Formulas for IPhO

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I Mathematics

1. Taylor series (truncate for approximations):

$$F(x) = F(x_0) + \sum F^{(n)}(x_0)(x - x_0)^n/n!$$

Special case — linear approximation:

$$F(x) \approx F(x_0) + F'(x_0)(x - x_0)$$

Some examples for $|x| \ll 1$:

$$\sin x \approx x,\ \cos x \approx 1 - x^2/2,\ e^x \approx 1 + x$$

$$\ln(1 + x) \approx x,\ (1 + x)^n \approx 1 + nx$$

2. Perturbation method: find the solution iteratively using the solution to the "non-perturbed" (directly solvable) problem as the 0^{th} approximation; corrections for the next approximation are calculated on the basis on the previous one.

3. Solution of the linear differential equation with constant coefficients $ay'' + by' + cy = 0$:

$$y = A\exp(\lambda_1 x) + B\exp(\lambda_2 x),$$

where $\lambda_{1,2}$ is the solution of the characteristic equation $a\lambda^2 + b\lambda + c = 0$ if $\lambda_1 \neq \lambda_2$. If the solution of the characteristic equation is complex, while a, b and c are real numbers, then $\lambda_{1,2} = \gamma \pm i\omega$ and

$$y = Ce^{\gamma x}\sin(\omega x + \varphi_0).$$

4. Complex numbers

$$z = a + \mathrm{b}i = |z|e^{\mathrm{i}\varphi},\ \bar{z} = a - \mathrm{i}b = |z|e^{-\mathrm{i}\varphi}$$

$$|z|^2 = z\bar{z} = a^2 + b^2,\ \varphi = \arg z = \arcsin\frac{b}{|z|}$$

$$\mathrm{Re}z = (z + \bar{z})/2,\ \mathrm{Im}z = (z - \bar{z})/2$$

$$|z_1 z_2| = |z_1||z_2|,\ \arg z_1 z_2 = \arg z_1 + \arg z_2$$

$$e^{\mathrm{i}\varphi} = \cos\varphi + \mathrm{i}\sin\varphi$$

$$\cos\varphi = \tfrac{e^{\mathrm{i}\varphi} + e^{-\mathrm{i}\varphi}}{2},\ \sin\varphi = \tfrac{e^{\mathrm{i}\varphi} - e^{-\mathrm{i}\varphi}}{2i}$$

5. Cross and dot products of vectors are distributive: $a(b + c) = ab + ac$.

$$\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a} = a_x b_x + a_y b_y + \ldots = ab\cos\varphi$$

$$|\vec{a}\times\vec{b}| = ab\sin\varphi;\quad \vec{a}\times\vec{b} = -\vec{b}\times\vec{a} \perp \vec{a}, \vec{b}$$

$$\vec{a}\times\vec{b} = (a_y b_z - b_y a_z)\vec{e}_x + (a_z b_x - b_z a_x)\vec{e}_y + \ldots$$

$$\vec{a}\times[\vec{b}\times\vec{c}] = \vec{b}(\vec{a}\cdot\vec{c}) - \vec{c}(\vec{a}\cdot\vec{b}).$$

Mixed prod. (volume of parallelep. def. by 3 vec.):

$$(\vec{a}, \vec{b}, \vec{c}) \equiv (\vec{a}\cdot[\vec{b}\times\vec{c}]) = ([\vec{a}\times\vec{b}]\cdot\vec{c}) = (\vec{b}, \vec{c}, \vec{a}).$$

6. Cosine and sine laws:

$$c^2 = a^2 + b^2 - 2ab\cos\varphi$$

$$a/\sin\alpha = b/\sin\beta = 2R$$

7. $\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$
$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$
$\tan(\alpha \pm \beta) = (\tan\alpha + \tan\beta)/(1 \mp \tan\alpha\tan\beta)$
$\cos^2\alpha = \frac{1 + \cos 2\alpha}{2}$, $\sin^2\alpha = \frac{1 - \cos 2\alpha}{2}$
$\cos\alpha\cos\beta = \frac{\cos(\alpha+\beta) + \cos(\alpha-\beta)}{2}, \ldots$
$\cos\alpha + \cos\beta = 2(\cos\frac{\alpha+\beta}{2} + \cos\frac{\alpha-\beta}{2}), \ldots.$

8. An angle inscribed in a circle is half of the central angle that subtends the same arc on the circle. *Conclusions:* hypotenuse of a right triangle is the diameter of its circumcircle; if the angles of a quadrilateral are supplementary, it is a cyclic quadrilateral.

9. Surface area of a triangle $= \frac{1}{2}ah_a = pr = \sqrt{p(p-a)(p-b)(p-c)} = abc/4R.$

10. Triangle's centroid: intersection point of medians, divides medians to 2:1.

11*. Vector approach to geometry problems.

12. Taking derivatives:

$$(fg)' = fg' + f'g,\ f[g(x)]' = f'[g(x)]g'$$

$$(\sin x)' = \cos x,\ (\cos x)' = -\sin x$$

$$(e^x)' = e^x,\ (\ln x)' = 1/x,\ (x^n)' = nx^{n-1}$$

$$(\arctan x)' = 1/(1 + x^2)$$

$$(\arcsin x)' = -(\arccos x)' = 1/\sqrt{1 - x^2}$$

13. Integration: the formulas are the same as for derivatives, but with swapped left-hand-side and rhs. (inverse operation!), e.g.

$$\int x^n \mathrm{d}x = x^{n+1}/(n + 1).$$

Special case of the substitution method:

$$\int f(ax + b)\mathrm{d}x = F(ax + b)/a.$$

14. Conic sections: $a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + a_1 x + a_2 y + a_0 = 0$ with $a_{11} = a_{22}$ — circle; with $a_{11}\cdot(a_{11}a_{22} - a_{12}^2) > 0$ — ellipse, $\ldots < 0$ — hyperbola, with $a_{11}a_{22} = a_{12}^2$ — parabola. Ellipse: $l_1 + l_2 = 2a$, $\alpha_1 = \alpha_2$, $A = \pi ab$; hyperbola: $l_1 - l_2 = 2a$, $\alpha_1 + \alpha_2 = 0$; parabola: $l + h = \text{const}$, $\alpha_1 = \alpha_2$.

15. *Numerical methods.* Newton's iterative method for finding roots $f(x) = 0$:

$$x_{n+1} = x_n - f(x_n)/f'(x_n).$$

Trapezoidal rule for approximate integration:

$$\int_a^b f(x)\mathrm{d}x \approx \frac{b - a}{2n}[f(x_0) + 2f(x_1) + \ldots + 2f(x_{n-1}) + f(x_n)]$$

16. Derivatives and integrals of vectors: differentiate/integrate each component; alternatively differentiate by applying the triangle rule for the difference of two infinitesimally close vectors.

II General recommendations

1. Check all formulas for veracity: a) examine dimensions; b) test simple special cases (two parameters are equal, one param. tends to 0 or ∞); c) verify the plausibility of solution's qualitative behaviour.

2. If there is an extraordinary coincidence in the problem text (e.g. two things are equal) then the key to the solution might be there.

3. Read carefully the recommendations in the problem's text. Pay attention to the problem's formulation — insignificant details may carry vital information. If you have solved for some time unsuccessfully, then read the text again — perhaps you misunderstood the problem.

4. Postpone long and time-consuming mathematical calculations to the very end (when everything else is done) while writing down all the initial equations which need to be simplified.

5. If the problem seems to be hopelessly difficult, it has usually a very simple solution (and a simple answer). This is valid only for Olympiad problems, which are definitely solvable.

6. In experiments a) sketch the experimental scheme even if you don't have time for measurements; b) think, how to increase the precision of the results; c) write down (as a table) all your direct measurements.

III Kinematics

1. For a point or for a translational motion of a rigid body (integral $\to$ area under a graph):

$$\vec{v} = \frac{\mathrm{d}\vec{x}}{\mathrm{d}t},\ \vec{x} = \int \vec{v}\mathrm{d}t \quad (x = \int v_x \mathrm{d}t \text{ etc.})$$

$$\vec{a} = \frac{\mathrm{d}\vec{v}}{\mathrm{d}t} = \frac{\mathrm{d}^2\vec{x}}{\mathrm{d}t^2},\ \vec{v} = \int \vec{a}\mathrm{d}t$$

$$t = \int v_x^{-1}\mathrm{d}x = \int a_x^{-1}\mathrm{d}v_x,\ x = \int \frac{v_x}{a_x}\mathrm{d}v_x$$

If $a = $ Const., then previous integrals can be found easily, e.g.

$$x = v_0 t + at^2/2 = (v^2 - v_0^2)/2a.$$

2. Rotational motion — analogous to the translational one: $\omega = \mathrm{d}\varphi/\mathrm{d}t$, $\varepsilon = \mathrm{d}\omega/\mathrm{d}t$;

$$\vec{a} = \vec{\tau}\mathrm{d}v/\mathrm{d}t + \vec{n}v^2/R$$

3. Curvilinear motion — same as point 1, but vectors are to be replaced by linear velocities, accelerations and path lengths.

4. *Motion of a rigid body.* **a)** $v_A\cos\alpha = v_B\cos\beta$; $\vec{v}_A$, $\vec{v}_B$ — velocities of pts. A and B; α, β — angles formed by $\vec{v}_A$, $\vec{v}_B$ with line AB. **b)** The instantaneous center of rotation ($\neq$ center of curvature of material pt. trajectories!) can be found as the intersection pt. of perpendiculars to $\vec{v}_A$ and $\vec{v}_B$, or (if $\vec{v}_A, \vec{v}_B \perp AB$) as the intersection pt. of AB with the line connecting endpoints of $\vec{v}_A$ and $\vec{v}_B$.

5. Non-inertial reference frames:

$$\vec{v}_2 = \vec{v}_0 + \vec{v}_1,\ \vec{a}_2 = \vec{a}_0 + \vec{a}_1 + \omega^2\vec{R} + \vec{a}_{Cor}$$

Note: $\vec{a}_{Cor} \perp \vec{v}_1, \vec{\omega}$; $\vec{a}_{Cor} = 0$ if $\vec{v}_1 = 0$.

6*. Ballistic problem: reachable region

$$y \leq v_0^2/(2g) - gx^2/2v_0^2.$$

For an optimal ballistic trajectory, initial and final velocities are perpendicular.

7. For finding fastest paths, Fermat's and Huygens's principles can be used.

8. To find a vector (velocity, acceleration), it is enough to find its direction and a projection to a single (possibly inclined) axes.

IV Mechanics

1. For a 2D equilibrium of a rigid body: 2 eqns. for force, 1 eq. for torque. 1 (2) eq. for force can be substituted with 1 (2) for torque. Torque is often better — "boring" forces can be eliminated by a proper choice of origin. If forces are applied only to 2 points, the (net) force application lines coincide; for 3 points, the 3 lines meet at a single point.

2. Normal force and friction force can be combined into a single force, applied to the contact point under angle $\arctan\mu$ with respect to the normal force.

3. Newton's 2^{nd} law for transl. and rot. motion:

$$\vec{F} = m\vec{a},\ \vec{M} = I\vec{\varepsilon} \quad (\vec{M} = \vec{r}\times\vec{F}).$$

For 2D geometry $\vec{M}$ and $\vec{\varepsilon}$ are essentially scalars and $M = Fl = F_t r$, where l is the arm of a force.

4. *Generalized coordinates.* Let the system's state be defined by a single parameter ξ and its time derivative $\dot{\xi}$ so that the pot. energy $\Pi = \Pi(\xi)$ and kin. en. $K = \mu\dot{\xi}^2/2$; then $\mu\ddot{\xi} = -\mathrm{d}\Pi(\xi)/\mathrm{d}\xi$. (Hence for transl. motion: force is the derivative of pot. en.)

5. If the system consists of mass points m_i:

$$\vec{r}_c = \sum m_i\vec{r}_i / \sum m_j,\ \vec{P} = \sum m_i\vec{v}_i$$

$$\vec{L} = \sum m_i\vec{r}_i \times \vec{v}_i,\ K = \sum m_i v_i^2/2$$

$$I_z = \sum m_i(x_i^2+y_i^2) = \int (x^2+y^2)\mathrm{d}m.$$

6. In a frame where the mass center's velocity is $\vec{v}_c$ (index c denotes quantities rel. to the mass center):

$$\vec{L} = \vec{L}_c + M_\Sigma \vec{R}_c \times \vec{v}_c,\ K = K_c + M_\Sigma v_c^2/2$$
$$\vec{P} = \vec{P}_c + M_\Sigma \vec{v}_c$$

7. Steiner's theorem is analogous (b — distance of the mass center from rot. axis): $I = I_c + mb^2$.

8. With $\vec{P}$ and $\vec{L}$ from pt. 5, Newton's 2^{nd} law:
$$\vec{F}_\Sigma = \mathrm{d}\vec{P}/\mathrm{d}t,\quad \vec{M}_\Sigma = \mathrm{d}\vec{L}/\mathrm{d}t$$

9*. Additionally to pt. 5, the mom. of inertia rel. to the z-axis through the mass center can be found as $I_{z0} = \sum_{i,j} m_i m_j[(x_i - x_j)^2 + (y_i - y_j)^2]/2M_\Sigma$.

10. Mom. of inertia rel. to the origin $\theta = \sum m_i \vec{r}_i^{\,2}$ is useful for calculating I_z of 2D bodies or bodies with central symmetry using $2\theta = I_x + I_y + I_z$.

11. Physical pendulum with a reduced length $\tilde{l}$:
$$\omega^2(l) = g/(l + I/ml),$$
$$\omega(l) = \omega(\tilde{l} - l) = \sqrt{g/\tilde{l}},\quad \tilde{l} = l + I/ml$$

12. Coefficients for the momenta of inertia: cylinder $\frac{1}{2}$, solid sphere $\frac{2}{5}$, thin spherical shell $\frac{2}{3}$, rod $\frac{1}{12}$ (rel. to endpoint $\frac{1}{3}$), square $\frac{1}{6}$.

13. Often applicable conservation laws:
energy (elastic bodies, no friction),
momentum (no net external force; can hold only along one axis),
angular momentum (no net ext. torque, e.g. the arms of ext. forces are 0 (can be written rel. to 2 or 3 pts., then substitutes conservation of lin. mom.).

14. Additional forces in non-inertial frames of ref.: inertial force $-m\vec{a}$, centrifugal force $m\omega^2\vec{R}$ and Coriolis force* $2m\vec{v}\times\vec{\Omega}$ (better to avoid it; being $\perp$ to the velocity, it does not create any work).

15. Tilted coordinates: for a motion on an inclined plane, it is often practical to align axes along and $\perp$ to the plane; gravit. acceleration has then both x- and y- components. Axes may also be oblique (not $\perp$ to each other), but then with $\vec{v} = v_x\vec{e}_x + v_y\vec{e}_y$, $v_x \neq$ to the x-projection of $\vec{v}$.

16. Collision of 2 bodies: conserved are a) net momentum, b) net angular mom., c) *angular mom. of one of the bodies with respect to the impact point*, d) total energy (for elastic collisions); in case of friction, kin. en. is conserved only along the axis $\perp$ to the friction force. Also: e) if the sliding stops during the impact, the final velocities of the contact points will have equal projections to the contact plane; f) if sliding doesn't stop, the momentum delivered from one body to the other forms angle $\arctan\mu$ with the normal of the contact plane.

17. Every motion of a rigid body can be represented as a rotation around the instantaneous center of rotation C (in terms of velocities of the body points). NB! Distance of a body point P from $C \neq$ to the radius of curvature of the trajectory of P.

18. Tension in a string: for a massive hanging string, tension's horizontal component is constant and vertical changes according to the string's mass underneath. Pressure force (per unit length) of a string resting on a smooth surface is determined by its radius of curvature and tension: $N = T/R$. Analogy: surface tension pressure $p = 2\sigma/R$; to derive, study the pressure force along the diameter.

19. Liquid surface takes equipot. shape (neglecting σ); in incompr. liquid, $p = p_0 - w$, w—vol. dens. of pot. en. (for a gas, see X-6).

20. Bernoulli law for incompr. fluid:
$$p + \frac{1}{2}\rho v^2 + \rho\varphi = \text{const};$$
in homog. field, the gravit. potential $\varphi = gh$. For gas of specific heat c_p [J/kg],
$$\frac{1}{2}v^2 + c_pT = \text{const}.$$

21*. Momentum continuity by straight streamlines: $p + \rho v^2 = \text{const}$.

22*. Adiabatic invariant: if the relative change of the parameters of an oscillating system is small during one period, the area of the loop drawn on the phase plane (ie. in p-x coordinates) is conserved with a very high accuracy.

23. For studying stability use a) principle of minimum potential energy or b) principle of small virtual displacement.

24*. Virial theorem for finite movement:
a) If $F \propto |\vec{r}|$, then $\langle K\rangle = \langle\Pi\rangle$ (time averages);
b) If $F \propto |\vec{r}|^{-2}$, then $2\langle K\rangle = -\langle\Pi\rangle$.

25. Tsiolkovsky rocket equation $\Delta v = u\ln\frac{M}{m}$.

V Oscillations and waves

1. Damped oscillator:
$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = 0\ (\gamma < \omega_0).$$
Solution of this equation is (cf. I.2.):
$$x = x_0 e^{-\gamma t}\sin(t\sqrt{\omega_0^2 - \gamma^2} - \varphi_0).$$

2. Eq. of motion for a system of coupled oscillators: $\ddot{x}_i = \sum_j a_{ij}x_j$.

3. A system of N coupled oscillators has N different eigenmodes when all the oscillators oscillate with the same frequency ω_i, $x_j = x_{j0}\sin(\omega_i t + \varphi_{ij})$, and N eigenfrequencies ω_i (which can be multiple, $\omega_i = \omega_j$). General solution (with $2N$ integration constants X_i and ϕ_i) is a superposition of all the eigenmotions :
$$x_j = \sum_i X_i x_{j0}\sin(\omega_i t + \varphi_{ij} + \phi_i)$$

4. If a system described with a generalized coordinate ξ (cf IV-2) and $K = \mu\dot{\xi}^2/2$ has an equilibrium state at $\xi = 0$, for small oscillations $\Pi(\xi) \approx \kappa\xi^2/2$ [where $\kappa = \Pi''(0)$] so that $\omega^2 = \kappa/\mu$.

5. The phase of a wave at pt. x,t is $\varphi = kx - \omega t + \varphi_0$, where $k = 2\pi/\lambda$ is a wave vector. The value at x,t is $a_0\cos\varphi = \Re a_0 e^{i\varphi}$. The phase velocity is $v_f = \nu\lambda = \omega/k$ and group velocity $v_g = d\omega/dk$.

6. For linear waves (electromagn. w., small-amplit. sound- and water w.) any pulse can be considered as a superpos. of sinusoidal waves; a standing w. is the sum of two identical counter-propagating w.:
$$e^{i(kx-\omega t)} + e^{i(-kx-\omega t)} = 2e^{-i\omega t}\cos kx.$$

7. Speed of sound in a gas
$$c_s = \sqrt{(\partial p/\partial\rho)_\text{adiab}} = \sqrt{\gamma p/\rho} = \bar{v}\sqrt{\gamma/3}.$$

8. Speed of sound in elastic material $c_s = \sqrt{E/\rho}$.

9. Sp. of waves in shallow ($h \ll \lambda$) water: $v = \sqrt{gh}$; in a string: $v = \sqrt{T/\rho_\text{lin}}$.

10. Doppler's effect: $\nu = \nu_0\frac{1+v_\parallel/c_s}{1-u_\parallel/c_s}$.

11. Huygens' principle: wavefront can be constructed step by step, placing an imaginary wave source in every point of previous wave front. Results are curves separated by distance $\Delta x = c_s\Delta t$, where Δt is time step and c_s is the velocity in given point. Waves travel perpendicular to wavefront.

VI Geometrical optics. Photometry.

1. Fermat's principle: waves path from point A to point B is such that the wave travels the least time.

2. Snell's law:
$$\sin\alpha_1/\sin\alpha_2 = n_2/n_1 = v_1/v_2.$$

3. If refraction index changes continuously, then we imaginarily divide the media into layers of constant n and apply Snell's law. Light ray can travel along a layer of constant n, if the requirement of total internal reflection is marginally satisfied, $n' = n/r$ (where r is the curvature radius).

4. If refraction index depends only on z, the photon's mom. p_x , p_y, and en. are conserved:
$$k_x, k_y = \text{Const.},\ |\vec{k}|/n = \text{Const.}$$

5. The thin lens equation (pay attention to signs):
$$1/a + 1/b = 1/f \equiv D.$$

6. Newton's eq. (x_1, x_2 — distances of the image and the object from the focal planes): $x_1x_2 = f^2$.

7. Parallax method for finding the position of an image: find such a pos. for a pencil's tip that it wouldn't shift with resp. to the image when moving perpendicularly the position of your eye.

8. Geometrical constructions for finding the paths of light rays through lenses:
a) ray passing the lens center does not refract;
b) ray $\parallel$ to the optical axis passes through the focus;
c) after refr., initially $\parallel$ rays meet at the focal plane;
d) image of a plane is a plane; these two planes meet at the plane of the lens.

9. Luminous flux Φ [unit: lumen (lm)] measures the energy of light (emitted, passing a contour, etc), weighted according to the sensitivity of an eye. Luminous intensity [candela (cd)] is the luminous flux (emitted by a source) per solid angle: $I = \Phi/\Omega$. Illuminance [lux (lx)] is the luminous flux (falling onto a surface) per unit area: $E = \Phi/S$.

10. Gauss theorem for luminous flux: the flux through a closed surface surrounding the point sources of intensity I_i is $\Phi = 4\pi \sum I_i$; single-source-case: at a distance r, $E = I/r^2$.

11. An experimental hint: if a grease stain on a paper is as bright as the surrounding paper, then the paper is equally illuminated from both sides.

VII Wave optics

1. Diffraction — method based on Huygens' principle: if obstacles cut the wavefront into fragment(s), the wavefront can be divided into small pieces which serve as imaginary point-like light sources; the wave amplitude at the observ. site will be the sum over the contributions of these sources.

2. Two slit interference (the slit width $d \ll a, \lambda$): angles of maxima $\varphi_{\max} = \arcsin(n\lambda/a)$, $n \in Z$; $I \propto \cos^2(k\frac{a}{2}\sin\varphi)$, where $k = 2\pi/\lambda$.

3. Single slit: angles of *minima* $\varphi_{\min} = \arcsin(n\lambda/d)$, $n \in Z$, $n \neq 0$. NB! the central maximum is double-wide. $I \propto \sin^2(k\frac{d}{2}\sin\varphi)/\varphi$.

4. Diffraction grating: the main maxima are the same as in pt. 2, the width of the main maxima — the same as for pt. 3 with d being the net grating length. Spectral resolving power $\frac{\lambda}{\Delta\lambda} = nN$, where n is the order number of the main max. and N — the number of slits.

5. Resolving power of a spectral device: $\frac{\lambda}{\Delta\lambda} = \frac{L}{\lambda}$, where L is the optical path difference between the shortest and longest beams.

6. Resolving power of a prism: $\frac{\lambda}{\Delta\lambda} = a\frac{dn}{d\lambda}$.

7. Angular distance when two pts. are resolved in an ideal telescope (lens): $\varphi = 1.22\lambda/d$. For that angle, the center of one point falls onto the first diffr. min. of the other point.

8. Bragg theory: a set of $\parallel$ ion planes of a crystal reflects X-rays if $2a\sin\alpha = k\lambda$; a — distance between neighb. planes, α — glancing angle.

9. Reflection from optically denser dielectric media: phase shift π. For semi-transparent thin films, $\phi_\rightarrow + \phi_\leftarrow = \pi$; $\phi_\rightarrow$ and $\phi_\leftarrow$ — phase shifts between reflected and trasmitted waves (arrows denote the direction of incidence).

10. Fabry-Pérot interferometer: two $\parallel$ semi-transp. mirrors with large reflectivity r ($1-r \ll 1$). Resolving power $\frac{\nu}{\Delta\nu} \approx \frac{2a}{\lambda(1-r)}$. Transmission spectrum can be found by introducing 5 plane waves (for left- and rightwards-propagating waves before the device, in the dev. and after the dev.) and tailoring these at the region boundaries.

11. Coherent electromagnetic waves: electric fields are added; vector diagram can be used, angle between vectors is the phase shift; NB! dispersion: $n = n(\omega) = \sqrt{\varepsilon(\omega)}$ (usually $\mu \cong 1$). Energy flux density (en. per unit area and time): $I = cn\varepsilon_0 E^2 = \frac{c}{n\mu_0}B^2$ (E, B — RMS values).

12. Malus' law: for linearly polarized light $I = I_0\cos^2\varphi$, where φ is the angle between the polarization planes.

13. $\lambda/4$-plate: phase shift $\pi/2$ between linearly polarized components.

14. Brewster's angle: reflected and refracted rays are $\perp$; reflected ray is completely polarized; incidence angle $\tan\varphi_B = n$.

15. Diffr. with optical elements: no need to calculate optical path lengths through lenses, prisms etc.: work simply with images. Particular conclusion: biprism gives the same diffr. as a double slit.

16*. Optical fibres: Mach-Zehnder interferometer is analogous to a double-slit diffraction; circular resonator — to Fabry-Pérot interferometer; Bragg filters work similarly to the X-ray case. Single-mode fibres: $\Delta n/n \approx \frac{1}{2}(\lambda/d)^2$.

VIII Circuits

1.
$$U = IR,\ P = UI$$
$$R_{\text{series}} = \sum R_i, \quad R_\parallel^{-1} = \sum R_i^{-1}$$

2. Kirchoff's laws:
$$\sum_{\text{node}} I = 0, \quad \sum_{\text{contour}} U = 0$$

3. To reduce the number of eqns. for pt 2: *method of node potentials; method of loop currents;* equivalent circuits (any 3-terminals $\Rightarrow \Delta$ or Y; 2-term. with emf $\Rightarrow r$ and $\mathcal{E}$ in series).

4. Resistance of infinite chain: use self-similarity; resistance between neighbour nodes of infinite grid: generalized method of electrical images.

5. AC: apply pts. 1–4 while substituting R with Z:
$$Z_R = R,\ Z_C = 1/i\omega C,\ Z_L = i\omega L;$$
$$\varphi = \arg Z,\ U_{\text{eff}} = |Z|I_{\text{eff}}$$
$$P = |U||I|\cos(\arg Z) = \sum I_i^2 R_i.$$

6. Characteristic times: $\tau_{RC} = RC$, $\tau_{LR} = L/R$, $\omega_{LC} = 1/\sqrt{LC}$. Relaxation to stationary current distribution exponential, $\propto e^{-t/\tau}$.

7. Energy conservation for electric circuits: $\Delta W + Q = Uq$, where q is charge which has crossed a potential drop U; work of emf is $A = \mathcal{E}q$.

8. $W_C = CU^2/2$, $W_L = LI^2/2$.

9. $\mathcal{E} = -d\Phi/dt = -d(LI)/dt$, $\Phi = BS$.

10. Nonlinear elements: graphical method — find the solution in U-I coordinates as an intersection point of a nonlinear curve and a line representing Ohm/Kirchoff laws. In case of many intersection points study stability — some solutions are usually unstable.

11. Make use of short- and long-time limits. For $t_{\text{observation}} \gg \tau_{RC}$ or τ_{LR}, quasiequilibrium is reached: $I_C \approx 0$ (wire is "broken" near C) and $\mathcal{E}_L \approx 0$ (L is effectively short-circuited). For $t_{\text{observation}} \ll \tau_{RC}$ or τ_{LR}, the charge leakage of C and current drop in L are small, $\Delta Q \ll Q$ and $\Delta I \ll I$: C is "short-circuited" and L is "broken".

12. If $L \neq 0$, then $I(t)$ is a continuous function.

13. Through a superconducting contour, magnetic flux $\Phi = \text{Const}$. In particular, with no external B, $LI = \text{Const}$.

14. Mutual inductance: magnetic flux through a contour $\Phi_1 = L_1 I_1 + L_{12} I_2$ (I_2 — current in a second contour). Theorems: $L_{12} = L_{21} \equiv M$; $M \leq \sqrt{L_1 L_2}$.

IX Electromagnetism

1. $F = kq_1q_2/r^2$, $\Pi = kq_1q_2/r$ — Kepler's laws are applicable (Ch. XII).

2. Gauss's law: $\oint \vec{B}d\vec{S} = 0$,
$$\oint \varepsilon\varepsilon_0 \vec{E}d\vec{S} = Q, \ \oint \vec{g}d\vec{S} = -4\pi GM.$$

3. Circulation theorem
$$\oint \vec{E}d\vec{l} = 0\ (= \dot{\Phi}), \ \oint \frac{\vec{B}d\vec{l}}{\mu\mu_0} = I, \oint \vec{g}d\vec{l} = 0.$$

4. Magnetic field caused by current element:
$$\mathrm{d}\vec{B} = \frac{\mu\mu_0 I}{4\pi}\frac{\mathrm{d}\vec{l} \times \vec{e}_r}{r^2};$$
hence, at the center of circular I: $B = \frac{\mu_0 I}{2r}$

5. $\vec{F} = e(\vec{v} \times \vec{B} + \vec{E})$, $\vec{F} = \vec{I} \times \vec{B}l$.

6. From the Gauss's and circulation laws: charged wire: $E = \frac{\sigma}{2\pi\varepsilon_0 r}$, DC: $B = \frac{I\mu_0}{2\pi r}$; charged surface $E = \frac{\sigma}{2\varepsilon_0}$, current sheet $B = \frac{\mu_0 j}{2}$;
inside a sphere (or infinite cylindrical surface) of homogeneous surface charge $E = 0$, inside a cylindrical surface current $\parallel$ to the axes $B = 0$, inside a ball ($d = 3$), cylinder ($d = 2$) or layer ($d = 1$) of homogeneous ρ or $\vec{j}$:
$$\vec{E} = \frac{\rho}{d\varepsilon_0}\vec{r}; \quad \vec{B} = \frac{1}{d\mu_0}\vec{j} \times \vec{r}$$
($\vec{r}$ — radius vector from the centre).

7. Long solenoid: inside $B = In\mu\mu_0$, outside 0; flux $\Phi = NBS$ and (with $n = \frac{N}{l}$) inductance $L = \Phi/I = Vn^2\mu\mu_0$. Short solenoid $B_\parallel = \frac{In\mu\mu_0\Omega}{4\pi}$ (Ω — solid angle).

8. Measuring magnetic field with a small coil and ballistic galvanometer: $q = \int \frac{\mathcal{E}}{R}\mathrm{d}t = NS\Delta B/R$.

9. Potential energy of a system of charges:
$$\Pi = k\sum_{i>j}\frac{q_i q_j}{r_{ij}} = \frac{1}{2}\int \varphi(\vec{r})\mathrm{d}q, \ \mathrm{d}q = \rho(\vec{r})\mathrm{d}V.$$

10. Force between parts of a uniformly charged sphere or cylindrical surface: substitute force due to charges with force due to hydrostatic pressure.

11. If all the charges are at the distance R (eg. at the center of an inhomogeneously charged sphere or ring), $\varphi = kQ/r$.

12. To find the net charge (or potential) induced by external charges, use the superpos. pr.: "smear" the charges to make the problem symmetric.

13. Conductor shields charges and electric fields, eg. charge distribution inside a hollow sphere cannot be seen from outside (it seems as if there is a conducting ball carrying a total charge Q)

14. Capacitances: $C = \varepsilon\varepsilon_0 S/d$ (plane), $4\pi\varepsilon\varepsilon_0 r$(sphere), $2\pi\varepsilon\varepsilon_0 l(\ln R/r)^{-1}$ (coaxial).

15. Dipole moment:
$$\vec{d}_e = \sum q_i\vec{r}_i = \vec{l}q, \ \vec{d}_\mu = I\vec{S}.$$

16. Energy and torque of a dipole:
$$W = -\vec{d}\cdot\vec{E}\ (\vec{B}), \ \vec{M} = \vec{d} \times \vec{E}\ (\vec{B}).$$

17. Dipole field: $\varphi = k\vec{d}\cdot\vec{e}_r/r^2$; $E, B \propto r^{-3}$.

18. Forces acting on a dipole: $F = (\vec{E}\vec{d}_e)'$, $F = (\vec{B}\vec{d}_\mu)'$; interaction between 2 dipoles: $F \propto r^{-4}$.

19. Point charge as a magn. dipole: $d_\mu \propto \Phi \propto v_\perp^2/B$ is an adiab. inv (see IV-20).

20. Electric and magnetic images: grounded (superconducting for magnets) planes act as mirrors. Field of a grounded (or isolated) sphere can be found as a field of one (or two) fictive charge(s) inside the sphere. The field in a planar waveguide (slit between metallic plates) can be obtained as a superposition of electromagnetic plane waves.

21. Ball's (cylinder's) polarization in homogeneous (electric) field: superpos. of homogeneously charged ($+\rho$ and $-\rho$) balls (cylinders), $d \propto E$.

22. Eddy currents: power dissipation density $\sim B^2v^2/\rho$; momentum given during a single pass: $F\tau \sim B^2a^3d/\rho$ (where d — thickness; a — size).

23. Inside a superconductor and for fast processes inside a conductor $B = 0$ and thus $I = 0$ (current flows in surface layer — skin effect).

24. Charge in homog. magnetic field $\vec{B} = B\vec{e}_z$ moves along a cycloid with drift speed $v = E/B = F/eB$; generalized mom. is conserved

$$p'_x = mv_x - Byq,\ p'_y = mv_y + Bxq,$$

as well as gen. angular mom. $L' = L + \frac{1}{2}Bqr^2$.

25. MHD generator (a — length along the direction of $\vec{E}$):

$$\mathcal{E} = vBa,\ r = \rho a/bc.$$

26. Hysteresis: S-shaped curve (loop) in B-H-coordinates (for a coil with core also B-I-coord.): the loop area gives the thermal energy dissipation density per one cycle).

27. Fields in matter: $\vec{D} = \varepsilon\varepsilon_0\vec{E} = \varepsilon_0\vec{E} + \vec{P}$, where $\vec{P}$ is dielectric polarization vector (volume density of dipole moment); $\vec{H} = \vec{B}/\mu\mu_0 = \vec{B}/\mu_0 - \vec{J}$, where $\vec{J}$ is magnetization vector (volume density of magnetic moment).

28. In an interface between two substances E_t, D_n $(= \varepsilon E_t)$, H_t $(= B_t/\mu)$ and B_n are continuous.

29. Energy density: $W = \frac{1}{2}(\varepsilon\varepsilon_0E^2 + B^2/\mu\mu_0)$.

30. For $\mu \gg 1$, fieldlines of B are attracted to the ferromagnetic (acts as a potential hole, cf. pt. 28).

31. Current density $\vec{j} = ne\vec{v} = \sigma\vec{E} = \vec{E}/\rho$.

32. Lenz's law: system responds so as to oppose to changes.

X Thermodynamics

1. $pV = \frac{m}{\mu}RT$

2. Internal energy of one mole $U = \frac{i}{2}RT$.

3. Volume of one mole at standard cond. is 22,4 l.

4. Adiabatic processes: slow as compared to sound speed, no heat exchange: $pV^\gamma = \text{Const.}$ (and $TV^{\gamma-1} = \text{Const.}$).

5. $\gamma = c_p/c_v = (i+2)/i$.

6. Boltzmann's distribution:

$$\rho = \rho_0 e^{-\mu gh/RT} = \rho_0 e^{-U/kT}.$$

7. Maxwell's distribution (how many molecules have speed v) $\propto e^{-mv^2/2kT}$.

8. Atm. pressure: if $\Delta p \ll p$, then $\Delta p = \rho g\Delta h$.

9. $p = \frac{1}{3}mn\bar{v}^2 = nkT$, $\bar{v} = \sqrt{3kT/m}$, $\nu = vnS$.

10. Carnot's cycle: 2 adiabats, 2 isotherms. $\eta = (T_1 - T_2)/T_1$; derive using S-T-coordinates.

11. Heat pump, inverse Carnot: $\eta = \frac{T_1}{T_1 - T_2}$.

12. Entropy: $dS = dQ/T$.

13. I law of thermodynamics: $\delta U = \delta Q + \delta A$

14. II law of thermodynamics: $\Delta S \geq 0$ (and $\eta_{\text{real}} \leq \eta_{\text{Carnot}}$).

15. Gas work (look also p. 10)

$$A = \int p\mathrm{d}V, \quad \text{adiabatic:} \quad A = \frac{i}{2}\Delta(pV)$$

16. Dalton's law: $p = \sum p_i$.

17. Boiling: pressure of saturated vapour $p_v = p_0$; at the interface betw. 2 liquids: $p_{v1} + p_{v2} = p_0$.

18. Heat flux $P = kS\Delta T/l$ (k — thermal conductivity); analogy to DC circuits (P corresponds to I, ΔT to U, k to $1/\rho$).

19. Heat capacity: $Q = \int c(T)\mathrm{d}T$. Solids: for low temperatures, $c \propto T^3$; for high T, $c = 3Nk$, where N — number of ions in crystal lattice.

20. Surface tension:

$$U = S\sigma,\ F = l\sigma,\ p = 2\sigma/R.$$

21. Stefan-Boltzmann law (gray body): $P = \varepsilon\sigma T^4$.

22. Wien's law: $f_{\max} = Ak_BT/h$ $(A \approx 2.8)$, $\lambda_{\max} = hc/A'k_BT$ $(A' \approx 5)$

XI Quantum mechanics

1. $\vec{p} = \hbar\vec{k}$ $(|\vec{p}| = h/\lambda)$, $E = \hbar\omega = h\nu$.

2. Interference: as in wave optics.

3. Uncertainty (as a math. theorem):

$$\Delta p\Delta x \geq \frac{\hbar}{2},\ \ \Delta E\Delta t \geq \frac{\hbar}{2},\ \ \Delta\omega\Delta t \geq \frac{1}{2}.$$

For qualitative estimates by non-smooth shapes, h serves better ($\Delta p\Delta x \approx h$ etc).

4. Spectra: $h\nu = E_n - E_m$; width of spectral lines is related to lifetime: $\Gamma\tau \approx \hbar$.

5. Oscillator's (eg. molecule) en. levels (with eigenfrequency ν_0): $E_n = (n + \frac{1}{2})h\nu_0$. For many eigenfrequencies: $E = \sum_i hn_i\nu_i$.

6. Tunnelling effect: barrier Γ with width l is easily penetrable, if $\Gamma\tau \approx \hbar$, where $\tau = l/\sqrt{\Gamma/m}$.

7. Bohr's model: $E_n \propto -1/n^2$. In a (classically calculated) circular orbit, there is an integer number of wavelengths $\lambda = h/mv$.

8. Compton effect — if photon is scattered from an electron, photon's $\Delta\lambda = \lambda_C(1 - \cos\theta)$.

9. Photoeffect: $A + mv^2/2 = h\nu$ (A - work of exit for electrons). I-U-graph: photocurrent starts at the counter-voltage $U = -(h\nu - A)/e$, saturates for large forward voltages.

10. Stefan-Boltzmann: $P = \sigma T^4$.

XII Kepler laws

1. $F = GMm/r^2$, $\Pi = -GMm/r$.

2. Gravitational interaction of 2 point masses (Kepler's I law): trajectory of each of them is an ellipse, parabola or hyperbola, with a focus at the center of mass of the system. Derive from R.-L. v. (pt 9).

3. Kepler's II law (conserv. of angular mom.): for a point mass in a central force field, radius vector covers equal areas in equal times.

4. Kepler's III law: for two point masses at elliptic orbits in r^{-2}-force field, revolution periods relate as the longer semiaxes to the power of $\frac{3}{2}$:

$$T_1^2/T_2^2 = a_1^3/a_2^3.$$

5. Full energy $(K + \Pi)$ of a body in a gravity field:

$$E = -GMm/2a.$$

6. For small ellipticities $\varepsilon = d/a \ll 1$, trajectories can be considered as having a circular shapes, with shifted foci.

7. Properties of an ellipse: $l_1 + l_2 = 2a$ (l_1, l_2 — distances to the foci), $\alpha_1 = \alpha_2$ (light from one focus is reflected to the other), $S = \pi ab$.

8. A circle and an ellipse with a focus at the circle's center can touch each other only at the longer axis.

9*. Runge-Lenz vector (the ellipticity vector):

$$\vec{\varepsilon} = \frac{\vec{L} \times \vec{v}}{GMm} + \vec{e}_r = \text{Const.}$$

XIII Theory of relativity

1. Lorentz transforms (rotation of 4D space-time of Minkowski geometry), $\gamma = 1/\sqrt{1 - v^2/c^2}$:

$$x' = \gamma(x - vt),\ y' = y,\ t' = \gamma(t - vx/c^2)$$
$$p'_x = \gamma(p_x - mv),\ m' = \gamma(m - p_xv/c^2)$$

2. Length of 4-vector:

$$s^2 = c^2t^2 - x^2 - y^2 - z^2$$
$$m_0^2c^2 = m^2c^2 - p_x^2 - p_y^2 - p_z^2$$

3. Adding velocities:

$$w = (u + v)/(1 + uv/c^2).$$

4. Doppler effect:

$$\nu' = \nu_0\sqrt{(1 - v/c)/(1 + v/c)}.$$

5. Minkowski space can be made Euclidean if time is imaginary ($t \to ict$). Then, for rot. angle φ, $\tan\varphi = v/ic$. Express $\sin\varphi$, and $\cos\varphi$ via $\tan\varphi$, and apply the Euclidean geometry formulae.

6. Shortening of length: $l' = l_0/\gamma$.

7. Lengthening of time: $t' = t_0\gamma$.

8. Simultaneity is relative, $\Delta t = -\gamma v\Delta x/c^2$.

9. $\vec{F} = d\vec{p}/dt$ $[= \frac{d}{dt}(m\vec{v})$, where $m = m_0\gamma]$.

10. Ultrarelativistic approximation: $v \approx c$, $p \approx mc$, $\sqrt{1 - v^2/c^2} \approx \sqrt{2(1 - v/c)}$.

11*. Lorentz tr. for E-B: $\vec{B}'_{||} = \vec{B}_{||}$, $\vec{E}'_{||} = \vec{E}_{||}$,

$$\vec{E}'_\perp = \gamma(\vec{E}_\perp + \vec{v} \times \vec{B}_\perp),\ \ \vec{B}'_\perp = \gamma(\vec{B}_\perp - \vec{v} \times \frac{\vec{E}_\perp}{c^2}).$$

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* marks an advanced material.

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