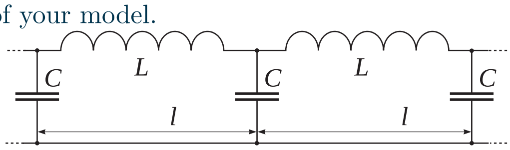
be short-circuited.

Keep also in mind that at a voltage resonance, a parallel $L-C$ connection has an infinite impedance, and can be cut off; similarly, at the current resonance, a series $L-C$ connection has a zero impedance, and can be short-circuited.

pr 91. In a black box with two ports, there are three components connected in series: a capacitor, an inductance, and a resistor. Devise a method to determine the values of all three components, if you have a sinusoidal voltage generator with adjustable output frequency ν, an AC-voltmeter and an AC-ammeter.

idea 51: Mathematically, electrical oscillations are analogous to the mechanical ones, which are usually studied by writing down differential equations. Method of impedances allows us to bypass this step. Based on the resonance condition $Z(\omega) = 0$, it is also possible to make a "reverse-engineering" and deduce the corresponding differential equation using a simple rule: a factor $i\omega$ corresponds to a time derivative (hence, a factor $-\omega^2$ corresponds to a second derivative). The safest method for verifying the equivalence of a mechanical system with an electrical one is to write down the differential equations (or systems of differential equations, if appropriate) describing the both systems and verify that these two are mathematically equivalent. The matching scheme is usually as follows: a capacitor's charge or loop current's time integral corresponds to a coordinate of a point mass; an inductance — to a mass; a capacitance — to a spring's stiffness.

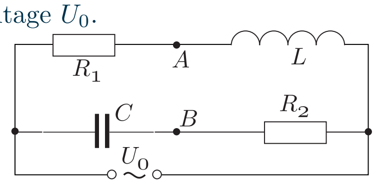
pr 92. [IPhO-1987] When sine waves propagate in an infinite LC-grid (see the figure below) the phase of the AC voltage across two successive capacitors differs by φ.
(a) Determine how φ depends on ω, L and C (ω is the angular frequency of the sine wave).
(b) Determine the velocity of propagation of the waves if the length of each unit is l.
(c) State under what conditions the propagation velocity of the waves is almost independent of ω. Determine the velocity in this case.
(d) Suggest a simple mechanical model which is an analogue to the above circuit and derive equations which establish the validity of your model.



In general, when dealing with two-dimensional problems, complex number analysis is a more powerful tool than a vector analysis, because everything what can be done with vectors, can be also done with complex numbers: addition and subtraction, multiplication with a constant, and even the scalar and vector products (although this is a little bit more tricky, see below), but there are many more things what can be done with the complex numbers (addition, division, taking powers and exponents, etc). A hint for the way of obtaining scalar- and vector products can be found in the idea 45: if we take two complex numbers z_1 and z_2, and consider the product $z_1\overline{z_2}$, then $\Re z_1\overline{z_2}$ equals to the scalar product of the respective vectors $\vec{z_1}$ and $\vec{z_2}$, and $\Im z_1\overline{z_2}$ equals to the z-component of the vector product $\vec{z_2} \times \vec{z_1}$ (assuming that the real axis corresponds to the x-axis, and the imaginary axis — to the y axis). However, regardless of what have been said, there are cases when it is more convenient to deal with vector diagrams of voltages and currents, rather than with the complex amplitudes.

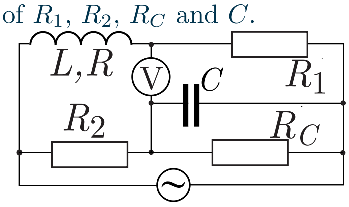
idea 52: If an AC-circuit problem turns out to be essentially a geometrical problem, it is better to use vector diagrams instead of complex amplitudes, i.e. to draw the vectors corresponding to the complex amplitudes, and to study the problem geometrically; keep in mind that using scalar product and rms. amplitudes, $P = \vec{U} \cdot \vec{I}$. Geometrical knowledge which can be useful: Thales theorem, inscribed angle theorem, laws of sines and cosines

pr 93. The circuit consists of a capacitor, inductance, and two resistors, see figure. The voltage on both resistors is 10 V, and the voltage between the leads A and B is also 10 V. Find the applied voltage U_0.

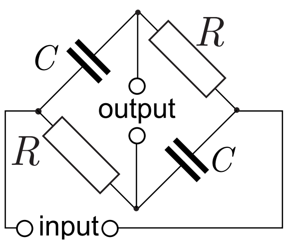


Problems involving ideas 44–52

pr 94. Consider a so-called *Maxwell's bridge* shown in figure below, which is used for measuring the inductance L and the ohmic resistance R of an inductor. To that end, the other parameters are adjusted so that the voltage reading will be zero. Assuming that such a state has been achieved, express L and R in terms of R_1, R_2, R_C and C.



pr 95. Below a circuit is given which makes it possible to adjust the phase of a voltage signal. Show that if the output current is negligibly small, its voltage amplitude will be the same as at the input leads, but with a different phase. Find the phase shift.

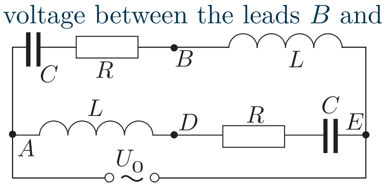


pr 96. A remote summer house receives electricity from a power station over a rather long cable. To check the status of the cables, the power meter can also measure the voltage supplied to the household. People left the summer house, and switched all the other electricity devices off, but forgot the

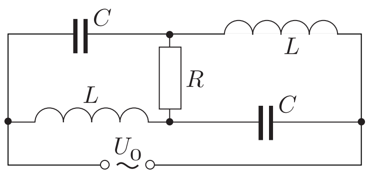
transformer under the voltage (the transformer was used for feeding low-voltage lamps). The transformer can be considered as a series connection of an inductance L and ohmic resistance r. Readings at the power meter of the house: when the transformer was switched on, voltage $U_1 = 234.0\,\mathrm{V}$ and power consumption $P_1 = 5\,\mathrm{W}$;
when everything was switched off, voltage $U_0 = 236.0\,\mathrm{V}$;
when the transformer is off, but an electric oven is switched on: electrical power $P_2 = 1200\,\mathrm{W}$, and voltage $U_2 = 219.6\,\mathrm{V}$ (oven is a purely ohmic resistance R).

You may assume that the voltage at the power station (to which the cables are connected) is always constant. Determine the power of electrical energy which was dissipated in the power cables (connecting the house with the power station).

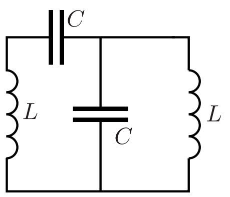
pr 97. A circuit consists of two identical inductances, two identical resistors, and two identical capacitors, see figure. The applied voltage $U_0 = 10\,\mathrm{V}$; the voltage on the lower inductance is $10\,\mathrm{V}$, and the voltage between the leads D and E is also $10\,\mathrm{V}$. Determine the voltage between the leads B and D.



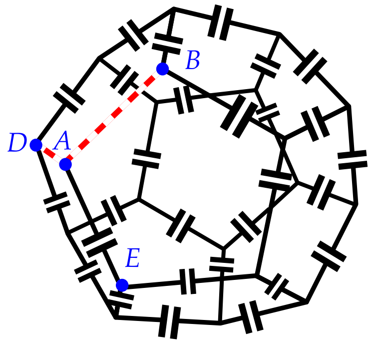
pr 98. A circuit consists of two identical inductances, two identical capacitors, and one resistor, see figure. The applied voltage is $U_0 = 10\,\mathrm{V}$, and the total current at the input leads is $I_0 = 1\,\mathrm{A}$; the voltage measured at the left capacitor is $10\,\mathrm{V}$, and $10\,\mathrm{V}$ is also measured at the left inductance. What is the active power dissipated in this circuit and what is the resistance of the resistor?



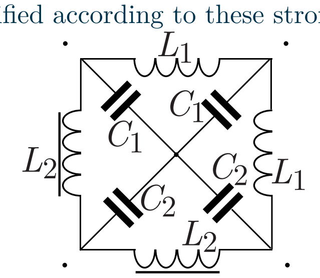
pr 99. Find the natural frequencies of the circuit given below.



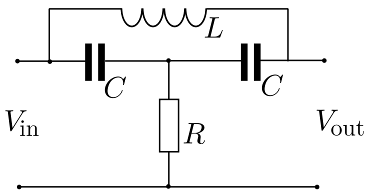
pr 100. Edges of a dodecahedron are made of wire of negligible electrical resistance; each wire includes a capacitor of capacitance C, see figure. Let us mark a vertex A and its three neighbours B, D and E. The wire segments AB and AD are removed. What is the capacitance between the vertices B and E?



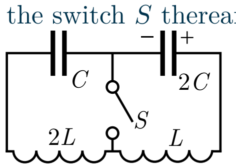
pr 101. Determine all the natural frequencies of the circuit shown in Figure. You may assume that all the capacitors and inductances are ideal, and that the following strong inequalities are satisfied: $C_1 \ll C_2$, and $L_1 \ll L_2$. Note that your answers need to be simplified according to these strong inequalities.



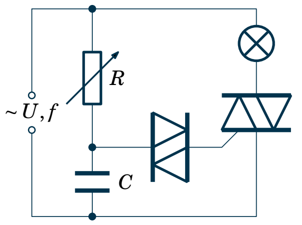
pr 102. [Adapted from IPhO-1984] An electronic frequency filter consists of four components as shown in figure: there are two capacitors of capacitance C, an inductor L, and a resistor R. An input voltage V_{in} is applied to the input leads, and the output voltage V_{out} is measured with an ideal voltmeter at the output leads, see figure. The frequency ν of the input voltage can be freely adjusted. Find the ratio of $V_{\text{out}}/V_{\text{in}}$ and the phase shift between the input- and output voltages for the following cases: (a) at the limit of very high frequencies; (b) at the limit of very low frequencies; (c) in the case of such a frequency ν_0 for which there is no voltage on the resistor; (d) in the case of such a frequency ν_1 for which the power dissipation in the circuit is maximal (assuming that the amplitude of the input voltage is kept constant). Find also the frequencies ν_0 and ν_1.

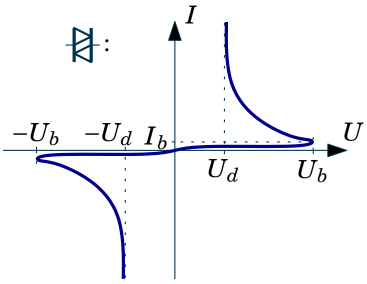


pr 103. [IPhO-2014] Initially: switch S in the circuit below is open; the capacitor of capacitance $2C$ carries electric charge q_0; the capacitor of capacitance C is uncharged; and there are no electric currents in either the coil of inductance L or the coil of inductance $2L$. The capacitor starts to discharge and at the moment when the current in the coils reaches its maximum value, the switch S is instantly closed. Find the maximum current $I_{\max}$ through the switch S thereafter.

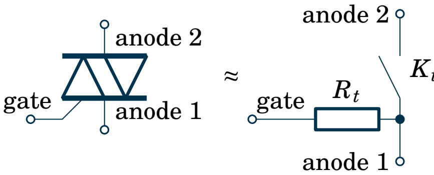


pr 104. [Est-Fin-2016] A dimmer for controlling the brightness of lighting consists of a rheostat, a capacitor, a diac and a triac, connected as in the circuit.





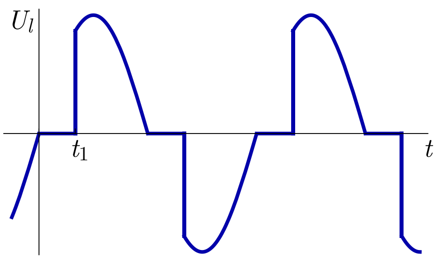
A diac is a component whose behaviour is determined by the voltage-current diagram shown above. A triac, on the other hand, can be thought of as a switch controlled by current — look at the following equivalent schematics.



The switch K_t is open as long as the current through the triac's gate stays under the threshold current I_t; closes when the threshold current is applied (in either direction) and stays closed while a current is flowing through the *switch* K_t (the gate current is irrelevant until the switch opens again).

(a) Assume that the resistance R_t is large enough that the charge moving through the diac can be neglected. Let the sinusoidal supply voltage have a maximum value of U and a frequency of f; the rheostat be set to the resistance R and the capacitor's capacitance be C. Find the maximum value of the voltage U_C on the capacitor, and its phase shift φ with respect to the supply voltage.

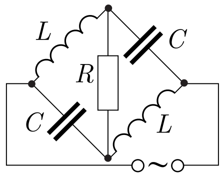
(b) What inequality should be satisfied by the diac's characteristic voltages U_b and U_d, triac's threshold current I_t and gate resistance R_t to ensure that when the diac starts to conduct (while the voltage on the capacitor rises), then the triac would also immediately start to conduct? You may assume that $I_b < I_t$ and that the diac's voltage at current I_t is U_d.



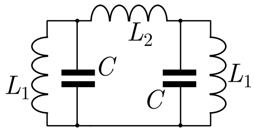
(c) The voltage U_l on the lamp follows the plot above. Let's assume that the assumption of part i) and the inequality of part ii) hold. Find the time t_0 during which the voltage on the lamp is zero.

(d) Express through t_0 and f, how many times the average power of the lamp is lower than the one of a lamp without a dimmer, assuming that the resistance of the lamp is unchanged.

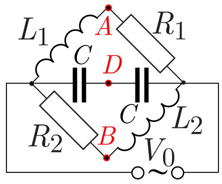
pr 105. In the circuit below, $R = \sqrt{L/C}$. A rectangular voltage waveform of period τ and amplitude V_0 is applied to the input ports (this means that during half of the period, the input voltage is V_0, and during the other half-period, the voltage is $-V_0$). Find the shape and amplitude of the current flowing through the input ports.



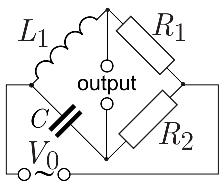
pr 106. Find the natural frequencies for the circuit below.



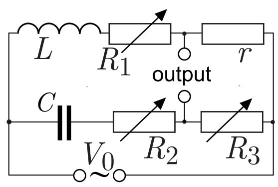
pr 107. For the circuit shown below, the frequency of the sinusoidal input voltage is unknown; given the capacitance C, inductances L_1, L_2, resistances R_1, R_2, amplitude of the input voltage V_0, and the phase shift φ between the currents through the nodes A and B, determine: (a) the amplitude of the voltage V_{AD} between the nodes A and D; (b) the phase shift between the voltages V_{AD} and V_{DB}.



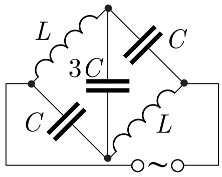
pr 108. For the circuit shown below, the frequency of the sinusoidal input voltage is unknown; given the capacitance C, inductance L, resistances R_1, R_2, amplitude of the input voltage V_0, and the phase shift φ between the inductor current and input voltage, what is the phase shift between the capacitor voltage and output voltage?



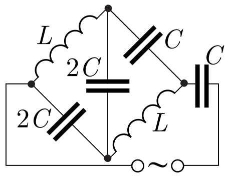
pr 109. In the circuit shown in the figure, the sinusoidal input voltage has a fixed amplitude V_0 and frequency f. What is the maximal amplitude of the output voltage, and for which values of the variable resistances R_1, R_2, and R_3 is the maximal amplitude achieved?



pr 110. Find such frequencies of the input voltage ω for which the circuit shown below has zero impedance.



pr 111. Find such frequencies of the input voltage ω for which the circuit shown below has zero impedance.



appendix 5: Euler's formula

The standard way of generalizing a function $F(x)$ from its real argument x to complex argument z is by using a Taylor expansion:

$$F(x) = F(x_0) + \sum F^{(n)}(x_0)(x-x_0)^n/n! :$$

in such a power series, we can just substitute x with z. Here, $F^{(n)}(x_0)$ stands for the n-th derivative of $F(x)$, calculated at $x = x_0$. The structure of this power series is quite easy to understand: if we truncate it by keeping only the first N terms, it approximates the function $F(x)$ with such a polynomial of N-th order for which the first N derivatives at $x = x_0$ are equal to those of the function $F(x)$. Furthermore, the thrown-away terms with $n > N$ are small if $|x-x_0|$ is not very large, because the denominator grows rapidly with n. By keeping more and more terms, the approximation becomes increasingly accurate, so that at the limit $N \to \infty$, the series becomes equal to the function.

With $x_0 = 0$, the Taylor series for the exponent, sine, and cosine functions are written as

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!},$$

$$\sin(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!},$$

$$\cos(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}.$$

Now, if we substitute in the expression of e^x the argument x with ix, we obtain

$$e^{ix} = \sum_{n=0}^{\infty} \frac{(ix)^n}{n!} = \sum_{k=0}^{\infty} \left[\frac{(ix)^{2k}}{(2k)!} + \frac{(ix)^{2k+1}}{(2k+1)!}\right] =$$

$$= \sum_{k=0}^{\infty} \left[\frac{i^{2k}x^{2k}}{(2k)!} + \frac{i \cdot i^{2k} \cdot x^{2k+1}}{(2k+1)!}\right].$$

Let us notice that $i^{2k} = (i^2)^k = (-1)^k$; if we compare now the series expansion for e^{ix}, and those of $\sin x$ and $\cos x$, we see that

$$e^{ix} = \cos x + i \sin x.$$

appendix 6: Exponent of a sum of two complex numbers

For real-valued arguments, the property $e^{a+b} = e^a \cdot e^b$ is an easy generalization from the same rule for integer arguments. This is a very useful property, and actually the only reason why the exponential function is easier to deal with than sine or cosine, but it is not obvious why it should held for complex-valued arguments. Since we generalized e^x to complex-valued arguments via the Taylor expansion, this series is the only thing we can use to prove the validity of this property. So, we start with

$$e^{a+b} = \sum_{n=0}^{\infty} \frac{(a+b)^n}{n!},$$

where a and b are complex numbers. Here we need the binomial theorem (see appendix 7),

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} a^k b^{n-k},$$

which leads us to

$$e^{a+b} = \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{n!}{k!(n-k)!}\frac{a^k b^{n-k}}{n!} == \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{a^k}{k!}\frac{b^{n-k}}{(n-k)!}.$$

Now, let us substitute $m = n-k$; in this double sum, as n goes from 0 to ∞, and k goes from 0 to n, the pair of numbers k, m takes all the possible integer-valued combinations, with both m and k varying from 0 to ∞:

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{a^k}{k!}\frac{b^{n-k}}{(n-k)!} = \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} \frac{a^k}{k!}\frac{b^m}{m!}.$$

This double sum can be factorised, because one factor doesn't depend on m, and the other one — on k, and constant terms (independent of the summation index) can be brought before the summation sign (before the braces):

$$\sum_{k=0}^{\infty}\sum_{m=0}^{\infty} \frac{a^k}{k!}\frac{b^m}{m!} = \sum_{k=0}^{\infty}\left(\sum_{m=0}^{\infty} \frac{a^k}{k!}\frac{b^m}{m!}\right) =$$

$$\sum_{k=0}^{\infty}\left[\frac{a^k}{k!}\left(\sum_{m=0}^{\infty}\frac{b^m}{m!}\right)\right] = \left(\sum_{k=0}^{\infty}\frac{a^k}{k!}\right)\left(\sum_{m=0}^{\infty}\frac{b^m}{m!}\right) = e^a \cdot e^b.$$

appendix 7: Newton binomial formula

If we open the n braces in the expression $(a+b)^n$, we'll have a sum of terms where each term is a product of n factors, each of which is either a or b. In that sum, $a^k b^{n-k}$ will arise as many times as many different possibilities there is for selecting exactly k braces out of the total n braces: from the "selected" braces we pick a as the factor entering a term in the de-factorised sum, and from the "non-selected" braces we pick b). This so-called number of k-combinations from a set of n elements is denoted by $\binom{n}{k}$.

In order to find the number of possibilities for selecting k objects out of n objects, let us enumerate all the objects with numbers from 1 till n. The number of permutations (different ways for ordering these enumerated objects) is $n!$ (there is n different ways for picking the first object, $n-1$ for picking the second, etc). In the case of each ordering of the objects, we "select" the first k ones. If we go through all the different orderings, we definitely obtain all the different ways of selecting k objects, but each selection will be obtained many times: as many times as we can re-order objects within a given selection. The selected object can be re-ordered in $k!$ different ways, and the non-selected objects — in $(n-k)!$ different ways. In order to obtain the number of different ways of selecting k objects, we divide the overall number of permutations by the number of different ways of re-ordering, which results in $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.

appendix 8: Basic properties of complex numbers

Complex numbers can be thought of as two-dimensional vectors: the real part of a complex number $z = x + \mathrm{i}y$ defines the x-coordinate of a vector, and the imaginary part — the y-coordinate. What differs complex numbers from vectors is that two complex numbers can be multiplied so that the result is still a complex number (there is a vector product of two vectors, but if we have two-dimensional vectors lying in $x-y$-plane, the resulting vector will no longer lie in that plane). Because of that, you can also divide two non-complex numbers with a uniquely-defined result — as long as the divisor is not zero (you cannot divide two non-parallel vectors!).

The modulus of a complex number is defined as the length of the corresponding vector, $|z| = \sqrt{x^2 + y^2}$. Bearing in mind the geometrical (vectorial) representation and using Euler's formula, we can write

$$z = |z|(\cos\alpha + \mathrm{i}\sin\alpha) = |z|\mathrm{e}^{\mathrm{i}\alpha},$$

where α is the angle between the vector and the x-axis; this is called the exponential form of a complex number, and α is called the argument (arg) of the complex number. Apparently,

$$\alpha = \arctan y/x = \arctan \Im z/\Re z.$$

Now, if we consider the product of two complex numbers,

$$z_1 \cdot z_2 = |z_1|\mathrm{e}^{\mathrm{i}\alpha_1}|z_2|\mathrm{e}^{\mathrm{i}\alpha_2} = |z_1||z_2|\mathrm{e}^{\mathrm{i}(\alpha_1+\alpha_2)}.$$

Here, the right-hand-side of the equality is an exponential representation of the complex number $z_1 z_2$,which means that

$$|z_1 z_2| = |z_1||z_2|,$$

and

$$\arg z_1 z_2 = \arg z_1 + \arg z_2.$$

Similarly, of course, $|z_1/z_2| = |z_1/z_2|$ and $\arg z_1/z_2 = \arg z_1 - \arg z_2$.

Here is a list of simple but sometimes useful formulae:

$$\Re z = \frac{1}{2}(z + \overline{z}),$$

where $z = x - \mathrm{i}y$ is called the complex conjugate of z;

$$|z|^2 = z\overline{z}.$$

Note that $\overline{z}$ is a vector symmetric to z with respect to the x-axis, and therefore

$$\overline{\mathrm{e}^{\mathrm{i}\alpha}} = \mathrm{e}^{-\mathrm{i}\alpha};$$

in particular, applying these two formulae for $z = \mathrm{e}^{\mathrm{i}\alpha}$ results in

$$\cos\alpha = \frac{\mathrm{e}^{\mathrm{i}\alpha} + \mathrm{e}^{-\mathrm{i}\alpha}}{2}, \quad \sin\alpha = \frac{\mathrm{e}^{\mathrm{i}\alpha} + \mathrm{e}^{-\mathrm{i}\alpha}}{2i}.$$

If you need to get rid of a complex number in a denominator of a fraction, you can use equality

$$\frac{z_1}{z_2} = \frac{z_1\overline{z_2}}{|z_2|^2}.$$

appendix 9: Determining the number of natural modes of electrical circuits

If we need to find the natural frequencies of a certain circuit, it can be done by using the method of loop currents: we need to select a full set of linearly independent current loops (cf. idea 13), consider the corresponding loop currents (which are now AC currents defined by complex amplitudes), and write down for each current loop the Kirchoff's voltage law. This will give us a set of N linear equations for finding N unknown currents, where N denotes the total number of current loops. For natural oscillation modes, there are no AC sources in the circuit, so this set of linear equations is a homogeneous system which has non-zero solutions (which are the natural oscillation modes) only if its determinant is zero. Equating the determinant of the system to zero gives us an algebraic equation for finding the oscillation frequencies ω. The degree of the equation is found as the sum of the degrees of the individual equations. An individual equation describing a certain current loop has degree 2 if it includes at least one capacitor and one inductor; it has degree 1 if it includes at least one resistor together with at least one capacitor or at least one inductor; finally, it has degree 0 if all its elements are of the same type (resistors, inductors, or capacitors). If we denote the number of such loop types as N_{LC}, $N_{\neq}$, and $N_{=}$ respectively, the degree of the equation is expressed as

$$N_d = 2N_{LC} + N_{\neq}.$$

According to the fundamental theorem of algebra, the number of complex-valued roots of such equation is also N_d. However, not all roots will correspond necessarily to different oscillation modes. Indeed, if there is a pair of solutions in the form $\omega = \pm\omega_0 - \gamma$, the corresponding current oscillations can be combined into one mode

$$I_1\mathrm{e}^{\mathrm{i}\omega_0 t-\gamma t} + I_2\mathrm{e}^{-\mathrm{i}\omega_0 t-\gamma t} = I_0\mathrm{e}^{-\gamma t}\cos(\omega_0 t + \varphi),$$

where φ is the oscillations phase. Conversely, if there is an oscillating mode (possibly decaying-in-time) $I_0\mathrm{e}^{-\gamma t}\cos(\omega_0 t + \varphi)$, there must be two corresponding solutions of the algebraic equation for ω (with equal imaginary parts and opposite real parts). To sum up, if we denote with N_o the number of oscillatory modes, and with N_e the number of such modes which decay without oscillations, the following equality holds:

$$2N_o + N_e = 2N_{LC} + N_{\neq}.$$

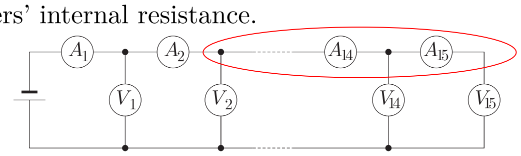
Finally, the number of zero-frequency modes equals to the number of such linearly independent loops which have only inductors in it, because in that case, with $\omega = 0$, the corresponding loop current is eliminated from all the equations so that it can take arbitrary value.

Hints

1. Determine the surface area under the graph(count the cells or approximate the shape with a set of trapezoids; pay attention to the units of your surface area (mm·m).
2. The sum of the voltages on R_1 and R_2 is constant, $I_1R_1 + I_2R_2 = \text{Const}$, hence one can find the change of current through R_2. Initially, $I_1 = I_2$; later, the difference of these two currents goes to the lamp.
3. Resistors 4 and 5 are connected in parallel between B and C.
4. Due to symmetry, there is no current through the bridge resistor, hence it can be removed (the both leads of it divide the overall voltage between the input leads of the circuit in 2:3-proportion).
5. Find the currents in the upper resistors (2 Ω and 3 Ω) by short-circuit the ammeter; the difference of these two currents goes to the ammeter.
6. For a voltmeter, the reading is proportional to the current through it. Hence, you can find the current through V_2;

use the Kirchoff's current law for finding the ammeter's current.

7. The sum of the voltmeters' readings is related to the sum of their currents: $\sum_{i=2}^{15} V_i = \sum_{i=2}^{15} rI_i = r\sum_{i=2}^{15} I_i$, which equals to A_2, as it follows from the idea 8 for the region marked with a red line in the figure below; here, r is the voltmeters' internal resistance.



8. The left three resistors form a Δ-connection which can be substituted by a Y-connection consisting of $1\,\Omega$-resistors.

10. Substitute the entire circuit in figure with an equivalent battery with $\mathcal{E}_{\text{eff}} = \mathcal{E}\frac{R_2}{r+R_1+R_2}$.

11. When solving using the idea 10: first find the internal resistance of an equivalent battery; the equivalent electromotive force will be found by comparing the currents of the two systems when the ports are short-circuited. When using the idea 11: substitute all the batteries with current sources; for a parallel connection of current sources, the currents are just added, so it is easy to substitute a set of current sources with a single equivalent current source (and at the final step, the current source with a battery).

12. Assume that $\mathcal{E}_1$ is short-circuited and calculate the currents through the batteries I_{1n}, $n = 1 \ldots 4$; then assume that $\mathcal{E}_2$ is short-circuited and calculate the currents through the batteries I_{2n} $n = 1 \ldots 4$. The final answers will be $I_{1n}+I_{2n}$, $n = 1 \ldots 4$. Alternatively, show that after mirroring the circuit with respect to the vertical axis, it remains identical to itself, but the currents through R_2 and R_4 will be reversed, hence these currents need to be zero, hence these resistors can be "cut" off.

15. As mentioned, the numerical values are such that in SI-units, each "resistance" of the dual circuit is numerically exactly 4 times smaller than the corresponding resistance of the original circuit. Therefore, the total "resistance" of the dual circuit must be also 4 times smaller, $4R^\star \cdot \Omega^2 = R$. On the other hand, the "resistance" of the dual circuit equals to the resistance of the original circuit, $R^\star = 1/R$; from here, we can immediately obtain the answer.

17. Assume that the circuit is equivalent to a battery of electromotive force $\mathcal{E}$ and internal resistance r; then write equations for $\mathcal{E}$ and r analogously to how it was done for problem 16.

18. If a current I is let into one of the leads, it is distributed equally between the three branches: $I/3$ flows in each. At the next junction, each of these currents is divided equally, again, so that the next wires have current equal to $I/6$.

19. For node-merging: merge B with F and C with E so that A will be connected with the merged BF node (and BF with CE) via a $R/2$-resistance. For edge-splitting: split OD into resistors OD' and OD'', each of resistance $2R$.

21. Consider symmetric current distributions: (A) I is driven into one vertex (P), and $I/19$ is driven out from all the other 19 vertices; (B) I is driven out from a neighbouring vertex of the vertex P, and $I/19$ is driven into all the other 19 vertices.

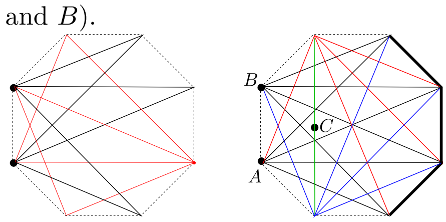
22. Reduce this problem to the previous one by representing the missing wire as a parallel connection of R and $-R$: the resistance of all the other resistors, except for the $-R$, is given by the answer of the previous problem.

23. For the first half of the process, the transitions of the states of the resistors will take place at the overall voltage values $V_1 = 1.5\,\text{V}$ and $V_2 = 5\,\text{V}$; for the second half, the respective transition voltages are $V_2' = 3\,\text{V}$ and $V_1' = 1.25\,\text{V}$.

24. The diode current can be expressed as $\mathcal{E} - IR$; draw this straight line onto the graph provided, and find the intersection point.

25. Find the tunnel diode current exactly in the same way as for problem 24; let the intersection point voltage be V_0. Find the cotangent $R_t = \Delta V/\Delta I$ of the $I(V)$-curve at that voltage (note that $R_t < 0$). Substitute the tunnel diode with a series connection of a battery of electromotive force V_0 and resistance R_t. If the input voltage changes by ΔV, the current will change by $\Delta I = \Delta V/(R + R_t)$, and the output voltage will change by $R\Delta I$, hence the amplification factor $n = R\Delta I/\Delta V$.

27. Keep 4 more wires(four red ones in the left figure below); short-circuit only four nodes as shown in figure; then, due to symmetry, the middle point C of the green wire can be also merged with the short-circuited nodes (because both will have the same potential if a voltage is applied between the leads A and B).



28. For the original polarity, the two leftmost diodes are open, i.e. these can be short-circuited (resulting in a parallel connection of the three leftmost resistors), and the third diode is closed, i.e. can be "cut" off. For the reversed polarity, the states of the diodes are reversed: the rightmost diode is open and short-circuits the four rightmost resistors (which can thus be removed), and the other diodes are closed so that the three leftmost resistors are connected in series.

29. The voltage on the first diode is $1\,\text{V}$, hence the other diodes have a lesser voltage applied (resistors take also some voltage), and are closed. As a result, the first diode can be replaced by a battery of $1\,\text{V}$, and the other diodes — "cut off". The power dissipation on the first diode is found as its voltage ($1\,\text{V}$) times the current through it.

30. Calculate the electromotive force and internal resistance of the battery which is equivalent to the set of four resistors and the battery; internal resistance calculation can be simplified, if the three Δ-connected resistors are substituted with a Y-connection.

31. Substitute the three resistors and the battery with an equivalent battery with an internal resistance r; thus the problem is reduced to Pr. 24.

32. Note that $I_A/I_B = R_B/R_A = 2$, hence the fuse A will melt first. Pay attention that in the second case, the fuse B tolerates more current than the overall current by which the fuse A melts.

33. The circuit breaks down into a combination of series- and parallel connections. Using the given voltage values, one can conclude that $\frac{Rr}{R+r} = \frac{2}{3}\frac{2Rr}{2R+r}$, where r is the voltmeter's resistance.

34. The radiated heat AT^4 equals to the electrical power VI; also, $R = V/I = BT$ (where A and B are constants).

35. First we apply the idea 6 and substitute the ammeters with wires; then we apply the idea 5: the two vertically positioned resistors connect directly the battery leads; the other four resistors form two pairs of parallel-connected resistors (these pairs are connected in series).

36. Apply the idea 6: the circuit becomes a combination of parallel and series connections; find ammeter current using Kirchoff's current law.

37. Apply the idea 6; find ammeter currents using Kirchoff's current law (keep in mind that due to symmetry, all the ammeters have the same current by modulus).

38. The voltage distribution between the voltmeters in this circuit is defined by the ratio of the resistor's resistance R and the voltmeter's internal resistance r. If we express the ratio $V_3/V_1 = 0.8$ in terms of r/R (the battery voltage cancels out from this equation), we obtain a square equation for $x = r/R$. Once we know r/R, we can also find V_2/V_1.

39. Apply the result of the problem 11 for the particular case of three batteries (one of which has $\mathcal{E} = 0$).

40. Since there is no current through the ammeter, it can be "cut" off. Additionally, since there was no current in the ammeter, the voltages on R_1 and R_2 are equal; owing to this, we can find the voltage V_2 on R_2. In order to find the voltage on R_3, we need its current, which goes also through R_2 and can be found as V_2/R_2.

41. Apply the idea 1 for the two cases (7 lamps and 8 lamps).

42. Solution is completely analogous to 35.

43. Apply the idea 9: it is not good to combine A_1, A_2 and A_3 into a Y, because we loose information supplied by the problem conditions (currents I_1 and I_2 would be merged into a single wire of the equivalent Y-connection). Therefore, we substitute A_2, A_3 and A_4 with a Δ-connection (with each resistor having a resistance of $3r$). We know the sum of currents I_2 in the two wires of the Δ-connection, and we can find the current through that $3r$-resistor which is parallel to A_1 as $I_1/3$ (see idea 1). Hence we can determine all the currents (knowing the currents, it is easy to find also R/r). Alternatively, the problem can be solved by making use of the idea 13.

44. For the first polarity, the leftmost diode short-circuits all the other resistors, except for the two at the left upper corner of the circuit. For the reversed polarity, the leftmost diode is closed (can be "cut" off), and the next diode short-circuits five resistors which remain rightwards from it.

45. Apply the solving technique of problem 20.

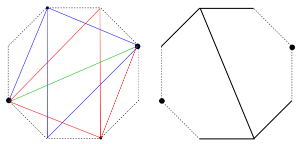
46. Apply the solving technique of problem 20.

47. Apply ideas 22 and 23 (use a negative resistor between B and C); calculate all the pair-wise resistances ($A-B$, $A-C$,and $B-C$) for the symmetric lattice; apply idea 9 to "cut off" the wire between B and C.

48. Apply the solving technique of problem 21.

49. Consider this decagon (with few missing sides) as the decagon of the problem 48, which has additionally negative resistances $-R$ connected parallel between the sides AB and BC (idea 23). Further, apply the generalized idea 9: represent the decagon of the problem 48 as a Y-connection of 10 identical resistors. Finally, calculate the resistance using the idea 1.

50. (I) leave the wires as shown below by red, blue and green lines; (II) short-circuit the nodes as shown by black lines below.



For a precise calculation, consider this side-less octagon as the octagon of the problem 48, which has additionally negative resistances $-R$ connected parallel to the sides (idea 23). Further, apply the generalized idea 9: represent the octagon of the problem 48 as a Y-connection of 8 identical resistors. Finally, calculate the resistance using the idea 1.

51. Note that removing the first cell of this infinite chain will increase its resistance exactly two times; use this fact to apply the idea 18.

52. Use the same approach as for problem 17.

53. Apply the idea 27: short-circuit (and later, cut off) all these wires which are not known to be intact; the ideas 20, 21, and 5 will be also useful.

54. Notice that a wheel graph is self-dual, so all we need to do is to use the idea 17. More specifically, assume that in the case of one of the spokes, there is additionally a battery connected in series to the resistor r. In the dual circuit, spokes become rim segments (of "resistance" $1/r$) and vice versa. So, the circuit is almost self-dual (with resistance-to-"resistance" ratio of Rr, just the "battery" is now on a rim segment (and not on a spoke as the original battery). There is also an alternative solution which exploits the result of the generalized problem 55.

55. First, we need to write down expression for the resistance between i-th node and j-th node *if there is a direct connection between these two*, and this can be done easily: $R_{ij} = (\varphi_i^{(ij)} - \varphi_j^{(ij)})/I$, where $\varphi_k^{(ij)}$ denotes the potential of the k-th node when a current I is driven into the i-th node, and driven out from the j-th node. Our plan is to add up all the equations, and because of that, we don't want to have in our expression "if"-conditions, because it is difficult to expect that such expressions will cancel out when a sum is taken. The solution here is to multiply this expression by the conductance σ_{ij} of a direct connection between the two nodes: if there is a direct connection, $\sigma_{ij} = 1/R$, and the result will be R_{ij}/R. If there is no direct connection, the result will be zero, and hence, there will be no contribution to the sum. The second issue is that if we keep using the potentials $\varphi_k^{(ij)}$, nothing will cancel out upon taking a sum: for each resistance R_{ij}, we introduce a new distribution of potentials (as indicated by the upper indices). We clearly need to reduce the number of potential distributions. Luckily, this can be done by using the superposition principle (similarly to the idea 22): we introduce a reference node, let it have index $i = 1$; we consider $n-1$ potential distributions $\varphi_j^{(k)}$ $(2 \le k \le n)$ when a current I is driven into the i-th node, and out of the 1st node. Then, $\varphi_k^{(ij)} = \varphi_k^{(i)} - \varphi_k^{(j)}$. What is left to do, is to take a sum $\sum_{i,j} R_{ij}$; note that the summation order can be chosen as needed, either $\sum_i \sum_j R_{ij}$ or $\sum_j \sum_j R_{ij}$; don't forget that with this sum, each node pair is counted twice.

56. According to the idea 31, perturbation current $\tilde{I}$ is described by a circuit which we obtain if we substitute the diode with a resistance R_{diff} and remove the electromotive force. For this equivalent circuit, use the fact 17.

58. Repeat the solution of pr 57 while using the ideas 32, 33, and 32 (which means that the energy loss at the diode equals to $V_d \Delta q$, where Δq is the charge change).

59. Use idea 32 to obtain an equation for the final voltage; apply also ideas 33, and 32, and notice that at the initial and final states, there is no current, hence no energy of the inductor.

61. Initially, all the capacitors are charged to the voltage $\frac{1}{3}\mathcal{E}$, i.e. $q = \frac{1}{3}C\mathcal{E}$. The total charge of the system "right plate of the 1st + left plate of the 2nd capacitor + right plate of the 3rd capacitor" (the system A) is conserved; at the final state, when there is no current through the resistor, the system A will be equipotential, and the applies to the system B (consisting of all the other capacitor plates), hence all the plates of the system A will have the same charge.

62. Part (a): apply ideas 32 and 35: for $\frac{dI}{dt} = 0$, there is no voltage of L, and hence no voltage on C_2, and hence a voltage $\mathcal{E}$ on C_1.
Part (b): use the same approach by noting that when there is an extremal voltage on C_1, there is also an extremal voltage on C_2 (as the sum of these voltages gives a constant $\mathcal{E}$). Hence, there is no current through the wires leading to C_1, and there is no current through the wires leading to C_2; from the Kirchoff's current law, there is also no current through L.

63. Apply idea 36: find the voltage on C_1 for the stationary state of the system (constant I implies no voltage on L, hence full $\mathcal{E}$ on C_1). Next, short-circuit $\mathcal{E}$ (C_1 and C_2 become parallel), and find the free voltage oscillations on C_1 in the form $V = V_0 \cos(\omega t + \varphi)$ (what is ω?) and determine V_0 and φ from the initial condition (i.e. initial values of V and $\frac{dV}{dt}$).

64. During the first half-period when K is closed, the diode receives the reverse voltage of the battery and, hence, is closed; according to the idea 37, the current through the diode grows linearly according to the applied voltage 5 V. At the beginning of the second half-period, when the key is opened, this current will be re-directed through the diode; the diode will open since this is a forward current. Now, the inductor will receive voltage −7 V, which corresponds to a linear decrease of the current. Once the current reaches 0 A, the diode will close again (a reverse voltage of −7 V will be applied to it). The average charging current is found as the charge through the battery (the surface area under the $I(t)$-graph for the second half-period), divided by the period.

65. (a) Initially, there was no charge nor current in the system; hence, immediately after the key is closed, there is still no current in L, according to the idea 38, we "cut" it off, and there is still no charge nor voltage on the capacitor, hence we short-circuit it; for such a simplified circuit, we can easily calculate the ammeter current.
(b) Once a new equilibrium is achieved, according to the idea 38, we "cut off" the capacitor, and short-circuit the inductance.
(c) For the final part, the inductor will retain the current of part (b), hence we substitute with the respective ideal current source; the capacitor will retain the voltage of the part (b), hence we substitute it with the respective ideal battery. The circuit is further simplified by using the idea 6.

66. (a) During each half-period, the capacitor will reach very fast (as compared with the period length) a stationary state: constant charge on the capacitor means no current, so that all the current flows through the resistor.
(b) According to the idea 38, capacitor remains essentially short-circuited, so that almost all the current goes to the capacitor plates (and nothing goes to the resistor). Hence, the charge q on the capacitor can be found using the idea 39; $V = q/C$.
(c) Use the graph from part (a); keep in mind that the amplitude of oscillations is half of the difference between the minimal and maximal values.
(d) There are still the same saw-tooth oscillations as in part (b), but the value V_0 around which the voltage oscillates changes slowly, until a completely periodic behaviour is reached; the stationary value of V_0 can be found by applying the idea 40. the mean current

67. Characteristic time L/R is much larger than the period: the current in the inductor will remain almost constant during a period. Suppose the AC input voltage is switched on; first, there is no current in inductor, it is as if "cut off". The diode is opened during a half-period: then, the diode works as a resistance-less wire, and the inductor's voltage equals to the input voltage; during the other half-period, the diode is closed, "cutting off" the right-hand-part of the circuit: since there is no current in the inductor, there is also no current in the resistor, and hence, no voltage on the inductor. This means that when averaged over the entire period, there is a positive voltage on the inductor: the inductor current starts slowly increasing. Constant inductor current means that the diode will remain open for a longer time than a half-period. Now, in order to apply the idea 40, sketch a graph for the inductor voltage as a function of time, and try to figure out, under which condition the average inductor voltage will become equal to zero.

68. Notice that the external magnetic field changes from $-NBS$ to $+NBS$ (or vice versa), and that the self-induced flux is 0 both at the beginning and at the end (although takes non-zero values in between); apply the idea 41.

69. (a) Notice that if $r = 0$, there is a superconducting loop containing L and r, which means that $I_2 =$ Const); use the Kirchoff's current law to obtain I_1 for $t < t_3$; notice that for $t = t_3$, $I_1 = 0$, so that even for $t > t_3$, there is no voltage on r, hence $I_2 =$ Const.

(b) At the moment $t = 1\,\text{min}$, the characteristic time of the circuit containing L will drop from infinity to a new value (use the idea 36 to find it); analyse now the problem using the idea 38 (the system will relax to the new equilibrium state with the above mentioned characteristic time).

(c) There are three stages: first, reduce the current in r (by increasing the total current) so that it falls below 0.5 A; second, switch it to a normal conduction state and while keeping I_1 small, reduce I_2 by reducing the total current I_2 (in order to keep I_1 small, $I_1 = L\frac{\mathrm{d}I_1}{\mathrm{d}t}r_n^{-1}$ needs to be small, i.e. the process needs to be slow); third, switch r back to superconducting state.

(d) First step is the same as for (c), second step is to increase the total current further (from $20\,\text{A} to 30\,\text{A}$) while $r = r_n$, third step is to make $r = 0$ and to decrease the total current back to zero.

70. (a) Use the idea 43-i: the magnetic flux $L(I_1 + I_2)$ in the ferromagnetic core (cf. fact 20) does not change instantaneously, and immediately before switching, the flux was zero (this gives us our first equation). Following the idea 43-ii, the inductive electromotive forces can be eliminated from the Kirchoff's voltage laws (written for the two loops), giving us a second equation.

(b) We just need to apply the idea 43-ii. More specifically, the Kirchoff's voltage laws for the two loops serve us as a system of differential equations for the two currents I_1 and I_2; if we multiply one equation by r, another one by R, and add the two equations, we end up in a single equation for $I = I_1 + I_2$; the solution of that equation can be found using the fact 17 and idea 36.

71. Apply the idea 38: immediately after the key is closed, all the inductor currents are the same as before, which you can find from the Kirchoff's current law; knowing the inductor currents, you can also find the currents in the lamps.

72. Notice that L_1 is short-circuited, hence its voltage is zero and its current is constant; keep this in mind while using the idea 32 to find heat; for finding the charge, consider the Kirhoff's voltage law for the loop involving the upper wire and inductance L_2, use the idea 41.

73. (a) Apply the idea 38: during each half-period, a new equilibrium will be reached, and the dissipated energy can be found using the idea 32.

(b) Apply the idea 39: the capacitor's voltage V_C remains essentially constant, hence the resistor's voltage will be $V_2 - V_C$, or $V_1 - V_C$, depending on the half-period. Upon long-term evolution, the average current through the capacitor will be equal to 0, hence $V_C = \frac{1}{2}(V_1 + V_2)$.

74. According to the idea 36, short-circuit the battery, upon which the parallel/series connection of the three resistors can be substituted by a single equivalent resistor.

75. To begin with, notice that RC needs to be large, so that the capacitor will keep almost a constant voltage during each period (otherwise there would be large current fluctuations). Next, the capacitor's voltage will be equal to the maximal voltage of the sinusoidal input voltage (the diode is opened once during each period, when the input voltage is maximal). Now, the resistance R can be found by combining the Kirchoff's voltage law (to obtain the resistor's voltage), and the Ohm's law. During each period when the diode is closed, the capacitor's voltage will decrease by the amount ΔV corresponding to the charge ΔQ which flew through the light emitting diodes. On the other hand, the allowed voltage variations can be expressed in terms of the allowed current variations by using the Ohm's law.

76. (a) Apply the ideas 38 (cut off the capacitors) and 6 (cut off the voltmeter).

(b) Use the same approach as in part (a): substitute capacitors with batteries and inductors — with current sources.

(c) Apply energy conservation law (notice that the circuit breaks down into two independent circuits, so that the power dissipation can be calculated separately for each of the circuits).

77. (a) Use the fact 5: $P(t) = V \cdot I(t)$, where V is the diode voltage, which is constant and can be brought before the braces (i.e. the averaging sign), $\langle P\rangle = V\langle I\rangle$; apply the idea 40 (together with the Kirchoff's current law) to conclude that the average current through the diode equals to the current through the load.

(b) Proceed in the same way as in Problem 75; do not forget the diode opening voltage 1 V, which needs to be subtracted from the AC voltage amplitude to obtain the maximal voltage of the capacitor.

(c) Proceed in the same way as in Problem 75.

(d) Apply the idea 29 to the first period when the capacitor is charged from zero volts up to the full working voltage.

78. (a) This problem is very similar to Pr 64: the inductor receives a constant voltage U_i when the key is closed, $U_0 - U_i$ when the key is opened and there is a forward current through the diode, and 0 V when the key is opened and there is no current. This corresponds respectively to a linearly increasing, to linearly decreasing, or to a constantly zero input current. Output current can be found from the Kirchoff's current law.

(b) See the hints of Pr 64.

(c) Apply the ideas 38, 40, and Kirchoff's current law to conclude the resistor current equals to the average output current from part (b); then apply the Ohm's law.

79. (a) Consider separately two cases: SG is open, we have a simple RC-circuit with a battery; SG is closed, essentially short-circuiting the capacitor and almost immediately discharging it.
(b) During the charging cycle, the capacitor current needs to remain almost constant; this current is defined by the voltage falling onto the resistor throughout the cycle.
(c) Charging current times T gives the break-down charge of the capacitor; equate this to the value defined by the break-down voltage and C.
(d) Look at your expression for T.
(e) Notice that amplitude equals to $V_0/2$. (f) Notice that the required waveform can be obtained from the waveform $V_b(t)$ from the question (b) as $V_0 - V_b(t)$; construct circuit which gives such an output.

80. (a) Apply fact 19; if $V = V_0\cos(\omega t)$, current can be expressed as $I = \frac{\mathrm{d}q}{\mathrm{d}t}$, where $q = VC$.
(b) Apply the ideas 29 and 36: consider separately the cases of forward and reverse currents; in both cases, there is a sinusoidal signal with a shifted symmetry axis.
(c) The system will stop if there is no current, i.e. when $-V_d < V < V_d$.
(d) Use your graph for question (b) to find the corresponding change of the capacitor's voltage ΔV (of course, $\Delta q = C\Delta V$).
(e) If q corresponds to the coordinate x of a spring-mass oscillator then to which physical quantity X will correspond a voltage applied to an inductor? If the current is positive then what can be said about the mechanical system? Zener diode provides a constant voltage if current is positive; what type of X would have an equivalent property?

81. (a) During the linear growth of B (from 0 ms to 10 ms), the coil serves as an ideal battery of emf. $NS\frac{\mathrm{d}B}{\mathrm{d}t}$. Estimate the characteristic times for two current loops: first involving the coil and C, and second, involving the coil and L; compare this with the time-scale of 10 ms. Apply the idea 38.
(b) Now, the coil operates essentially as a wire (as its inductance is negligible); the current in R_1 is defined by the voltage obtained by the capacitor during the first 10 ms, and the current in R_2 is defined by the current induced in the inductor during the first 10 ms.
(c) apply the idea 41 for the current loop consisting of the coil, L, and R_2.

82. (a) According to the idea 31, perturbation current $\tilde{I}$ is described by a circuit which we obtain if we substitute the diode with a resistance R_{diff} and remove the electromotive force. Further we apply idea 48; for instance, we can "cut" the circuit near the battery and equate the impedance of the resulting circuit segment to zero, $Z(\gamma) = 0$ with $\gamma = -i\omega$. According to the idea 31, the system is stable if all the solutions are stable, so all the roots for γ must have a non-negative real part.
(b) For very fast perturbations, the impedance of the inductance L can be considered infinitely large, hence no current can enter the inductor L and resistor R: this part of the circuit can be "cut off". Similarly, for such fast processes, the impedance of the capacitor C is negligibly small, so it can be short-circuited in the equivalent circuit. Otherwise, the analysis repeats the steps of task (a).

83. (a) resistances are found straightforwardly as the cotangents of the line segment slopes;
(b) use the idea 24 and determine the number of solutions as the number of intersection points between the $I-U$-curve and the straight line describing the Kirchoff's voltage law;
(c) apply the ideas 31 and 26 (cf. problem 56);
(d) apply the idea 29 together with the fact 9: notice that as long as the thyristor operates at the lower branch of the V-I-curve, there will be always a positive capacitor current charging it, and as long as the thyristor operates at the upper branch, there is a is negative capacitor current discharging it;
(e) apply the idea 29 together with the facts 12 and 13;
(f) since were are asked about an estimate only, there are many methods what can be applied, e.g. calculating the minimal and maximal instantaneous dissipation power values based on the cycle drawn for task (d) and approximating the average power with the average of the minimal and maximal values;
(g) with battery voltage $\mathcal{E}'$, there is a stationary state for the system at the lower branch of the $V-I$-curve, but not at the upper branch; if the battery voltage is increased, the stationary state disappears and the system starts evolving as described for task (f); the behavour depends how fast we switch the voltage back: was there enough time for the system to jump over to the upper branch or not [for more detailed analysis, use the same methods as in the case of the task (d)];
(h) write down the qualitative criterion found for task (g) using the facts 12 and 13.

84. (a) Write down the Kirchoff's current law for the loop containing L and $\mathcal{E}$ and keep in minde idea 37;
(b) when K_1 opens, the current flows in loop L-R: study the behaviour of current in time, keep in mind the idea 17 and pay attention to the fact that $L/R \ll \tau_K \ll \tau_L$;
(c) recall the intermediate result of the previous question — how behaves $I(t)$ on the inductor;
(d) use the idea 29; charge can be found by either using the idea 41, or direct integration of the $I(t)$-dependance (cf. fact 18);
(e) apply the idea 32: compare the energy released by the inductor and the heat dissipation by keeping in mind that in average, the capacitor's energy remains constant;
(f) study, how large charge is lost on capacitor when the diode is closed: keep in mind the idea 39.

85. (a) Apply the idea 43-i: the fluxes of parallel currents add up destructively, hence, at the first moment, the fluxes in the two coils must be equal and parallel; keep in mind that the total flux in both coils are equal, hence their voltages are also equal.
(b) Follow the idea 43-ii: write down Kirchoff's voltage

laws, eliminate current derivatives, express one current through the other and substitute this expression back to one of the Kirchoff's equations, solve this differential equation for the initial conditions found in task (a), cf. fact 17.
(c) Note that a current flowing along the loop defined by the two coils does not cause any flux, hence it can be switched on instantaneously without causing any voltage on the inductors.

86. Apply ideas 44 and 45: calculate the ohmic resistance R of the gun from the nominal values, express the new power dissipation as $P_1 = |I_1|^2 R$, where $I_1 = V/Z$, with $Z = R + 1/i\omega C$.

87. (a) calculate the ohmic resistance of the lamp as U'/I, and apply the idea 44 (express the total impedance containing L as an unknown, and take the modulus from the Ohm's law, written for the entire circuit, to obtain equation for finding L).
(b) Apply idea 44 (the formula $\varphi = \arg Z$).
(c) Apply the idea 45 (keep in mind that we are dealing with the rms amplitudes here).
(d) Apply the idea 38 (for a brief instance, the inductor will act as a constant current source).
(e) Express the instantaneous power as a function of time by substituting $I = I_0 \cos(\omega t)$ into the Joule's law.
(f) Ignition is needed when the gas is insulating, i.e. when there are almost no ions in the gas.
(g) Calculate the new power dissipation; does it change? Compare the magnitudes of the active and reactive powers of this device (see fact 22).

88. Notice that each of the halves of the coil have inductance $\frac{L}{4}$, and due to the fact 21, $M = \frac{L}{4}$. (This can be understood if we cut off the capacitor and consider a current I through the inductor: let the inductance of one half be L' so that also $M = L'$; then each of the halves will have voltage $L'\frac{dI}{dt} + M\frac{dI}{dt} = 2L'\frac{dI}{dt}$, i.e. the total voltage on the full inductor would be $4L'\frac{dI}{dt}$.) Let us use clock-wise current loops I_1 and I_2 in the both halves of the circuit; then the voltage on a half of the inductor will depend on $I_1 + I_2$ (use idea 46!) and voltage on the capacitor — on $I_1 - I_2$; write down the two equations given by the Kirchoff's voltage law, first express both $I_1 - I_2$ and $I_1 + I_2$ in terms of V_0, and from these expressions find I_2.

90. (a) Apply the ideas 44 and 45. Note that for the frequencies ν_+ and ν_-, you'll have a fourth order equation with breaks easily down into two quadratic equations, one of which has roots ν_+ and $-\nu_-$, and the other one — $-\nu_+$ and ν_-. Indeed, both negative and positive frequencies must be the solutions, because they provide physically the same signal, $\cos(\omega t) = \cos(-\omega t)$, and each of these quadratic equations have roots which are clearly different by modulus. Because of that, the difference $\nu_+ - \nu_-$ is actually the sum of the two roots of a quadratic equation, which can be found using the Vieta's formula.
(b) Apply the idea 47 to conclude that there is only one non-zero natural frequency, and the idea 48 to find it (equate the impedance between the left- and right-hand-sides of the circuit to infinity).
(c,d) Note that there is a zero-frequency-mode: a constant current can circulate in the loop formed by the two inductors. The total current is the sum of such a constant current, and a sinusoidally oscillating current; use the initial conditions (the values of i_{01}, i_{02}, and U_0), together with the Kirchoff's current law, and the fact that the ratio of capacitor currents equals to the ratio of the capacitances, to find the respective amplitudes.

91. Study the behaviour of the impedance of the black box as a function of frequency, and pay attention to the low-frequency asymptotics, to the high-frequency asymptotics, and to the minimum of the modulus of the impedance.

92. (a) Apply the idea 44, together with either the method of loop currents, or node potentials; keep in mind that each next node has potentials and currents phase-shifted by φ, for instance, $\phi_{j+1} = \phi_j e^{i\varphi}$, where ϕ_j is the potential of the j-th node (we assume that the lower wire has a zero-potential); once you eliminate potentials (or currents) from your system of equations, you should obtain an equation relating ω to φ .
(b) Keep in mind that the phase speed of a wave is $v_p = \omega/k$, and the phase shift is related to the wave vector via equality $\varphi = kl$.
(c) Study the low-frequency limit of your result for part (b).
(d) Use the idea 51 to compare two systems: an infinite chain of springs and masses, and the given circuit; now it is more convenient to work with the loop currents, and write differential equations for charges passing through inductors. This is because for the mechanical system, the second derivative of each coordinate enters only into one of the differential equations (the Newton's II law); meanwhile, when using the node potential method and capacitors' charges, the inductors' currents (terms giving rise to second derivatives) are expressed from the Kirchoff's current law and would involve the charges of all the capacitors).

93. Apply the idea 52. More specifically, notice that the four vectors form a quadrilateral, opposing angles A and B of which are right angles, hence this is an inscribed quadrilateral, and the other diagonal (other than AB) is the diameter. Pay attention to the fact that the quadrilateral is not convex, because the direction of the voltage vector on L is obtained from that of R_1 by a 90°-counter-clockwise rotation (multiplication by $iL\omega/R_1$), and the direction of the voltage vector on R_2 is obtained from that of C by the same rotation (multiplication by $iC\omega R_2$. The problem simplifies further owing to the fact that two sides and one diagonal of the inscribed quadrilateral are all equal to each other.

94. Apply the idea 44, together with the idea 6: the circuit breaks down into parallel and series connections. Express the voltage on the voltmeter and equate it to zero; pay attention to the fact that a complex number is zero if both real and imaginary parts are zeros, i.e. one equation for complex numbers gives actually two equations for the real-valued quantities.

95. You can solve it either by straightforward calculations by applying idea 44, or geometrically (idea 52).

96. Knowing P_2, U_2, and U_0, one can easily find the resistance of the power lines R_l. Further there are two options. First, one can proceed via a brute-force approach and using the idea 44: with the known voltage at the power station U_0, the values U_1 and P_1 yield two equations for finding two unknowns, r and L.
Another and mathematically easier way is to use the idea 52: apply the cosine theorem to express the known voltage at the power station via the voltage on the power lines $U_l = IR_l$, and the voltage on the transformer. This equation can be solved directly with respect to $P_l = I^2 R_l$ once you notice that the term with cosine can be written as $2PR_l$ and hence, is already known.

97. Proceed similarly to the problem 93 (though, the quadrilateral is not inscribed): show that the diagonal AE divides the quadrilateral of voltages into two equilateral triangles. More specifically, notice that lower and upper branches of the circuit have identical impedances and hence, there is no phase shift between the currents in them; because of that, the voltage vector on AB is equal (hence also parallel) to the one on DE; the same applies to the voltages on BE and AD. Therefore, the triangle formed by voltage vectors on AB, BE, and AE is equilateral; the same applies to the remaining triplet of voltages.

98. Apply the idea 52. Show that similarly to the problem 97, the voltage vectors form two equilateral triangles. More specifically, use the symmetry to show that the voltage vectors on the two capacitors are equal to each other, and the voltage vectors on the two inductors are equal to each other. Details of exploiting the symmetry are as follows. Rotate the circuit by 180°, upon which the capacitors (and inductors) are swapped, and the applied voltage becomes negative; further, rotate the input voltage vector by 180°: the new and old circuits become identical, hence, all the corresponding voltages are equal. In particular, if originally, the left capacitor had voltage $\vec{U_{C1}}$ (from the input lead towards the resistor), the originally right capacitor has now also voltage $\vec{U_{C1}}$, i.e. before the input voltage reversal, it had voltage $-\vec{U_{C1}}$ (towards the resistor), which means that originally, it had voltage $\vec{U_{C1}}$, as measured from the resistor to the input lead. Next, study the quadrilateral of the voltage vectors: its one diagonal gives the input voltage, and the other one — the voltage on the resistor. While the current vector of the resistor is parallel to the voltage vector, in the case of a capacitor it is rotated by 90° clockwise, and in the case of an inductor — by 90° counter-clockwise; use this observation when writing down the Kirchoff's current law (for the node where R, L, and C meet each other) to conclude that the currents in C, L, and R are all equal by modulus. Keep in mind that while the difference of the current vectors of L and C gives the resistor's current, the sum of those gives the input current. Finally, find the power dissipation and the resistor's resistance by using the resistor's voltage and current values.

99. Proceed according to the idea 48: "cut" the circuit near one of the inductors and equate the impedance of the resulting circuit to zero.

100. Apply the idea 44 to reduce the problem to a problem of resistances, very similar to the problem 22. Similarly to that problem, you need to apply the ideas 23 and 22. Notice that the segment AE is also essentially broken, and once AB and AE are broken, we can keep DA because if the output leads are B and E, there is no current in the segment DA (due to symmetry); breaking BA and AE is the same as connecting respective negative resistances; with these negative resistances, we can perform a node splitting at A, so that a negative resistance is connected only between B and E.

101. First, count the number of degrees of freedom (i.e. the number of natural frequencies). Further, notice that there is one loop current, which involves only inductors through which a permanent current can circulate (this yields one frequency). Next, apply the idea 50: two limit cases are obtained: one circuit contains only L_1 and C_1 (use the result of problem 99!), and the other — L_2 and C_2.

102. Use the idea 38: for (a), inductors can be "cut off", and the capacitors short-circuited; for (b) it is vice versa. For (c), the impedance of the connection of inductors and capacitors needs to be inifinite (there is a voltage resonance), and since there is no voltage on the resistor, it can be short-circuited. For (d), the voltage on R needs to be maximal, hence the modulus of the overall impedance — minimal; this means that the impedance of the connection of inductors and capacitors is zero (current resonance).

103. Keep in mind the idea 34. Find the maximal current I_x when the switch is opened — either using the idea 35, or using the idea 36 with fact 19. When the switch is closed, we have two independent LC-circuits with the same frequency, so the current in the switch is found as the difference between the two currents in these LC-circuits. Amplitude (and hence, the maximal value) can be found using phasor diagram, phase shift is to be found from the initial charges on capacitors and initial currents in inductors when the switch was closed.

104. (a) diac current can be neglected, hence R and C are connected in series directly to the AC voltage source so that we can apply the idea 44;
(b) we need to study, how will change the diac's current while the capacitor voltage grows, to that end we use the idea 24 and fact 9: if we start with a small voltage, there is only one solution for the current, but with a small enough R_t, at a certain voltage two more solutions appear, and at an even larger voltage (around U_b), two smaller solutions disappear; at that moment, the diac's current is forced to jump to the only remaining solution (now it becomes also clear what does mean "diac starts conducting"); the condition "triac starts immediately conducting" means that the new diac's current must be larger than I_t;
(c) according to (a), the capacitor voltage lags behind the overall voltage and hence, behind the lamp's current; when the lamp's current goes through zero (at $t = 0$ on the

graph) and triac closes, the capacitor's and diac's voltage is still negative; the triac will open again when the capacitor's voltage becomes positive and equal to U_b at $t = t_1$; (d) the energy disspated on the lamp is calculated as $\int R^{-1}[U_l(t)]^2 \mathrm{d}t$.

105. Notice that the circuit is self-dual. Calculate the impedance for sinusoidal voltage input using the idea 15. Since the impedance is independent of the sinusoidal circular frequency ω, it is equivalent to an active resistor. NB! The analysis would have been much more difficult if the overall impedance were to depend on ω as the impedances cannot be applied directly in the case of non-sinusoidal signals.

106. The number of degrees of freedom is two, so is the number of natural oscillation modes. According to the ideas 48 and 49, consider voltage resonance ($Z_{AB} = \infty$) for fictitious terminals at the symmetry axis (A — middle point of the inductor L_2 dividing it into two inductances, $L_2/2$ each, and B — the middle point of the lower wire); the impedance Z_{AB} simplifies due to symmetry to such a degree that the condition $Z_{AB} = \infty$ gives us only one frequency. According to the idea 49, the missing frequency is found by short-circuiting nodes A and B, which makes C connected to a parallel connection of L_1 and $L_2/2$.

107. According to the idea 52, consider phasor diagram; according to the Thales' theorem, the voltages (with respect to the one of the input terminals) of the nodes A and B lie on circle drawn around the input voltage vector as its diagonal; the voltage of the node D is the circle's centre. Keep in mind that due to the Thales' theorem, the median of a right triangle equals to half of its hypotenuse; don't forget that central angle is twice as large as the corresponding inscribed angle.

108. Similarly to the problem 107, the voltages (with respect to one of the input terminals) of the output nodes lie on circle drawn around the input voltage vector as its diagonal; apply the inscribed angle theorem.

109. Proceed similarly to the problem 108; the only difference is that the relevant voltages do not form right angles, but angles $\geq \pi/2$; correspondingly, the voltages of the output terminals lie inside the circle. Note that the segment which lies completely inside a circle cannot be longer than the diameter, and can be equal in length to the diameter if its endpoints lie on the circle.

110. Reversing the idea 48, we find these frequencies as the natural frequencies of the circuit for which the terminals are short-circuited. We result in a symmetric circuit which has two parallel connections of L and C, connected in series with $3C$. We represent the capacitor $3C$ as a series connection of two capacitances $6C$, and use the idea 49 together with the voltage reonance condition ($Z_{AB} = \infty$, where A and B are the symmetrical nodes of the circuit) to deduce the first natural frequency. Finally, we short-circuit nodes A and B to find the missing natural frequency.

111. We start solving the same way as in the case of problem 110. Once we short-circuit the input terminals, we obtain a symmetric circuit which has three linearly independent loops, but there is one loop which consists of only capacitors, so there are two natural oscillation modes. We apply ideas 48 and 49 by splitting the circuit at the position of one of the inductors and requiring $Z = 0$. Due to the symmetry of the obtained circuit we can remove the bridge connection without affecting its impedance; the condition $Z = 0$ yields us one frequency. The missing frequency is obtained as the natural frequency of the simplified circuit: the circuit where the "cut" point is left disconnected.

Answers

1. $R \approx 14\,\Omega$

2. $I = 0.5\,\mathrm{A}$

3. $R = 0.5\,\Omega$

4. $R = 2.5\,\Omega$

5. $I = \frac{3}{22}\,\mathrm{A}$

6. $I = 196\,\mu\mathrm{A}$

7. $V_\Sigma = 78\,\mathrm{V}$

8. $I = \frac{21}{19}\,\mathrm{A}$

9. $I_4 = 3\,\mathrm{A}$, $I_3 = 2\,\mathrm{A}$

10. $P = \frac{1}{4}\frac{R_2}{(r+R_1+R_2)(R_1+r)}\mathcal{E}^2$

11. $r = (\sum_{i=1}^n r_i^{-1})^{-1}$, $\mathcal{E} = r\sum_{i=1}^n \mathcal{E}_i r_i^{-1}$

12. $I_2 = I_4 = 0$, $I_1 = I_3 = \mathcal{E}/R$

15. $R = 2\Omega$

16. $R = \frac{R_1}{2}\left(1 + \sqrt{1 + 4R_2/R_1}\right)$

17. $r' = \frac{R}{2}\left(1 + \sqrt{1 + 4R/r}\right)$, $\mathcal{E}' = \mathcal{E}$

18. $R = \frac{5}{6}\Omega$

19. $R_{AO} = \frac{9}{20}R$.

21. $r = \frac{19}{30}R$

22. $r = \frac{19}{11}R$

23. Straight lines connecting the following points: (0 s, 0 A); (1.5 s, 1 A); (1.5 s, 0.6 A); (5 s, 2 A); (5 s, $\frac{5}{3}$A); (10 s, $\frac{10}{3}$A); (17 s, 1 A); (17 s, 1.2 A); (18.75 s, 0.5 A); (18.75 s, $\frac{5}{6}$A); (20 s, 0 A)

24. $I \approx 8\,\mathrm{mA}$

25. approximately -1.4 times

27. $\frac{2}{29}R \approx 0.414R < r < \frac{4}{9}R \approx 0.444R$.

28. increases $\frac{16}{9}$ times

29. 0.75 mW, 0 W, 0 W.

30. $r' = R\frac{3R+2r}{5R+3r}$, $\mathcal{E}' = \mathcal{E}\frac{R}{5R+3r}$, $P_{\max} = \frac{\mathcal{E}^2 R}{4(5R+3r)(3R+2r)}$

31. $I \approx 3\,\mathrm{mA}$

32. $I_1 = 1.5\,\mathrm{A}$; $I_2 = 1.7\,\mathrm{A}$

33. $R = 40\,\mathrm{k}\Omega$

34. $I \propto V^{0.6}$ or equivalently $V \propto I^{5/3}$

35. $I_1 = 0$, $I_2 = 3\mathcal{E}/R$, $I_3 = I_4 = 1.5\mathcal{E}/R$

36. 4 A

37. all ammeters: 0.75 A.

38. $V_2 \approx 8.65\,\mathrm{V}$

39. $V=(\mathcal{E}_1R_2-\mathcal{E}_2R_1)/(R_1+R_2+\frac{R_1R_2}{R_3})$

40. $V_3=1\,\mathrm{V}$

41. increases ≈ 1.14 times

42. 3 mA, 6 mA, 7 mA, and 14 mA.

43. $R/r=9$

44. $I_1=0.7I_0$

45. $R_{AB}=R/3$

46. $r=R$

47. $r=\frac{3}{8}R$

48. $r=2R/n$

49. $r=\frac{67}{315}R$

50. $\frac{R}{3}<r<\frac{5R}{11}$; $r=\frac{6}{17}R$

51. $r=R\frac{1}{4}(3+\sqrt{17})$

52. $\mathcal{E}'=\mathcal{E}\left(1+\frac{r}{r'}\right)$ with $r'=\frac{R}{2}\left(1+\sqrt{1+4\frac{R}{r}}\right)$

53. $\frac{40}{87}R\le r\le\frac{47}{87}R$

54. $R(1-\frac{R_2}{r})$

56. $R+R_{\mathrm{diff}}>0$

58. $\frac{1}{2}(V_0-V_d)^2C$

59. $2(\mathcal{E}-V_d)$

61. $\frac{2}{27}C\mathcal{E}^2$

62. $C_1\mathcal{E}/\sqrt{L(C_1+C_2)}$, $\mathcal{E}\left(1+\frac{C_1}{C_1+C_2}\right)$

63. Sinusoid with minima at $V=\mathcal{E}\frac{C_1}{C_1+C_2}$ and maxima at $V=V_{\max}$; $\omega=1/\sqrt{(C_1+C_2)L}$

64. 8.9 mA

65. 0 mA, 0 mA, and 0 mA

66. $V=IR$ for the first half-period, and $V=-IR$ for the second half-period [more precisely, for each half-period, the asymptotic values are reached exponentially, $V=\pm I_1R(1-2\mathrm{e}^{-\Delta t/RC})$, where Δt is the time elapsed since the beginning of the half-period]; saw-tooth profile which grows linearly from 0 to I_1T/C, and decreases linearly down to 0 during the second half-period; $(I_2+I_1)R/2$, $(I_2-I_1)R/2$; $(I_2+I_1)R/2$, $(I_2-I_1)T/8C$

67. $I_R=V_0\cos(2\pi\nu t)/R$, $I_D=V_0[\cos(2\pi\nu t)+1]/R$ (if we don't use approximation $L\omega\gg R$ then $I_D=V_0\sqrt{R^{-2}+(2\pi\nu L)^{-2}}[\cos(2\pi\nu t+\varphi)+1]$).

68. $2BNS/R$

69. See at the website of IPhO

70. (a) Both by modulus $\mathcal{E}/(R+r)$;
(b) $\frac{\mathcal{E}}{R+r}\mathrm{e}^{-t/\tau}$ and $\frac{\mathcal{E}}{R}(1-\frac{r}{R+r})\mathrm{e}^{-t/\tau}$.

71. $2I$, I, and I

72. $\frac{1}{2}L_2\frac{\mathcal{E}^2}{r^2}$; $\frac{L_2\mathcal{E}}{rR}$

73. $P=(U_2-U_1)^2/4R$, and $P=C(U_2-U_1)^2/T$

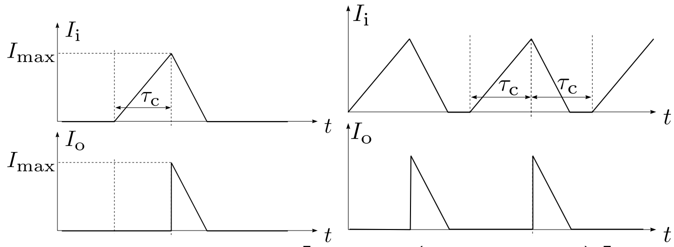
74. $\tau=\left(R_1+\dfrac{R_2R_3}{R_2+R_3}\right)C.$

75. 8.06 kΩ, > 50 µF

76. $\mathcal{E}$; $-2\mathcal{E}$; $L\mathcal{E}^2/(2R^2)$, $C\mathcal{E}^2/2+L\mathcal{E}^2/(2R^2)$.

77. $P=2\,\mathrm{mW}$; $U_0=21\,\mathrm{V}$; $C\ge 200\,\mathrm{\mu F}$; $P_1=200\,\mathrm{mW}$

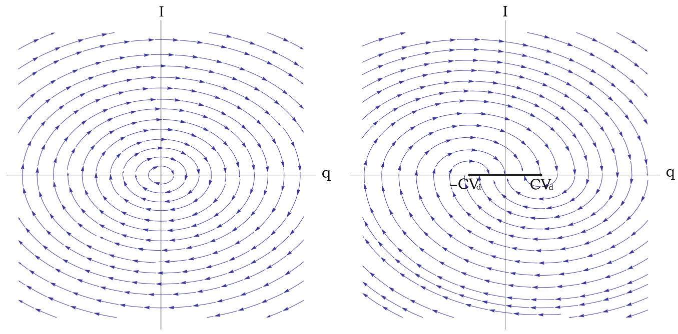
78.



$J=\frac{\tau_c}{4L}\frac{U_i^2}{U_0-U_i}$; $U_0=\max\left[2U_i,\ \frac{U_i}{2}\left(1+\sqrt{1+\frac{\tau_cR}{L}}\right)\right]$

79. sawtooth profile consisting of curve segments $V_0=V_i(1-\mathrm{e}^{-t/RC})$ (from $V_0=0$ till $V_0=V_f$) and vertical line segments (from $V_0=V_f$ till $V_0=0$); $V_i\gg V_f$; $T=V_fRC/V_i$; R; both; for instance: use tha same circuit, but connect another battery of emf. V_i-V_f and opposed polarity to the node between the battery and the resistance, and take output signal between the '−'-lead of the new battery, and the top lead of the *SG* (many other solutions are possible).

80.



$-CV_d<q<CV_d$; $\Delta q=-4CV_d$, $t=N\pi\sqrt{LC}$, $N=\left\lfloor\frac{|q_0|-CV_d}{2CV_d}\right\rfloor$; a mass-spring system where the mass is subject to a dry friction force.

81. $I_1\approx 0.33\,\mathrm{A}$, $I_2=5\,\mathrm{mA}$; $I_1'\approx 5.6\,\mathrm{mA}$, $I_2'=U\tau/L=10\,\mathrm{mA}$; $\Delta q=LI_2'/R_2=3.3\,\mathrm{mC}$.

82. (a) $r+R_{\mathrm{diff}}<0$ and $\frac{L}{C}<R|R_{\mathrm{diff}}|+r|R+R_{\mathrm{diff}}|$;
(b) additionally, $L_d<r|R_{\mathrm{diff}}|C_d$

83. (a) $R_{\mathrm{off}}=10\,\Omega$, $R_{\mathrm{on}}=1\,\Omega$, $R_{\mathrm{int}}=2\,\Omega$, $I_0=6\,\mathrm{A}$; (b) 3 Ω: always one state; 1 Ω: 1, 2 or 3 states; (c) 3 A, 6 V, stable; (d) from 4 V to 10 V moves along the lower branch, jumps from 1 A to 10 A, returns down to 4 V along the upper branch, and completes the cycle by jumping down to 0.4 A; (e) 2.41 µs, 3.71 µs, 6.12 µs; (f) $P\sim 20\,\mathrm{W}$; (g) $\tau<\tau_{\mathrm{crit}}$: relaxation towards a higher stationary current, followed by a relaxation towards the original stationary current; $\tau>\tau_{\mathrm{crit}}$: relaxation towards a higher stationary current along the lower branch, followed by a jump to the upper branch, current decrease along the upper branch, jump to the lower branch, and relaxation towards the original stationary state; (h) 0.936 µs.

84. a) $\tau_L=LI_0/\mathcal{E}$; b) $V_{\max}=RI_0$; c) $P=\frac{V_0\mathcal{E}}{R}$; d) $V_{av}=\sqrt{\frac{\mathcal{E}I_0R}{2}}$; e) $U_0=\frac{I_0L}{2C}\sqrt{\frac{I_0}{2R\mathcal{E}}}$.

85. (a) $\frac{2}{5}\frac{\mathcal{E}}{R}$ and $\frac{1}{5}\frac{\mathcal{E}}{R}$; (b) $\frac{2}{5}\frac{\mathcal{E}}{R}\mathrm{e}^{-t/\tau}$, $\tau=\frac{5L}{R}$; (c) constant, equal to $\mathcal{E}/R$

86. 2.8 µF

87. 1.09 H; 64.1° 59.9 W; to create huge voltage to ionize the gas, graph $\propto 1+\cos(2\pi\nu t)$ [or slightly raised: $a+\cos(2\pi\nu t)$ with $a > 1$]; recombination time is large enough to keep vapors in the plasma state; the current is almost the same as before, the phase $-63.6°$, this is to reduce the reactive power if two lamps are in parallel

88. $V_0(\frac{1}{\omega L} + \frac{\omega C}{4}$.

90. $R\sqrt{C/L}$; $1/\sqrt{LC}$ with $L = \frac{L_1 L_2}{L_1+L_2}$ and $C = C_1+C_2$; 0.1 A; 0.2 A

91. $C = 1/\omega \lim_{\omega\to 0} |Z(\omega)|$, $L = \lim_{\omega\to\infty} |Z(\omega)|/\omega$, $R = \min_\omega |Z(\omega)|$

92. $\varphi = 2\arcsin\left(\frac{1}{2}\omega\sqrt{LC}\right)$; $\omega l/\varphi$; $\varphi \ll 1$ when $v_0 = l/\sqrt{LC}$; infinite chain of masses connected by springs

93. 20 V

94. $L = R_1 R_2 C$, $R = R_1 R_2 / R_C$

95. $2\arctan(\omega RC)$

96. ≈ 300 W

97. $10\sqrt{3}$V

98. 10 W, 30 Ω

99. $\omega = \frac{\sqrt{5}\pm 1}{2\sqrt{LC}}$

100. $\frac{11}{18}C$

101. 0; $\frac{2}{\sqrt{L_2 C_2}}$; $\frac{\sqrt{5}\pm 1}{2\sqrt{L_1 C_1}}$

102. 1, 0; 1, 0; 1, π, $\nu_0 = 4\pi/\sqrt{2LC}$; $\sqrt{1+\frac{L}{R^2C}}$, $\arctan\frac{L}{R^2C}$, $\nu_1 = 2\pi/\sqrt{LC}$

103. $q_0/\sqrt{2LC}$.

104. (a) $U_C = U|k| = U/\sqrt{1+(2\pi fRC)^2}$ and $\varphi = \arg k = -\arctan(2\pi fRC)$; (b) $R_t I_t < U_b - U_d$; (c) $t_0 = [\arcsin(U_b/U_C) - \varphi]/(2\pi f)$; (d) $\left[1 - 2ft_0 + \frac{\sin(4\pi f t_0)}{2\pi}\right]^{-1}$.

105. rectangular waveform of amplitude $V_0\sqrt{\frac{C}{L}}$.

106. $\omega_1 = 1/\sqrt{L_1 C}$; $\omega_2 = \sqrt{\frac{2L_1+L_2}{L_1 L_2 C}}$.

107. $V_0/2$; 2φ.

108. φ

109. V_0; $R_1 = R_2 = 0$, $R_3 = L/Cr$.

110. $1/\sqrt{LC}$; $1/\sqrt{7LC}$.

111. $\sqrt{2/3LC}$; $\sqrt{3/4LC}$.