PROBLEMS ON KINEMATICS

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1 INTRODUCTION

For a majority of physics problems, solving can be reduced to using a relatively small number of ideas (this also applies to other disciplines, e.g. mathematics). In order to become good at problem solving, one must learn these ideas. However, it is not enough if you only *know* the ideas: you also need to learn how to *recognize* which ideas are to be used for a given problem With experience it becomes clear that usually problems actually contain hints about which ideas need to be used.

This text attempts to summarise the main ideas encountered in solving kinematics problems (though, some of these ideas are more universal, and can be applied to some problems of other fields of physics). For each idea, there are one or several illustrative problems. First you should try to solve the problems while keeping in mind those ideas which are suggested for the given problem. If this turns out to be too difficult, you can look at the hints — for each problem, rather detailed hints are given in the respective section. It is intentional that there are no full solutions: just reading the solutions and agreeing to what is written is not the best way of polishing your problem solving skills. However, there is a section of answers — you can check if your results are correct. There are also revision problems for which there are no suggestions provided in the text: it is your task to figure out which ideas can be used (there are still hints).

Problems are classified as being simple, normal, and difficult (the problem numbers are coloured according to this colour code). Please keep in mind that difficulty levels are relative and individual categories: some problem marked as difficult may be simple for you, and vice versa. As a rule of thumb, a problem has been classified as a simple one if it makes use of only one idea (unless it is a really tricky idea), and a difficult one if the solution involves three or more ideas.

It is assumed that the reader is familiar with the concepts of speed, velocity and acceleration, radian as the measure for angles, angular speed and angular acceleration, trigonometric functions and quadratic equations. In few places, derivatives and differentials are used, so a basic understanding of these concepts is also advisable (however, one can skip the appropriate sections during the first reading).

2 VELOCITIES

idea 1: Choose the most appropriate frame of reference. You can choose several ones, and switch between them as needed. Potentially useful frames are where:

⋆ some bodies are at rest;
⋆ some projections of velocities vanish;
⋆ motion is symmetric.

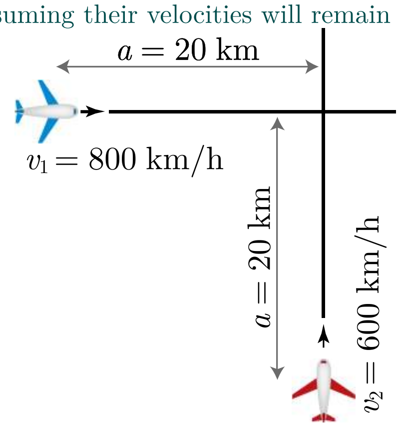
It is recommended to investigate process in all potentially useful frames of reference. As mentioned above, in a good frame

of reference, some velocity or its component (or acceleration or its component) vanishes or two velocities are equal. Once a suitable frame of reference has been found, we may change back into the laboratory frame and transform the now known velocities-accelerations using the rule of adding velocities (accelerations). NB! the accelerations can be added in the same way as velocities *only if* the frame's motion is translational (i.e. it does not rotate).

pr 1. On a river coast, there is a port; when a barge passed the port, a motor boat departed from the port to a village at the distance $s_1 = 15\,\mathrm{km}$ downstream. It reached its destination after $t = 45\,\mathrm{min}$, turned around, and started immediately moving back towards the starting point. At the distance $s_2 = 9\,\mathrm{km}$ from the village, it met the barge. What is the speed of the river water, and what is the speed of the boat with respect to the water? Note that the barge did not move with respect to the water.

Here, the motion takes place relative to the water, which gives us a hint: let us try solving the problem when using the water frame of reference. If we look at things closer, it becomes clear that this is, indeed, a good choice: in that frame, the speed of the boat is constant, and barge is at rest, i.e. the motion of the bodies is much simpler than in the coastal frame of reference.

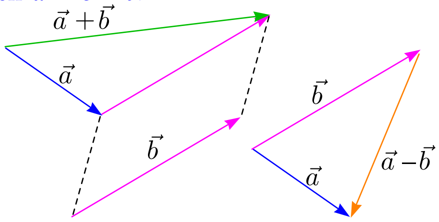
pr 2. Two planes fly at the same height with speeds $v_1 = 800\,\mathrm{km/h}$ and $v_2 = 600\,\mathrm{km/h}$, respectively. The planes approach each other; at a certain moment of time, the plane trajectories are perpendicular to each other and both planes are at the distance $a = 20\,\mathrm{km}$ from the intersection points of their trajectories. Find the minimal distance between the planes during their flight assuming their velocities will remain constant.



The idea 1 advises us that we should look for a frame where some bodies are at rest; that would be the frame of one of the planes. However, here we have a two-dimensional motion, so the velocities need to be added and subtracted vectorially.

def. 1: A scalar quantity is a quantity which can be fully described by a single numerical value only; a vector quantity is a quantity which needs to be described by a magnitude (also referred to as modulus or length), and a direction. The sum of two vectors $\vec{a}$ and $\vec{b}$ is defined so that if the vectors are interpreted as displacements (the modulus of a vector gives the distance, and its direction — the direction of the displacement) then the vector $\vec{a} + \vec{b}$ corresponds to the net displacement as a result of two sequentially performed displacements $\vec{a}$ and $\vec{b}$.

This corresponds to the triangle rule of addition, see figure. Subtraction is defined as the reverse operation of addition: if $\vec{a}+\vec{b}=\vec{c}$ then $\vec{a}=\vec{c}-\vec{b}$.



After having been introduced the concept of vectors, we can also fix our terminology.

def. 2: *Velocity* is a vectorial quantity which can be defined by the projections to the axes $\vec{v}=(v_x,v_y,v_z)$; *speed* is the modulus of a vector, $v=|\vec{v}|=\sqrt{v_x^2+v_y^2+v_z^2}$. Similarly, *displacement* is a vector pointing from the starting point of a body to its final position; travelled *distance* is the sum of the moduli of all the elementary displacements (the curve length).

For vectorial addition, there are two options. First, we can select two axes, for instance x and y, and work with the respective projections of the velocity vectors. So, if our frame moves with the velocity $\vec{u}$ and the velocity of a body in that frame is $\vec{v}$ then its velocity in the lab frame is $\vec{w}=\vec{u}+\vec{v}$, which can be found via projections $w_x=v_x+u_x$ and $w_y=v_y+u_y$. Alternatively, we can approach geometrically and apply the triangle rule of addition, see above.

Once we have chosen the reference frame of one of the planes, the problem 2 can be solved by using the following idea.

idea 2: For problems involving addition of vectors (velocities, forces), the problems can be often reduced to the application of simple geometrical facts, such as (a) the shortest path from a point to a line (or plane) is perpendicular to the line (plane); (b) among such triangles ABC which have two fixed side lengths $|BC|=a$ and $|AC|=b<a$, the triangle of largest $\angle ABC$ has $\angle BAC=90°$.

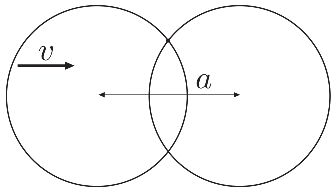
The next problem requires the application of several ideas and because of that, it is classified as a difficult problem. When switching between reference frames, the following ideas will be useful.

idea 3: Try to reveal hidden symmetries, and make the problem into a symmetric one.

idea 4: It is possible to figure out everything about a velocity or acceleration once we know one of its components and the direction of the vector.

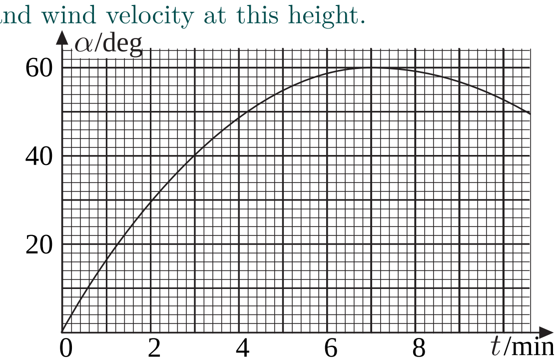
Mathematicians' way of stating it is that a right-angled triangle is determined by one angle and one of its sides. For example, if we know that velocity is at angle α to the horizontal and its horizontal component is w then its modulus is $w/\sin\alpha$.

pr 3. One of two rings with radius r is at rest and the other moves at velocity v towards the first one. Find how the velocity of the upper point of intersection depends on a, the distance between two rings' centres.



The idea 4 can be used again in the following problem:

pr 4. Balloons with constant ascending velocity can be used to investigate wind velocities at various heights. The given graph of elevation angle against time was obtained by observing a such balloon. The balloon was released at distance $L=1\,\text{km}$ from the point of observation and it seemed to be rising directly upwards. Knowing that wind velocity near the ground was zero, find the balloon's height at time $t=7\,\text{min}$ after its start and wind velocity at this height.



In order to answer to the first question here, we need also the following idea.

idea 5: If a graph of y versus x is given, quite often some tangent line and its slope $\frac{dy}{dx}$ turn out to be useful. In such cases, unless it is obvious, you have to show that the derivative $\frac{dy}{dx}$ is related to a physical quantity z relevant to the solution of the problem. To this end, you need to express z in terms of small (infinitesimal) increments dx and dy, and manipulate mathematically until these increments enter the expression only via the ratio $\frac{dy}{dx}$.

To be mathematically correct, there are two options. First, in simpler cases, you can say that the increments are infinitely small (infinitesimal), and denote these via *differentials* dx and dy. In more complex cases it may be more convenient to start with small but finite increments Δx and Δy, make your calculations while keeping only the leading terms (e.g. for $\Delta x+\Delta x\cdot\Delta y$, the second term is a product of two small quantities and can be neglected as compared with the first one), and finally go to the limit of infinitely small increments, $\Delta x\to dx$, $\Delta y\to dy$.

In order to answer the second question, we need one more idea.

idea 6: There are calculations which cannot be done in a generic case, but are relatively easy for certain special values of the parameters. If some unusual coincidence stands out in the problem (in this case the slope of the tangent is zero at the given time) then it is highly probable that this circumstance has to be used.

idea 7: If friction affects the motion then usually the most appropriate frame of reference is that of the environment causing the friction.

pr 5. A white piece of chalk is thrown onto a black horizontal board moving at constant velocity. Initially, the chalk's velocity was perpendicular to the board's direction of motion. What is the shape of the chalk's trace on the board?

To solve the next problem, in addition to the previous idea we also need to use 2, which can be rephrased in a slightly more general (but less specific) way: some minima and maxima can be found without taking any derivatives, in fact the solution without a derivative can turn out to be much simpler. For this problem, an even more narrowed down formulation would be the following.

idea 8: If one of two vectors is constant and the direction of the other is fixed then the modulus of their sum is minimal if they form a right-angled triangle.

pr 6. A block is pushed onto a conveyor belt. The belt is moving at velocity $v_0 = 1\,\mathrm{m/s}$, the block's initial velocity $u_0 = 2\,\mathrm{m/s}$ is perpendicular to the belt's velocity. During its subsequent motion, what is the minimum velocity of the block with respect to the ground? The coefficient of friction is large enough to prevent the block from falling off the belt.

The next problem is slightly unusual, specific comments will be given after the problem. To tackle such situation one can give seemingly trivial but very often an overlooked advice.

idea 9: Read carefully the problem text, try to understand the meaning of every statement, don't make hasty assumptions by yourself.

For a well-written problem, there are no redundant sentences. Things become more troublesome if that is not the case. Sometimes the problem author wants to educate you more than just by giving you the very problem, and tells you many things (such as historical background) which are definitely interesting but unrelated to the solution of the problem. It is OK if you are solving the problem as an exercise at home and you have plenty of time. However, you need to develop skills of parsing fast through such paragraphs at competitions under time pressure: you need to make sure that there are really no important hints hidden inside.

pr 7. After being kicked by a footballer, a ball started to fly straight towards the goal at velocity $v = 25\,\mathrm{m/s}$ making an angle $\alpha = \arccos 0.8$ with the horizontal. Due to side wind blowing at $u = 10\,\mathrm{m/s}$ perpendicular the initial velocity of the ball, the ball had deviated from its initial course by $s = 2\,\mathrm{m}$ by the time it reached the plane of the goal. Find the time that it took the ball to reach the plane of the goal, if the goal was situated at distance $L = 32\,\mathrm{m}$ from the footballer.

A typical problem gives all the parameter values describing a system and then asks about its behaviour. Here, the system might seem to be over-described: why do we need the value of s, couldn't we just use the initial velocity to determine the flight time to deduce $t = \frac{L}{v\cos\alpha}$? Such a question might arise, first of all, because you are used to ignoring air friction. However, no-one mentioned that you can neglect it here! Furthermore, it is even evident that the air drag cannot be neglected, because otherwise the ball would not depart from its free-fall trajectory! It would be a very difficult task (requiring a numerical integration of a differential equation) to estimate the trajectory of the ball subject to a turbulent air drag. However, this is not what you need to do, because the air drag is not described by a formula for the drag force, but instead, by the final departure from the corresponding free-fall-trajectory.

So, with the help of idea 9 we conclude that the air drag cannot be neglected here. Once we have understood that, it becomes evident that we need to apply the idea 7. However, even when equipped with this knowledge, you might run into mathematical difficulties as there is no direct way of expressing the flight time t in terms of the given quantities. Instead, you are advised to write down an equation containing t as an unknown, and then to solve it.

idea 10: It is often useful first to write down an equation (or a system of equations) containing the required quantity as an unknown, instead of trying to express it directly (sometimes it is necessary to include additional unknowns that later get eliminated).

Furthermore, unlike the problems we had thus far, this problem deals with a 3-dimensional geometry, which makes it difficult to draw sketches on a 2-dimensional sheet of paper. Thus we need one more simple idea.

idea 11: It is difficult to analyse three-dimensional motion as a whole, so whenever possible, it should be reduced to two dimensions (projecting on a plane, looking at planes of intersection).

The next problem illustrates

idea 12: An elastic collision is analysed most conveniently in the centre of mass frame of the process.

Let us derive from this idea a ready-to-use recipe when a ball collides with a moving wall. First, since the wall is heavy, the system's centre of mass coincides with that of the wall, hence we'll use the wall's frame. In the frame of the centre of mass, if the collision is elastic and there is no friction then due to the energy and momentum conservation, the bodies will depart with the same speed as they approached, i.e. the normal component of the ball's velocity is reversed. If we apply the addition of velocities twice (when we move to the wall's frame, and when we switch back to the lab frame), we arrive at the following conclusion.

idea 13: For an elastic bouncing of a ball from a wall which moves with a velocity $\vec{u}$ in the direction of the surface normal, the normal component $\vec{v}_n$ of the ball's velocity $\vec{v}$ is increased by $2\vec{u}$, i.e. $\vec{v}_n{}' = -\vec{v}_n + 2\vec{u}$.

For this problem we must also remember

fact 1: Angle between velocity vectors depends on the frame of reference!

pr 8. A tennis ball falls at velocity v onto a heavy racket and bounces back elastically. What does the racket's velocity u have to be to make the ball bounce back at a right angle to its initial trajectory and not start spinning if it did not spin before the bounce? What is the angle β between $\vec{u}$ and the normal of the racket's plane, if the corresponding angle for $\vec{v}$ is

α?

If we keep in mind the idea 9 and read the text carefully, we notice that the racket is *heavy* so that we can use the idea 13. Also, pay attention that the ball will not rotate after the collision — this is important for finding the parallel (to the racket's plane) component of the velocity.

Earlier we mentioned that vectors can be dealt with either geometrically (e.g. by applying the triangle rule for a sum of vectors and solving a trigonometrical problem), or algebraically using projections. Quite often, geometrical approach provides shorter solutions, but not always; this observation leads us to the following recommendation.

idea 14: For vectorial calculations, prefer geometrical approach, but if it seems unreasonable (e.g. some of the conditions are formulated through the projections of the vectors) switch to the algebraic approach and write expressions down in terms of components.

For the algebraic approach, **optimal choice of axes** is very important. "Optimal" means that the conditions are written in the simplest possible way. Sometimes it may happen that the most useful coordinate axes are not even at right angles.

For the problem 8, geometrical solution turns out to be simpler, but more difficult to come up with. This is quite typical: algebraic approach leads to a brute-force-solution when it is clear from the beginning what you need to do, but the calculations are mathematically long. Still, there are no fundamental difficulties and you just need to execute it. As long as the mathematical part will not be *unreasonably long* or leading to fundamental difficulties (such as unsolvable equations), brute force approach is still OK: figuring out an elegant solution can also take some time.

Typically, the geometrical solutions of physics problems represent very simple geometrical tasks and hence, finding these shorter-than-algebraic solutions is also quite easy. In this case, however, the geometrical task turns out to be quite a tricky problem. While the idea 14 suggests that the algebraic approach is good for problem 8 (the no-rotation-requirement gives us a condition for the parallel component of the velocity), it is recommended that you try both methods here. In both cases you need one more mathematical idea.

idea 15: Two vectors $\vec{a} = (a_x, a_y, a_z)$ and $\vec{b} = (b_x, b_y, b_z)$ are perpendicular if their scalar product is zero, $a_xb_x + a_yb_y + a_zb_z = 0$. (This assumes that the axes x, y and z are perpendicular to each other.)

idea 16: For trigonometric problems involving right triangles keep in mind that the circumcentre of a right triangle is at the centre of the hypotenuse, hence the median drawn from the right angle divides the triangle into two isosceles triangles, and the right angle into the angles equal to the acute angles of the triangle.

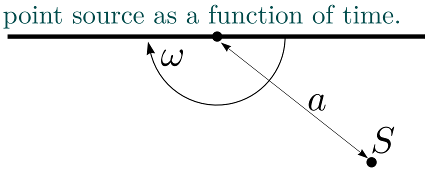
Idea 1 told us to make use of switching between different frames of reference. This idea can be also used when dealing with rotational motion.

def. 3: Angular velocity $\vec{\omega}$ equals by modulus to the rotation angle (in radians) per unit time, and is parallel to the rotation axis, the direction being given by the screw rule (if the screw is rotated in the same way as the body, the vector points in the direction of the screw movement).

idea 17: When switching between rotating frames of reference, angular velocities are to be added in the same as translational velocities in the case of translationally moving frames of reference. NB! This remains valid even if the angular velocities are not parallel (although non-small rotation angles can be added only as long as the rotation axis remains unchanged).

This idea is illustrated by a relatively simple problem below.

pr 9. Vertical mirror with two reflecting surfaces (front and back) rotates around a vertical axis as shown in figure, with angular speed ω. There is an unmoving point source of light S at a distance a from the rotation axis. Find the speed of the image of the point source as a function of time.



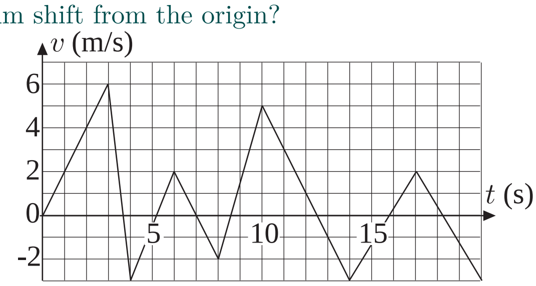
3 ACCELERATIONS, DISPLACEMENTS

Thus far we dealt with instantaneous or constant velocities, and in few cases we applied a simple formula $s = vt$ for displacements. In general, when the velocity $\vec{v}$ is not constant, the displacement is found as the curve under the graph of the velocity as a function of time. For instance, the displacement along x-coordinate Δx is surface area under the graph $v_x = v_x(t)$; mathematically we can write it via integral $\Delta x = \int v_x(t)dt$. You don't need to know more about integrals right now, just that it represents surface areas under graphs.

idea 18: Calculation of many physical quantities can be reduced (sometimes not in an obvious way) to the calculation of surface areas under a graph (i.e. to an integral). In particular: distance is the area under a $v-t$ curve (velocity-time), velocity the area below an $a-t$ curve etc.

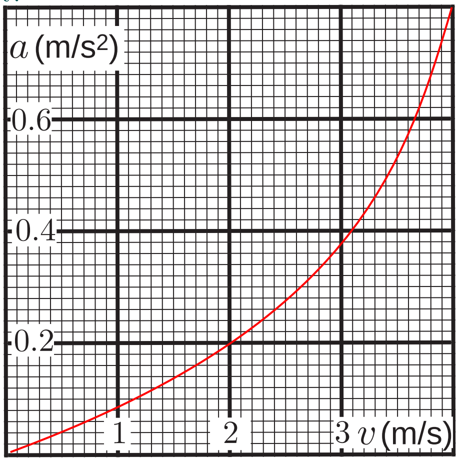
Note that drawing a graph is not always absolutely necessary (if you are skilled with integrals, formulae can be derived analytically, without drawing graphs), but doing it helps to imagine the process. Visualisation of this kind is always beneficial, it simplifies finding the solution and reduces the chances of making mistakes.

pr 10. A particle starts from the origin of coordinates; the figure shows its velocity as a function of time. What is its maximum shift from the origin?

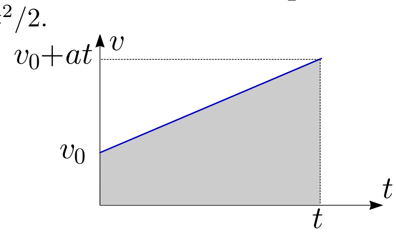


The next problem is much more difficult, although it is also reduced to finding a surface area; due to difficulty, the full solution (apart from replotting the graph and numerical calculations) is given under "hints".

pr 11. The acceleration of a boat depends on its speed as shown in graph. The boat is given initial speed $v_0 = 4\,\mathrm{m/s}$. What is the total distance travelled until the boat will almost come to rest?



The idea 18 can be also used to derive basic formulae for displacement in the case of a motion with constant acceleration. Suppose that a body has initial speed v_0 and moves with constant acceleration a; we want to know to which distance it travels by the moment of time t. The surface area under the graph is a right trapezoid (see figure), with surface area equal to the product of the median $v_0 + \frac{1}{2}at$, and the height t, i.e. $s = v_0t + at^2/2$.



Alternatively, if we are given the initial and final velocities (v_0 and v_1) instead of the travel time, the median of the trapezoid is expressed as $\frac{1}{2}(v_0 + v_1)$, and the height as $t = (v_1 - v_0)/a$. This leads us to $s = (v_0 + v_1)(v_1 - v_0)/2a = (v_1^2 - v_0^2)/2a$. If we rewrite it as $as = \frac{1}{2}(v_1^2 - v_0^2)$, we can call it the energy conservation law for unit mass if the free fall acceleration is a.

fact 2: If a body moves with initial speed v_0, final speed v_1 and constant acceleration a during time t, the distance travelled

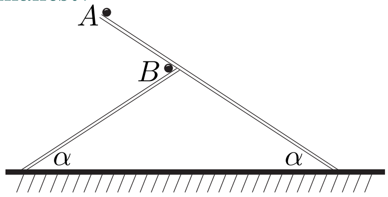
$$s = v_0t + \frac{1}{2}at^2 = \frac{v_1^2 - v_0^2}{2a}.$$

The next problem can be solved in various ways, but the simplest solution involves the following idea.

idea 19: Sometimes it is useful to change into a non-inertial frame of reference: velocities are added just in the usual way, $\vec{v}_{\mathrm{lab}} = \vec{v}_{\mathrm{rel}} + \vec{v}_{\mathrm{fr}}$, where $\vec{v}_{\mathrm{lab}}$ is the velocity in the lab frame, $\vec{v}_{\mathrm{rel}}$ — velocity in the moving frame, and $\vec{v}_{\mathrm{fr}}$ — the speed of that point of the moving frame where the body is at the given moment. If the frame moves translationally (without rotations) then the accelerations can be added in the same way, $\vec{a}_{\mathrm{lab}} = \vec{a}_{\mathrm{rel}} + \vec{a}_{\mathrm{fr}}$[1]. In particular, if a problem involves two or more free-falling bodies then using a free-falling frame simplifies calculations significantly.

It should be emphasized that if the frame rotates, the formula for acceleration obtains additional terms.

pr 12. Two smooth slides lie within the same vertical plane and make angles α to the horizontal (see the figure). At some moment, two small balls are released from points A and B and they start sliding down. It took time t_1 for the first ball that started from point A to reach the ground; for the second one the time of descent was t_2. At what time was distance between the balls the smallest?



idea 20: Sometimes, it is possible to separate two- or three-dimensional motion of a body into independent motions in perpendicular directions: (a) motion along x is independent from the motion along y for 2D geometry; (b) motion along x is independent from the motion along y, which is independent from the motion along z; (c) motion along x is independent from the motion in $y-z$-plane. In particular, this can be done for frictionless collisions from a plane[2]: if the axis x lies in the plane, and y is perpendicular to it, you can study separately motion along x and motion along y.

The simplest application of this idea is provided by a two-dimensional motion of a body in an homogeneous gravity field, which is studied in every textbook on kinematics: horizontal and vertical motions are decoupled, because vertical acceleration g does not depend on the horizontal coordinate x and horizontal velocity v_x, and body moves with a constant speed, and horizontal acceleration (0) does not depend on the vertical coordinate x and vertical velocity v_y. As a result, we obtain $x = v_{0x}t$ and $y = v_{0y}t - gt^2/2$ (where v_{0x} and v_{0y} are the respective initial velocity components); the respective trajectory in $x-y$-plane is a parabola which we obtain if we eliminate t from the second equation by substituting $t = x/v_{0x}$.

fact 3: Free fall trajectory of the centre of mass a body in homogeneous gravity field g is a parabola, parametrically given as $x = v_{0x}t$ and $y = v_{0y}t - gt^2/2$.

Let us discuss in more details how to apply the idea 20 to frictionless interactions (collisions or sliding) of a body with a plane. If the plane is inclined, we need to take the axes to be inclined as well; then, the gravitational acceleration will have a non-zero component along both axes, i.e. motion will have acceleration in both directions.

fact 4: Free fall problems can be also analysed when using inclined system of axis (this might be useful because of idea 20); then, the free fall acceleration is decomposed into two re-

[1] This can be easily seen if we relate the respective radius vectors as $\vec{r}_{\mathrm{lab}} = \vec{r}_{\mathrm{rel}} + \vec{r}_{\mathrm{fr}}$ and take twice the time derivative: the derivative of a sum is the sum of derivatives, even if we deal with vectors (this can be understood if we work with projections, e.g. the addition rule for the x-components of the accelerations can be obtained by taking twice the time derivative of the equality relating the x-coordinates, $x_{\mathrm{lab}} = x_{\mathrm{rel}} + x_{\mathrm{fr}}$).

[2] As long as there is no other mechanism (such as the Lorenz force) which couples the motions in different directions

spective perpendicular components, $\vec{g} = \vec{g}_x + \vec{g}_y$ with $g_x = \sin\alpha$ and $g_y = \cos\alpha$, α being the angle between the surface and the horizon.

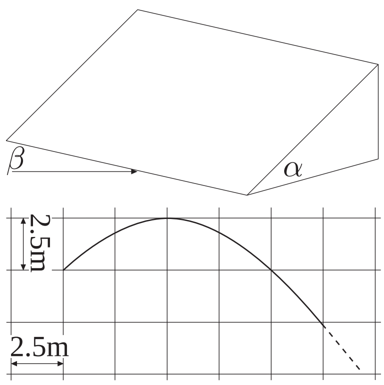
During a collision, as there is no friction force, v_x (parallel to the surface velocity component) does not change, i.e. it does not "notice" that there was a collision: in order to analyse the evolution of the x-coordinate, we can completely forget about the changes of the y-coordinate (and vice versa).

If the surface is curved, generally such a separation is no longer possible. Indeed, previously x was independent of y because the dependence of the acceleration on the coordinates is introduced by the normal force, which is a function of y only, and has no x-component. If the surface is curved, it is impossible to have the x-axis to be everywhere parallel to the surface: the acceleration due to the normal force has both non-vanishing x and y components, and depends both on x and y coordinates. However, in the case of side surfaces of cylinders, prisms and other generalized cylinders[3], it is still possible to find one axis x which is everywhere parallel to the surface and hence, motion along x can be separated from the motion in $y-z$-plane.

pr 13. An elastic ball is released above an inclined plane (inclination angle α) at distance d from the plane. What is the distance between the first bouncing point and the second? Collisions occur without friction.

The next problem makes also use of the idea 20; however, one more idea is needed, see below.

pr 14. A puck slides onto an icy inclined plane with inclination angle α. The angle between the plane's edge and the puck's initial velocity $v_0 = 10\,\mathrm{m/s}$ is $\beta = 60°$. The trace left by the puck on the plane is given in the figure (this is only a part of the trajectory). Find α under the assumption that friction can be neglected and that transition onto the slope was smooth.



The last sentence here is very important: if the transition is *sharp*, the puck approaches the inclined plane by sliding along the horizontal and collides with it — either elastically in which case it jumps up, or plastically. In particular, if the collision is perfectly plastic then that part of the kinetic energy which is associated with the motion along the surface normal of the inclined plane is lost. More specifically, if we introduce perpendicular coordinates so that the x-axis is along the contact line of the two surfaces and y-axis lays on the inclined surface, x, y, and z-motions are all separated; at the impact, v_z goes to zero, and due to the absence of friction, v_x and v_y are preserved.

In this problem, however, the transition from one surface to the other is smooth: around the line separating the two flat surfaces, there is a narrow region where the surface has a curvature. Within this narrow region, the motion in y- and z-directions cannot be separated from each other, and we need one more idea.

idea 21: If a force is perpendicular to the direction of motion (normal force when sliding along a curved surface, tension in a rope when a moving body is attached to an unstretchable rope fixed at the other end, force on a charge in magnetic field) then the velocity vector can only turn, its modulus will not change.[4]

pr 15. Three turtles are initially situated in the corners of an equilateral triangle at distances 1 m from one another. They move at constant velocity 10 cm/s in such a way that the first always heading towards the second, the second towards the third and the third towards the first. After what time will they meet?

Two approaches are possible here: first, we may go into the frame of reference rotating with the turtles, in which case we apply the following idea.

idea 22: Sometimes even a reference frame undergoing very complex motion can be useful.

Alternatively, we can use

idea 23: Instead of calculating physical velocities, it is sometimes wise to look at the rate of change of some distance, the ratio of two lengths, etc.

The following problem requires integration[5], so it can be skipped by those who are not familiar with it.

pr 16. An ant is moving along a rubber band at velocity $v = 1\,\mathrm{cm/s}$. One end of the rubber band (the one from which the ant started) is fixed to a wall, the other (initially at distance $L = 1\,\mathrm{m}$ from the wall) is pulled at $u = 1\,\mathrm{m/s}$. Will the ant reach the other end of the band? If yes then when will it happen?

Here we need to apply the

idea 24: For some problems, optimal choice of parametrization can simplify mathematical calculations significantly. An incomplete list of options: Cartesian, polar, cylindrical, and spherical coordinates; travel distance; *Lagrangian coordinates* (i.e. for fluids flow using the initial coordinate of a fluid particle instead of its current coordinate); relative position of a particle according to a certain ranking scheme, etc.

Here, the problem itself contains a hint about which type of parametrization is to be used. It is clear that the Cartesian coordinate of the ant is not good: it does not reflect the progress of the ant in advancing along the rubber band. In order to describe such a progress, we can use the relative position on the band: which fraction k of the rubber is left behind; the ant

[3] Surfaces with constant cross-sections.
[4] This is the energy conservation law using the fact that forces perpendicular to the velocity will not perform work.
[5] You may find helpful to know that $\int \frac{dx}{ax+b} = a^{-1}\ln(ax+b) + C$

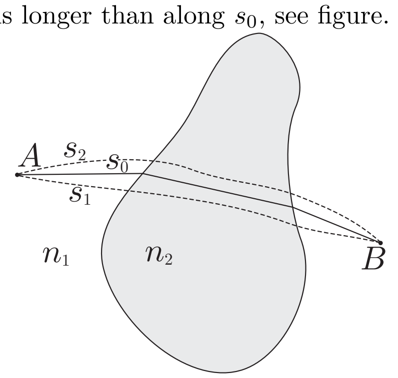
starts with $k = 0$, and $k = 1$ corresponds to the ant reaching the end of the band. The parameter k is essentially a Lagrangian coordinate: it equals to the initial coordinate of the current rubber point in the units of the initial rubber length.

4 OPTIMAL TRAJECTORIES

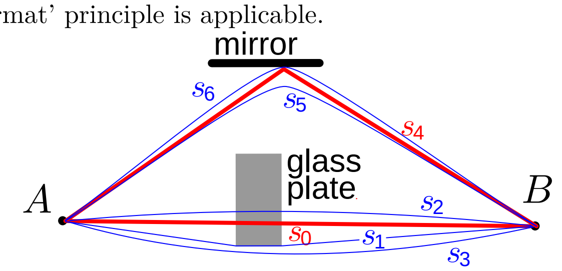
Majority of the kinematical optimal trajectory problems fall into two categories: the problems of finding the trajectories of shortest travel time, and the problems of finding the smallest initial speeds of a free-falling body.

idea 25: In those kinematics problems where velocities in various environments are given and the quickest way from point A to point B is asked, Fermat's principle (formulated for geometrical optics) can be of help.

Namely, if we have a configuration of bodies with different indices of refraction, and if a ray of light originating from point A passes through point B then the actual path of the ray is the quickest way for light to reach point B from point A (as a reminder, if the index of refraction of some environment is n then the travelling speed of light is c/n). Therefore time along path s_1 or s_2 is longer than along s_0, see figure.



We must clarify that the Fermat' principle applies to a local minimum: the travel time along the path s_0 needs to be shorter than for any other path which departs from the path s_0 but remains in its immediate neighbourhood. Furthermore, it is required that for small path variations, the travel time variations remain also small. The following figure clarifies in which cases the Fermat' principle is applicable.

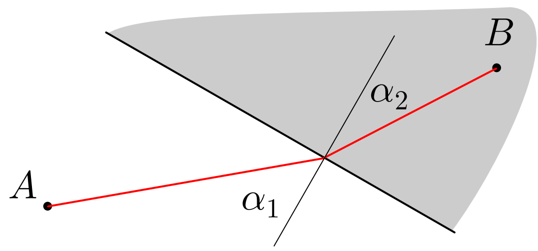


As the propagation speed in the glass plate is smaller than in the air, the global minimum of the travel time is achieved for the path s_1. However, such arbitrarily small variations of the path s_1 which go thorough the glass plate have a non-small change in the travel time, hence s_1 is not a valid light beam path (the path can be deformed downwards, e.g. into s_3, but upwards deformations incur a jump in travel time due to passing through the glass plate). Next, the path s_0 provides a good local minimum: the travel time along s_0 is smaller than along any small variation of the path s_0, and if the path variation is small, the time variation remains also small. Hence, the Fermat' principle can be applied: the path s_0 provides a valid light beam path. Finally, in the case of reflections, we need to compare only those paths which include similar reflections. So, the path s_5 is faster than the path s_4, but the former does not involve reflections and cannot be included into the set of reference paths. Among those paths which include one reflection from the mirror and represent a small variation of the path s_4 (such as the path s_6), the path s_4 is the fastest one and hence, represents a valid light beam path.

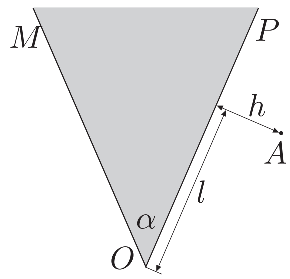
If light can travel from one point to the other along several different paths (e.g. from some point through a lens to the optical image of that point) then time along all these paths is exactly the same.

In order to apply the idea 25 to kinematics problems, we often need the Snell's law.

fact 5: Let a point A be situated in a medium where the light propagation speed is v_1, and point B — in a medium where the speed is v_2. Then, the light propagates from A to B according to the Snell's law: it refracts at the interface so that the angle between the surface normal and the path forms angles α_1 and α_2 (see figure) satisfying equality $\sin\alpha_1/\sin\alpha_2 = v_1/v_2$.



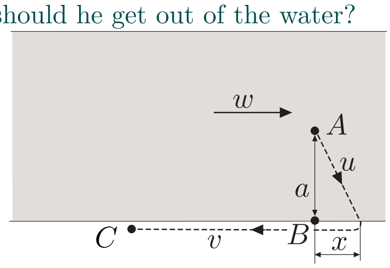
pr 17. A boy lives on the shore OP of a bay MOP (see the figure). Two shores of the bay make an angle α. The boy's house is situated at point A at distance h from the shore and $\sqrt{h^2+l^2}$ from point O. The boy wants to go fishing to the shore OM. At what distance x from point O should be the fishing spot, so that it would take as little time as possible to get there from the house? How long is this time? The boy moves at velocity v on the ground and at velocity $u < v$ when using a boat.



Here we can use a small addition to the last idea: if the quickest way to a plane (in a 3-d problem) or to a line (in 2-d) is asked then this plane or line can be substituted with a point very far (at infinity) in the perpendicular direction to it. The reason for that is quite simple: it takes the same amount of time to reach any point on the plane (line) from that very-very distant point. If we think about this in terms of geometrical optics then it means that a set of light rays normal to the surface falls onto the plane (line).

pr 18. A boy is situated at point A in a river, at distance a from the riverbank. He can swim at speed u or run at speed

$v > u$ on the shore; water flows in the river at velocity $w > u$. The boy wants to reach the point C upstream on the riverbank with minimal time. At what distance x from point B aligned with point A should he get out of the water?



Here we have two options: first, to use a brute force approach and express the travel time t as a function of x, and then equate $\frac{dt}{dx} = 0$. The second option is to apply the methods of geometrical optics. However, notice that in the lab frame, the speed in water depends on the direction of swimming, and in the water's frame, the starting and destination points are moving.

warning: Fermat's principle can be applied only if velocities are the same in all directions and initial and final points are at rest.

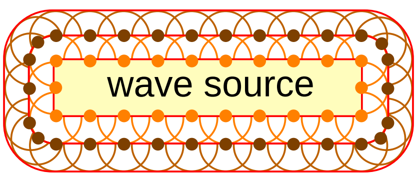
Now we have two sub-options. First, we can try to modify the problem so that while the answer remains the same, the Fermat's principle becomes applicable; try to do this. The second option is to use the Huygens' method of building wavefronts; let us consider this approach in more details

idea 26: When studying a reversible process, sometimes it is easier to analyse the reverse process.

Notice that in the case of the problem 18, the process can be reversed: if we make all the velocities opposite then the river flows from right to left, the boy starts running from point C, and wants to reach the point A in the river as fast as possible. Obviously, if a certain forward-process-trajectory is the fastest among all the alternatives then the same applies to the respective reverse process.

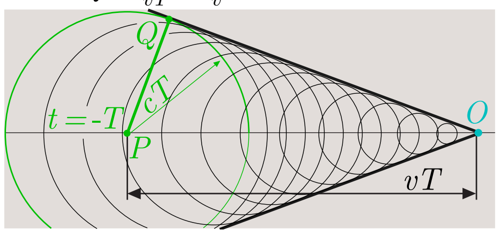
idea 27: For the fastest path problems in kinematics, the approach based on the Huygens principle can be used.

For wave propagation problems, according to the Huygens principle, wave fronts can be constructed step-by-step, by putting a series of fictitious light sources at a previous wave front. Then, after a short time period t, around each fictitious light source a circular wavefront of radius ct is formed (where c stands for the speed of light); the overall new wavefront is the envelope of all the small wavefronts, see figure (orange dots are the first generation Huygens sources, and the orange circles are the respective wavefronts; red lines are the overall wavefronts, and brown colour corresponds to the second generation Huygens sources).

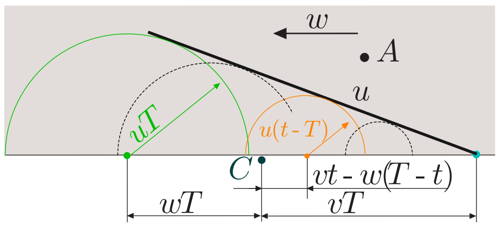


In the case of light waves, once we have the pattern of wavefronts, the rays of light can be found as such curves which are everywhere perpendicular to the wavefronts.

As a simple demonstration of how the Huygens principle can be applied for calculations, let us express the angle of the so-called *Mach cone* in terms of wave speed c and wave source speed u. If the wave source moves faster than the waves, it gives rise to what is known as the *Cherenkov radiation*, see below[6]. Let us consider a boat moving along a straight line and construct the Huygens wavefront as discussed earlier, see the figure below. We have drawn a series of circles corresponding to disturbances created my the moving source along its trajectory at a series of moments of time. The envelope of the wavefronts is a straight line, because the ratio of a circle's radius is proportional to the distance of its centre from the current position O of the wave source. The angle $\angle POQ$ is referred to as half of the *Mach cone angle* — it is called *cone* because in three-dimensional geometry, the circles become spheres and the envelope of the wave fronts becomes a cone. It is easy to see that $\sin \angle POQ = \frac{cT}{vT} = \frac{c}{v}$.



Continuing with the problem 18 (with the reversed velocities as discussed above), we need to build "wavefronts" as the sets of farthest points which the boy can reach for a given moment of time T once departing from point C and starting swimming at an arbitrary intermediate moment of time $t < T$. The construction of such a wavefront is depicted in the figure below.

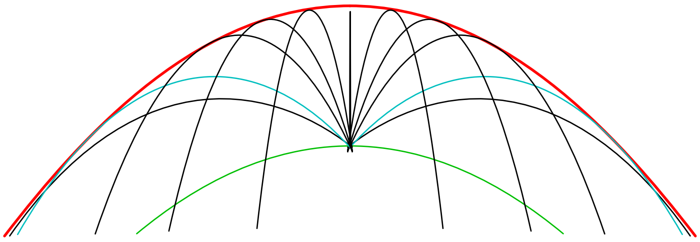


Here, the green circle corresponds to the set of farthest points the boy can reach if he starts swimming immediately, and the cyan dot depicts his position if he continues running along the coast; the bold black line shows the overall wavefront. If we let the wavefront evolve, it propagates towards the point A and reaches it at a certain moment T. Our procedure essentially tests all the swimming strategies and hence, T equals to the shortest travel time; what is left to do, is to trace back, which Huygens sources created that part of the wavefront which met the point A, i.e. what would be the optimal trajectory of the boy.

idea 28: Questions involving optimal ball-throwing can be often reduced to the *ballistic range problem*: a cannon can shoot projectiles with a fixed launching speed; in which range can the targets be hit? Therefore, it is useful to know the answer:

[6] Classically, Cherenkov radiation is used for radiation created by superluminal charges: in dielectric medium, the speed of light waves is reduced n times, where n is the coefficient of refraction, and relativistic particles can move faster than that; however, *sonic booms* (shock wave caused by a supersonic flight) and waves behind fast boats are caused by the same physical phenomenon.

the targets should be within a *paraboloidal region*, and the canon is *at its focus*; this paraboloid is the envelope of all the possible projectile trajectories, see figure (red curve is the envelope, the cyan curve — trajectories for 45°-launching angles, green curves — trajectories for 0°-launching angles).



In order to be able to use the full potential of the idea 28, the following simple facts need to be kept in mind.

fact 6: The black and cyan curves (in the figure above) represent the optimal trajectories for hitting targets at the red parabola: for these trajectories, the projectile's launching speed at the origin is minimal.

Indeed, for a target inside the region $\mathcal{R}$, the region can be made smaller by reducing the launching speed in such a way that the target would still remain inside the shooting range.

fact 7: When a target is shot with the smallest possible launching speed, the trajectory and the shooting range boundary (corresponding to the launching speed) are tangent to each other at the target's position.

If you trust what has been stated above (or have proved it once and now want to use the fact), it is easy to figure out the parameters of the parabola: first, the trajectory for 90°-launching angle meets the tip of the parabola, and the red curve needs to have the same shape as the green curve, because the green one represents the optimal trajectory for targets at very low altitudes beneath the horizon.

Trust, but verify: let us solve the following problem.

pr 19. A cannon is situated in the origin of coordinate axes and can give initial velocity v_0 to a projectile, the shooting direction can be chosen at will. What is the region of space $\mathcal{R}$ that the projectile can reach?

This question is an example of a class of problems that seem easy, but the solution can get very long if brute force is applied. This can lead to mistakes or giving up on the problem altogether.

warning: If equations get tediously long then it is the right time to pause and think whether there can be an alternative way to reach the answer. If one exists, it pays to quit and try out the other path and see if it is shorter.

In such cases, before actual calculations, you should outline a strategy for tackling the problem: you should see in your mind a "road", a sequence of calculations which you hope you are able to perform, and which, if successful, lead to the answer.

Here a strategy which comes to mind is compiling an equation for the launching angle α required for hitting a target at the coordinates (x, z); if there are solutions to this equation, the target lies within the region $\mathcal{R}$, and if there are no solutions then it lies outside. Then, formula describing the region comes from the condition (inequality) which needs to be satisfied for the existence of solutions. Furthermore, it is quite easy to figure out that if the point $T = (x, z)$ lies inside the region $\mathcal{R}$, there should be actually two solutions for the launching angle. Indeed, the target can be hit both at the rising leg of the projectile's trajectory, and at the descending leg of it[7] So, we expect that the equation has two solutions (T is within $\mathcal{R}$), one solution (T is at the boundary of $\mathcal{R}$), or no solutions (T is outside of $\mathcal{R}$). Such a behaviour is consistent with the quadratic equation, so we can hope that with a good parametrization (c.f. idea 24), we obtain a quadratic equation. To summarize, let us formulate

idea 29: If it is asked to find the region in which a solution exists to a certain problem then the boundary of this region can often be found as a curve for which some discriminant vanishes.

The solution of problem 19 can be also used to derive a simple particular conclusion which we formulate as a fact.

fact 8: If the target is at the same level as the canon, the optimal shooting angle (corresponding to the smallest launching speed) is 45°.

Indeed, from the solution of problem 19 we have a quadratic equation for the shooting angle, where we can put $z = 0$; the required result is immediately obtained if we use the fact that for optimal shooting, the discriminant of the equation is zero.

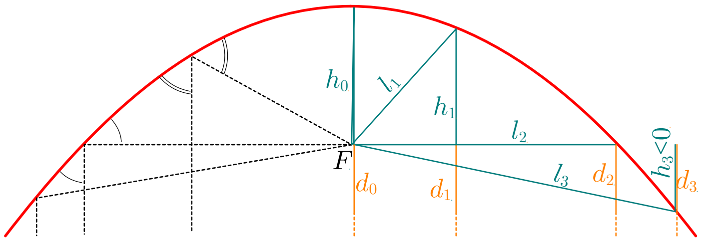
pr 20. Under the assumptions of the problem 19, and knowing that the boundary of the region $\mathcal{R}$ is a parabola, show that the cannon is at the focus of the parabola.

While there is an extremely simple solution to this problem, it might not be easy to come up with it, because we need

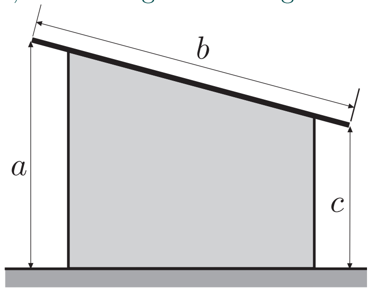
idea 30: For several problems of kinematics, geometrical solutions making use of the properties of a parabola are possible; typically, such solutions are considerably shorter than the alternatives.

fact 9: Each parabola has a focus, the properties of which are most easily expressed in terms of geometrical optics: if the parabola reflects light, all those rays which are parallel to the symmetry axis are reflected to the focus (dashed lines in the figure); due to the Fermat' principle, this means also that for each point on the parabola the distance to the focus plus the distance to the infinitely distant light source are equal, $l_1 + h_1 + d_1 = 2h_0 + d_0 = \ldots$; since $d_0 = d_1 = \ldots$, we have $2h_0 = l_1 + h_1 = l_2 = l_3 + h_3$ (see the figure).

[7] This is somewhat simplifying statement; to be more rigorous, we need to consider the two intersection points of the projectile's trajectory with a horizontal line $z = z_0$, and how these intersections $x = x_1$ and $x = x_2$ (with $x_2 > x_1$) move when the launching angle is changed from 90° to 0°: for $\alpha = 90°$, they are both at $x = 0$, and start moving to larger values of x with increasing α. For a certain value of α, x_2 reaches its maximal value $x_{\max}$ and starts decreasing; the two solutions merge and disappear when the trajectory is tangent to the line $z = z_0$. Hence, during this process, each point on the segment $0 < x < x_{\max}$ is passed exactly two times by one of the intersection points (either x_1 or x_2), and the target at that point (x, z_0) can be hit by the two corresponding values of α.



pr 21. What is the minimum initial velocity that has to be given to a stone in order to throw it across a sloped roof? The roof has width b, its two edges have heights a and c.



This problem has two solutions, a brute force one, and a short but tricky one. Both solutions, however, start in the same way, by using

idea 31: To find a minimum (or a maximum), we have to vary free parameters (in this case the throwing point and the angle) by infinitesimally small increments and see what happens to the quantity of interest. If it increases for all allowed variations, we have found a minimum.

You are supposed to use this method for showing that the stone touches both edges of the roof.

Both solutions share also the next step, reducing the problem to the case $c = 0$. This reduction would be very useful because then we know the optimal throwing point — the right edge of the roof. For this step you need

idea 32: For a free-fall of a body, there is an integral of motion (quantity which is conserved), $\frac{1}{2}v^2 - gh$, where v is an instantaneous speed, and h is the current height of the body.

This, of course, is a particular form of the energy conservation law, which will be considered in many more details in the other sections (mainly in "Mechanics").

Next we need to apply the idea 28; while in the case of the brute force approach, this is a straightforward mathematical application of the idea, the trickier solution makes use of the idea 30 (and of the fact 9).

Let us consider one more problem which illustrates the usage of the idea 30.

pr 22. A target is shoot with the smallest possible launching speed; show that the launching velocity is perpendicular to the terminal velocity (i.e. at the target).

Note that this problem can be also solved without using the idea 30. Then, instead, you should use the idea 19 by considering the relative motion of two projectiles which are shoot simultaneously at slightly different angles (still, very close to the optimal angle) and with the same speed. You also need to make use of mathematical observations, formulated below as a facts.

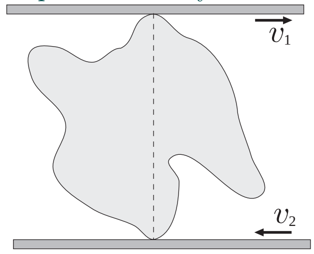
fact 10: If a function $f(x)$ has a minimum or maximum at $x = x_0$ then for small departures Δx from x_0, the variation of the function value $\Delta f = f(x+\Delta x) - f(x)$ remains quadratically small[8] and can be often neglected.

fact 11: As it follows from the vector addition rule (c.f. def. 1), if the difference of two vectors of equal moduli is very small, it is almost perpendicular to each of them.

Try to solve the problem 22 using this alternative approach, as well!

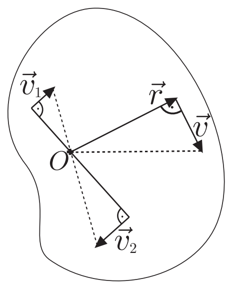
5 RIGID BODIES, HINGES AND ROPES

pr 23. A rigid lump has been squeezed between two plates, one of which is moving at velocity v_1 and the other at v_2. At the given moment, velocities are horizontal and the contact points of the lump and plates are aligned. In the figure, mark all points of the lump with velocity modulus equal to v_1 or v_2.



This question is entirely based on

idea 33: The motion of a rigid body can always be considered as rotation about an instantaneous axis of rotation (for 2D geometry, about a rotation centre) with a certain angular speed.



This centre of rotation can be reconstructed if

(a) we know the directions of velocities of two points and these directions are not parallel - it is where perpendicular lines drawn from these points intersect;

(b) we know the velocities of two points, and the vectors are parallel and perpendicular to the line connecting these points - we find the intersection of the line connecting the points and the line connecting the tips of velocity vectors (see the figure)

Note that it is also possible that the velocities of two points are equal (by modulus and by direction), in which case the rotation centre is at infinity, i.e. the body moves translationally. All the other combinations not covered by options (a) and (b) are impossible for a rigid body.

[8] This is the consequence of the Taylor expansion, to be discussed in more details in other booklets.

Once the centre of rotation has been found then the velocity of any point can be found as a vector perpendicular to the line drawn from the centre of rotation with modulus proportional to distance r from the centre of rotation (see the figure), according to the formula $v = \omega r$, where ω is the instantaneous angular speed.

The last formula can be generalized:

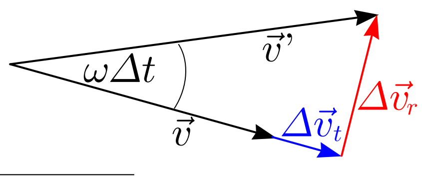
fact 12: if a vector $\vec{a}$ rotates with an angular speed $\vec{\omega}$ so that the rotation axis is perpendicular to $\vec{a}$ then the time derivative $\frac{d\vec{a}}{dt}$ is perpendicular to $\vec{a}$ and $|\frac{d\vec{a}}{dt}| = \omega a$

This formula is derived similarly to the fact 11: we need to apply the definition 1 and consider a small time increment Δt. According to the definition of vector derivatives, $\frac{d\vec{a}}{dt}$ equals to $\frac{\Delta\vec{a}}{\Delta t}$ at the limit of infinitesimally small increments $\Delta\vec{a}$ and Δt. The increment $\Delta\vec{a}$ is calculated using the definition 1: this is the base of the isosceles triangle, the equal sides of which are formed by the initial and final positions of the vector $\vec{a}$. Since the vector rotates at the angular speed ω, the apex angle (in radians) is $\omega\Delta t$; therefore, the base length equals to $2a\tan(\omega\Delta t/2) \approx a\omega\Delta t$[9] Thus, $\left|\frac{\Delta\vec{a}}{\Delta t}\right| = \frac{a\omega\Delta t}{\Delta t} = a\omega$[10]. Note that the formula can be rewritten via a vector product as $\frac{d\vec{a}}{dt} = \vec{\omega}\times\vec{a}$. Within this booklet, vector products will not be needed; however, it is still useful to know that a vector product of two vectors $\vec{a}$ and $\vec{b}$ equals by modulus to the surface area $S = ab\sin\alpha$ of the parallelepiped built on the vectors $\vec{a}$ and $\vec{b}$ (α is the angle between $\vec{a}$ and $\vec{b}$); the vector is perpendicular to both $\vec{a}$ and $\vec{b}$, the direction being given by the screw rule (rotate the screw from the first vector to the second one).

Now we can apply the fact 12 to a rotating velocity vector: let a point rotate with constant angular speed ω around the origin so that its distance to the origin remains constant. According to the formula we obtain $v = \left|\frac{d\vec{r}}{dt}\right| = \omega r$ and $a = \left|\frac{d\vec{v}}{dt}\right| = \omega v = \omega^2 r$. The last equation can be also written for the acceleration vector as $\vec{a} = -\omega^2\vec{r}$; the minus sign means that the acceleration is directed towards the origin; because of this it is called *centripetal acceleration*. If the angular velocity is not constant, the point will obtain also a *tangential acceleration* (tangent to the trajectory).

def. 4: Angular acceleration is defined as the time derivative of the angular velocity, $\vec{\varepsilon} = \frac{d\vec{\omega}}{dt}$.

Let us consider the case when the rotation axis remains fixed; then, both $\vec{\varepsilon}$ and $\vec{\omega}$ have a fixed direction and hence, are essentially scalar quantities, so that the vector signs can be dropped. In order to derive an expression for the tangential acceleration, let us consider a small time increment Δt; this corresponds to a small angular speed increment $\Delta\omega$, as well as to a small velocity increment $\Delta\vec{v}$. Let us decompose the velocity increment into radial and tangential components, $\Delta\vec{v} = \Delta\vec{v}_r + \Delta\vec{v}_t$. Now, the velocity vector changes both its direction and length, see figure below.



Note that in the figure, the increments are exaggerated, actually these are very small. It is easy to see from the figure that for small angle approximation, $\Delta v_r = \omega\Delta t \cdot v$; with this we recover the expression for the radial (centripetal) acceleration $a_r = \omega v$: we just need to divide the equality by Δt. Meanwhile, due to the small angle approximation, $|\vec{\Delta}v_t| \approx |\vec{v}'| - |\vec{v}| = \omega' r - \omega r = \Delta\omega r = \varepsilon\Delta t r$. Hence, at the limit of infinitesimal increments, $a_t = \frac{|\Delta v_t|}{\Delta t} = \varepsilon r$. This leads us to

fact 13: If a rigid body is rotating about a *fixed axis* then the acceleration of any of its points has two components: centripetal acceleration $\omega^2 r = v^2/r$ directed towards the axis of rotation and a component perpendicular to it, the tangential acceleration ϵr. NB! the formula cannot be applied if we deal with an instantaneous rotation axis[11] (more precisely, the formula can be used if the acceleration of the instantaneous rotation axis is zero). The formula v^2/r can be also used for the perpendicular to the motion acceleration of a point along a curved trajectory; then, r is the curvature radius of the trajectory.

pr 24. Cycloid is a curve which can be defined as a trajectory of a point marked on the rim of a rolling wheel or radius R. Determine the curvature radius of such curve at its highest point.

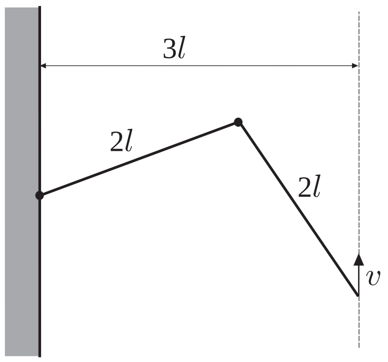
idea 34: For kinematics problems, often the tangential acceleration is not known initially, but the rotation speed is known and hence, the centripetal acceleration can be easily calculated using the fact 13. You may be able figure out the direction of the acceleration using other arguments, in which case you can recover the whole acceleration by using the idea 4.

The next problem will illustrate this idea. However, the following idea will be also useful.

idea 35: Since the distance between any two points is fixed in a rigid body, the projections of velocities of both points on the line connecting them are equal. NB! the respective projections of accelerations are not necessarily equal due to the centripetal acceleration; when dealing with accelerations you need to use the fact 13 in one point's frame of reference, instead.

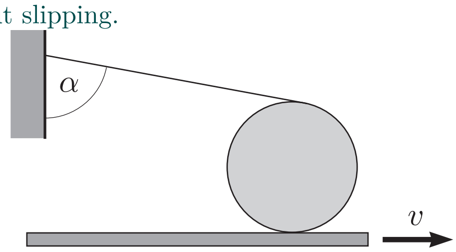
pr 25. A hinged structure consists of two links of length $2l$. One of its ends is attached to a wall, the other is moving at distance $3l$ from the wall at constant vertical velocity v_0. Find the acceleration of the hinge connecting the links when a) the link closer to the wall is horizontal b) the velocity of the connection point is zero.

[9] Here we have used approximate calculation for small angles: $\sin\alpha \approx \tan\alpha \approx \alpha$, where $\alpha \ll 1$ is measured in radians.
[10] We have replaced approximate equality by strict equality because the tangent is taken from an infinitely small angle
[11] For instantaneous axis, r would not be constant, which makes the derivation void.



Now, let us continue with rolling spool problems.

pr 26. Some thread has been wound around a cylinder, the other end of the thread is fixed to a wall. The cylinder lies on a horizontal surface that is being pulled with horizontal velocity v (perpendicular to the axis of the cylinder). Find the velocity of the cylinder's axis as a function of α, the angle that thread makes with the vertical. The cylinder rolls on the surface without slipping.



This problem can be solved using the idea 33. However it can be also solved using the following idea (you are encouraged to try both methods). and the spool is rolling along some surface. In these cases, we need

idea 36: Problems involving spools and rope (un)wounding can be typically solved by writing down the "rope balance" equation, i.e. by relating the rope unwinding rate to the velocity of the spool.

When writing down such equations, it is useful to notice that as long as the unwound rope does not rotate, the rate of unwinding equals to ΩR, where Ω is the spool's angular speed, and R — its radius; if the unwound rope rotates, idea 17 can be used to conclude that now the spool's angular speed Ω needs to be substituted with the difference of two angular velocities.

In order to relate the rope unwinding rate to the motion of the spool, the following very general idea (with a wide application scope) is helpful.

idea 37: Draw two very close (infinitesimally close) states of the system and examine the change in the quantity of interest (in this case, the length of the rope).

When doing that, we should not forget that the change was infinitesimal, so that we can simplify our calculations (e.g. two subsequent states of the rope can be considered parallel). Note that we have actually already used this method when deriving the fact 13.

pr 27. One end of a rod of length l is supported by the floor and the other leans against a vertical wall. What is the velocity u and acceleration a of the lower end at the moment when the upper end is sliding downwards at a constant velocity v and the angle between the floor and the rod is α?

[12]The continuity law plays an important role in physics, in general.

Here the velocity can be found using the idea 33 and the acceleration — using the idea 34; however, please solve it now using the following idea (its full solution is given in the section of hints).

idea 38: If parts of a system are connected by some ties with fixed length then one way of calculating velocities and accelerations is to write out this relation in terms of coordinates and take the time derivative (twice for acceleration) of the whole expression.

6 MISCELLANEOUS TOPICS

idea 39: Some kinematics problems are based on the continuity law[12]: if there is a stationary flow of something and we consider a certain region of space, as much as flows in, flows also out.

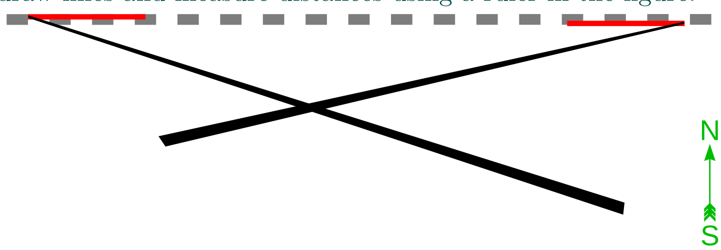
Let us illustrate the continuity law with the following problem.

pr 28. A single-lane road is full of cars which move at the speed $v = 90\,\mathrm{km/h}$. The average distance between neighbouring cars is such that a standing observer would measure a time lapse $\tau = 2\,\mathrm{s}$ from the head of the first car until the head of the second car. One car has to stop due to malfunctioning and a queue of standing cars starts forming behind it. Find the speed u with which the queue length is growing if the average distance between the centres of neighbouring cars in the queue is $l = 6\,\mathrm{m}$?

Pay attention that the flow of cars is not stationary in the lab frame of reference; hence, first you need to use the idea 1 to make it stationary.

idea 40: In the case of problems involving moving objects and moving medium (air, water), if the moving objects leave trails, and you are asked to analyse a sketch (a photo), pay attention to the fact that if the objects met, the meeting point corresponds to an intersection point of the trails.

pr 29. Figure below is copied from an aerophoto: there are two trains (depicted in red) which both travel with the speed of $v = 50\,\mathrm{km/h}$ along a railway (grey dashed line). Their engines emit fume, the trails of which are depicted by black lines. Determine the direction and speed of wind (express the direction of wind as a clock-wise rotation angle from north). You may draw lines and measure distances using a ruler in the figure.



idea 41: Sometimes it is useful to include a time axis in addition to spatial coordinates and analyse graphs even if the problem's text does not mention time dependence explicitly. Thus

we'll be studying two-dimensional graphs for one-dimensional problems and three-dimensional graphs for a planar motion.

Try to apply this idea to the following problem. Keep in mind the following stereometric facts: 3 points always lie in a plane; a straight line and one point determine a plane (unless the point lies on the line).

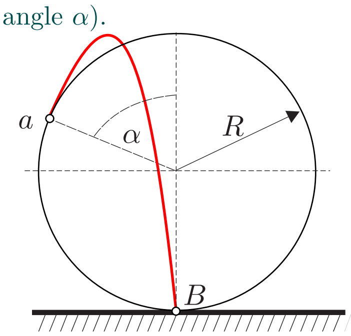
pr 30. Points A, B and C lie on a straight line, so that B is situated between A and C. Three bodies a, b and c start from these points, moving at constant (but different) velocities. It is known that i) if c was missing, a and b would collide and ii) if b was missing, a and c would collide and it would happen before than in i). Would b and c collide if a was missing?

In mathematics, for geometrical problems, there is a technique called auxiliary constructions: the solution of the problem can be significantly simplified if you draw an additional line(s). While in physics, such constructions are less often used, but in some cases auxiliary constructions are still useful (in fact, idea 41 makes use of such construction). The following idea represents a recipe for one more auxiliary construction.

idea 42: Sometimes it is useful to consider instead of a single particle, a fictitious ensemble of auxiliary particles.

While the following problem can be solved via a brute force approach, introducing an ensemble of auxiliary particles (released all over the wheel at the same time) simplifies your calculations.

pr 31. A wheel with radius R is situated at height R from the ground and is rotating at angular velocity Ω. At some point A, a drop of water separates from the wheel and reaches the ground at point B situated directly below the wheel's axle (see the figure). Find the falling time of the drop and the location of point A (i.e. angle α).



In addition to the previous ones, there is an idea that helps solving this problem. However, it is very non-standard, i.e. there are almost no other problems where to use this idea. To apply the idea elsewhere, it has to be formulated in a very general form: More specifically for this question: imagine that at the same time with the given drop, small droplets separated from all the other points of the wheel as well. It is clear that in a freely falling frame of reference, this set of droplets forms a circle at all times; it should not be too difficult to find the radius of that circle as a function of time. The first droplet touching the ground is the one that we are interested in. We can express time from the equation describing the condition of touching (the height of the lowest point of the circle becomes zero) - idea 10.

idea 43: For some problems, the main difficulty is understanding, what is going on; once you understand, the calculations are typically quite easy. Keep your mind calm: in physics competitions, you are not asked to do something impossible: act as a detective by investigating step-by-step what is going on, and narrowing down possibilities by the exclusion method.

And here is a test case for the detective inside you!

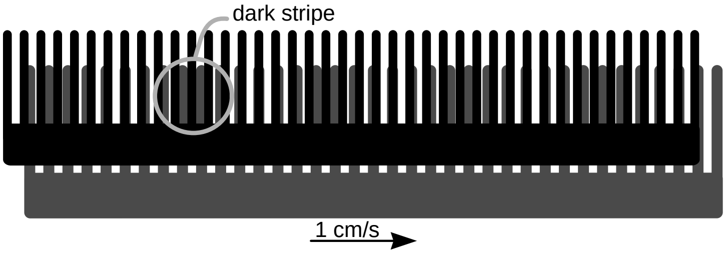
pr 32. This photo of a rotating propeller of a plane is taken with a phone camera. For such a camera, the image is scanned line-by-line: at first, the leftmost column of image pixels is read, followed by the second column of pixels, etc.
i) in which direction does the propeller rotate as seen by the photographer (clockwise, or counterclockwise)?
ii) How many blades does the propeller have?
iii) How many rotations does the propeller make in one minute if the total scanning time of this image was $\frac{1}{8}$ seconds?

The first part of the next problem can be also solved using the idea 43. The second part, however, becomes mathematically challenging; it would be much easier to solve it using the following idea.

idea 44: Wave propagation problems can be often conveniently analysed using the wave vectors: using orthogonal coordinates (x, y, z), and time t, a sinusoidal wave $a(x, y, z, t)$ can be represented as $\sin(\vec{k}\vec{r} - \omega t)$, where the scalar product of the wave vector k and radius vector $\vec{r}$ equals to $\vec{k}\vec{r} = k_x x + k_y y + k_z z$. Here ω is the angular frequency of the wave and the wave's propagation speed $v = \omega/k$[13].

pr 33. **i)** There are two combs which are depicted in the figure below. The figure renders the proportions of the combs correctly; the scale of the figure is unknown. The grey comb moves with a speed of $v = 1\,\text{cm/s}$ (the direction of its motion is shown in the figure); the black comb is at rest. Find the speed and direction of motion of the dark stripes.

[13] Sinusoidal waves are studied in more details in the booklet "Wave optics".



ii) Now, we have the same situation (and the same question) as before, except that the black comb is under a small angle $\alpha \ll 1$ with respect to the grey comb (for numerics, use $\alpha = 0.1\,\mathrm{rad}$).

How to apply the idea 44 to the problem 33? Pay attention to the following. Instead of a moving comb, we can take a wave $a(x,y,t) = \sin(\vec{k}\vec{r} - \omega t)$; then, centres of the teeth correspond to points where $a(x,y,t) = 1$. If we have two overlapping combs, the "transparent regions" where the centres of the teeth of both combs are at the same position can be found as the points where $a_1(x,y,t)a_2(x,y,t) = 1$, and the regions where the teeth are almost at the same position can be found as the points where $a_1(x,y,t)a_2(x,y,t) \approx 1$. Now we have a product of two sinusoidal waves, so we can say that the moiré pattern (this is how these dark stripes are called) is due to a nonlinear interaction of waves. Further, in order to answer the question about what is the propagation speed of the moiré pattern, we need to decompose this product of sinusoids into a sum of sinusoids (for a sinusoid, we can calculate the speed as the ratio of the angular frequency and the modulus of the wave vector).

idea 45: Well-known conservation laws are the ones of energy, momentum, and angular momentum[14]. However, sometimes additional quantities can be conserved (then, of course, you need to show that it is conserved), which makes otherwise mathematically very difficult problems easily solvable.

What can be hints that a non-trivial conservation law is valid for a problem? Well, if you understand clearly that the difficulty is mathematical (you understand what is going on, and are able to write down the equations) and what you have at hand are differential equation(s)[15] then that might be the case. For instance, let us consider the following problem.

pr 34. A dog is chasing a fox running at constant velocity v along a straight line. The modulus of the dog's velocity is constant and also equal to v, but the vector $\vec{v}$ is always directed towards the fox. When the dog noticed the fox and started chasing, the distance between them was L and at the first moment, their velocity vectors formed a right angle. What is the minimal distance between them during the chase?

Here we can express the velocity components of the dog in terms of its coordinates (using also the coordinates of the fox, which are known functions of time); these are differential equations. Since you are not supposed to be able to solve differential equations for this booklet, this should not be the way to go (furthermore, unless properly parametrized, c.f. idea 24, these equations are actually not easy to solve). Note that at physics competitions, you should always keep in mind, what are those mathematical skills with which you are assumed to be familiar with: when not certain which solving route to take, this can help you narrowing down the options[16]. While extra skills are sometimes useful, these can also easily lead you to a wrong (mathematically unnecessarily complex or even unsolvable) path; therefore, apply your extra skills only if you can see in your mind the whole path to the answer, i.e. when you are sure that you will not run into mathematical difficulties.

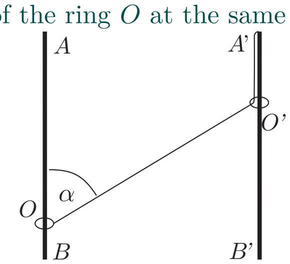
Note that such nontrivial conservation laws can include all these quantities which enter the differential equations, except for those derivatives for which these equations are written. For instance, here, the equations express velocities in terms of coordinates, so velocities cannot be included. If the equations were expressing accelerations in terms of velocities and coordinates then both velocities and coordinates could enter the law (but not the accelerations). The conservation law is typically proved by showing that its time derivative is zero. For instance, if our conservation law would be a sum of two distances, we would need to show that the changing rates of those two distances are equal by modulus and have opposite signs.

7 CONCLUSION

Some of the ideas presented here (in particular, 1, 4, 5, 10, 14, 18, 19, 31, 37) are more universal than others. In any case, they are all worth remembering. Note that it is always useful to summarise the idea(s) in your mind after finding a solution to a new problem; from time-to-time you may be able to come up with some entirely new ideas! For further practice, here are some additional problems based on aforementioned ideas.

pr 35. A boy is swimming in a fast-flowing river of width L; the water speed is $u = 2\,\mathrm{m/s}$, and the boy can swim with speed $v = 1\,\mathrm{m/s}$. While being at point A near one coast, he wants to reach such a point B near the other coast which is directly across the river ($\vec{AB} \perp \vec{u}$). Since the river is too fast, he cannot avoid being carried downstream to a certain distance a from point B; what is the smallest possible value of a which he can achieve for the optimally chosen swimming direction?

pr 36. Rings O and O' are slipping freely along vertical fixed rails AB and $A'B'$ (see the figure); the distance between the rails is b. Some unstretchable rope has been tied to ring O and pulled through ring O'. The other end of the rope is fixed to point A'. At the moment when $\angle AOO' = \alpha$, the ring O' is moving downwards at a constant velocity v. Find the velocity and acceleration of the ring O at the same moment.

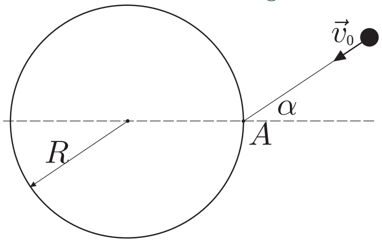


[14] These are typically not applicable to the kinematics problem.

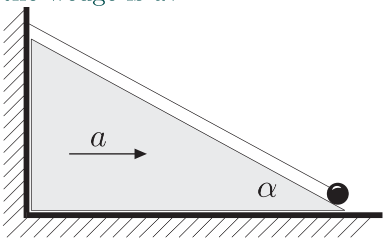
[15] These are equations relating derivatives to coordinates (typically, velocities or accelerations as functions of coordinates and/or velocities and which define the evolution of the system.

[16] Of course, such arguments are cannot be used for real-life problems.

pr 37. A small ball is moving at velocity v_0 along a smooth horizontal surface and falls into a cylindrical vertical well at point A, after which it starts bouncing elastically against the wall and the smooth horizontal bottom. The well has depth H and radius R; the angle between $\vec{v}_0$ and the well's diameter drawn through point A is α. What condition between R, H, v_0 and α must be satisfied for the ball to exit from the well again? Rotation of the ball can be neglected.

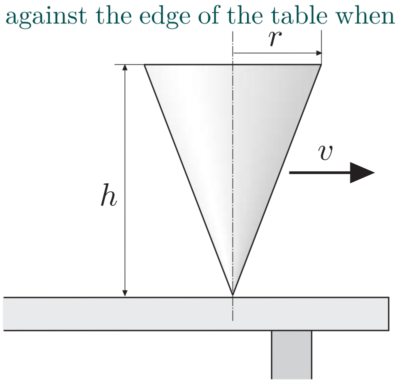


pr 38. A ball A lies on a wedge with angle α. It is also tied to an unstretchable string, the other end of which is attached to a vertical wall at point B (see the figure). What will be the trajectory of the ball? What is its acceleration if the acceleration of the wedge is a?



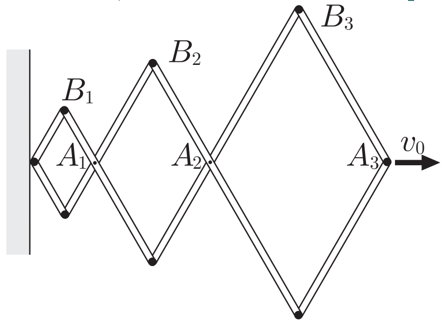
pr 39. A dog is chasing a fox running at constant velocity v_1 along a straight line. The modulus of the dog's velocity is constant and equal to v_2, but the vector $\vec{v}$ is always directed towards the fox. The distance between the animals was l at the moment when their velocity vectors were perpendicular. What was the acceleration of the dog at that moment?

pr 40. A spinner having the shape of a cone (height h, radius r) is moving along a smooth table and spinning rapidly. What does its translational velocity v have to be in order to avoid bumping against the edge of the table when it gets there?



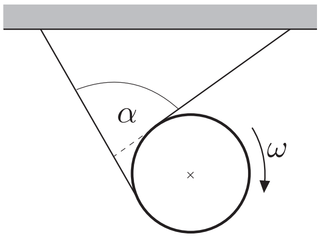
pr 41. A uniform rope has been manufactured from an explosive material, combustion travels along the rope at velocity v. The velocity of a shock wave in the air is c, with $c < v$. Along which curve should the rope be laid out to make the shock wave reach a given point at the same time from all points of the rope? (Finding a quantitative formula for the shape requires solving a very simple differential equation.)

pr 42. A hinged structure consists of rhombi with side lengths l, $2l$ and $3l$ (see the figure). Point A_3 is moving at constant horizontal velocity v_0. Find the velocities of points A_1, A_2 and B_2 at the moment when all angles of the structure are equal to 90°. Also, find the acceleration of point B_2.

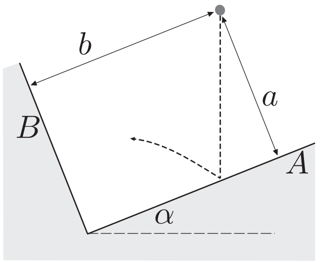


pr 43. Two motorboats simultaneously depart from two harbours (A and B) at distance l from one another, velocities of the boats are v_1 and v_2, respectively. The angles between their velocities and the line connecting A and B are α and β, respectively. What is the minimum distance between the boats?

pr 44. A heavy disk of radius R is rolling downwards, unwinding two strings in the process. The strings are attached to the ceiling and always remain under tension during the motion. What was the magnitude of velocity of the disk's centre when its angular velocity was ω and the angle between the strings was α?

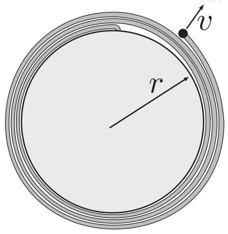


pr 45. Two boards have been placed at right angles to one another. Their line of touching is horizontal and one of them (A) makes an angle α to the horizontal. An elastic ball is released at a point at distance a from plane A and b from B. On the average, how many times does the ball bounce against wall B for each time it bounces against wall A? Collisions are absolutely elastic.

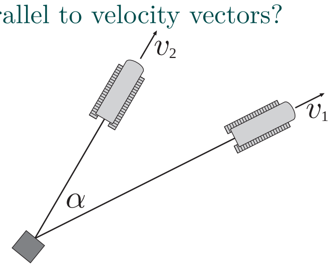


pr 46. One end of a string of a negligible mass has been attached to the side of a cylinder, not far from the ground. The cylinder itself has been fixed on smooth slippery horizontal surface, with its axis vertical. The string has been wound k times around the cylinder. The free end of the string has been tied to a block, which is given a horizontal velocity v directed along the radius vector drawn from the cylinder's axis. After what

time will the string be fully wound around the cylinder again, this time the other way round? [This problem leads to a very simple differential equation; if you don't know how to solve it, the following equality can be helpful: $l\frac{dl}{dt} = \frac{1}{2}\frac{d(l^2)}{dt}$.]



pr 47. A heavy box is being pulled using two tractors. One of these has velocity v_1, the other v_2, the angle between velocities is α. What is the velocity of the box, if we assume that the ropes are parallel to velocity vectors?

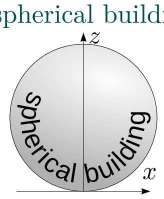


pr 48. A boy is running on a large field of ice with velocity $v = 5\,\mathrm{m/s}$ toward the north. The coefficient of friction between his feet and the ice is $\mu = 0.1$. Assume as a simplification that the reaction force between the boy and the ice stays constant (in reality it varies with every push, but the assumption is justified by the fact that the value averaged over one step stays constant).

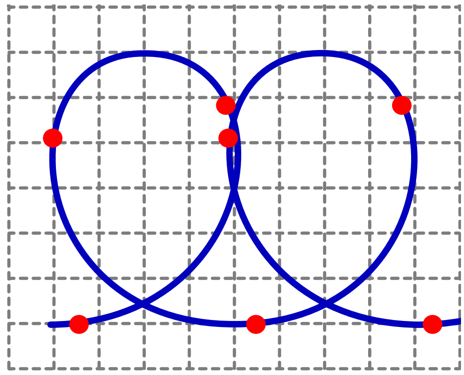
i) What is the minimum time necessary for him to change his moving direction to point towards the east so that the final speed is also $v = 5\,\mathrm{m/s}$?

i) What is the shape of the optimal trajectory called?

pr 49. A ball thrown with an initial speed v_0 moves in a homogeneous gravitational field of strength g; neglect the air drag. The throwing point can be freely selected on the ground level $z = 0$, and the launching angle can be adjusted as needed; the aim is to hit the topmost point of a spherical building of radius R (see fig.) with as small as possible initial speed v_0 (prior hitting the target, bouncing off the roof is not allowed). Sketch qualitatively the shape of the optimal trajectory of the ball. What is the minimal launching speed $v_{\min}$ needed to hit the topmost point of a spherical building of radius R?



pr 50. The figure represents a photo which was taken using a very long exposure time (camera was pointing directly down). What you can see is a trace of a blue lamp which burned continuously, but also flashed periodically with a red light (after each $t = 0.1\,\mathrm{s}$. The lamp was fixed to the surface of a solid disk, at the distance $a = 4.5\,\mathrm{cm}$ from its symmetry axis. The axis was vertical, and the disk slid and rotated freely on a horizontal smooth ice surface. What was the speed of the centre of the disk? You can take measurements from the figure using a ruler.



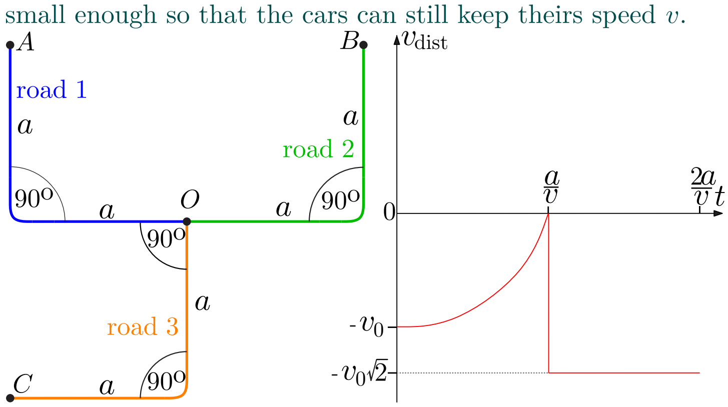
pr 51. There is a capital O and three cities A, B and C, connected with the capital via roads 1, 2, and 3 as shown in the left figure. Each road has length $2a$. Two cars travel from one city to another: they depart from their respective starting points simultaneously, and travel with a constant speed v. The figure on the right depicts the increasing rate of the distance between the cars (negative values means that the distance decreases) as measured by the GPS devices of the cars. The turns are taken by the cars so fast that the GPS devices will not record the behaviour during these periods.

i) Which cities were the starting and destination points of the cars? Motivate your answer.

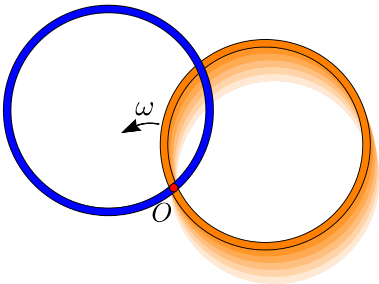
ii) What is the surface area between the v_{dist} -graph and the t-axis for the interval from $t = 0$ to $t = a/v$?

iii) Now let us consider a case when three cars (denoted by A, B, and C) depart simultaneously from their cities (A, B, and C, respectively) towards the capital; all the cars travel with a constant speed v. Sketch the graphs for the distance changing rate for the following car pairs: $A - B$ and $B - C$.

iv) Suppose that now the GPS-devices are good enough to record the periods of taking the turns. Sketch a new appropriate graph for the pair of cars $B - C$. The curvature of the turns is small enough so that the cars can still keep theirs speed v.



pr 52. Consider two rings with radius r as depicted in the figure: the blue ring is at rest, and the yellow ring rotates around the point O (which is one of the intersection points of the two rings) with a constant angular speed ω. Find the minimal and maximal speeds $v_{\min}$ and $v_{\max}$ of the other intersection point of the two rings.

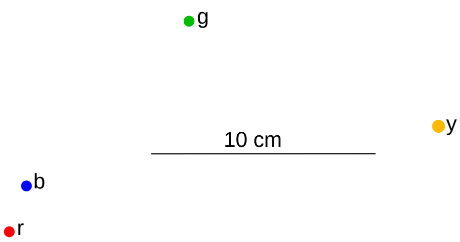


pr 53. A moving bicycle is photographed using a relatively long exposure time with a motionless camera. As a result, the bike appears smudged on the photo; however, certain points of certain spokes appear sharp. Determine the shape of the curve upon which the sharp points of spokes lie.

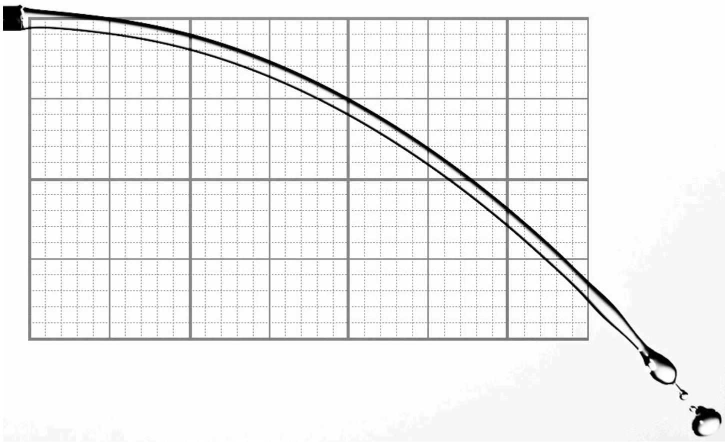
pr 54. A lamp is attached to the edge of a disk, which moves (slides) rotating on ice. The lamp emits light pulses: the duration of each pulse is negligible, the interval between two pulses is $\tau = 100\,\text{ms}$. The first pulse is of orange light, the next one is blue, followed by red, green, yellow, and again orange (the process starts repeating periodically). The motion of the disk is photographed using so long exposure time that exactly four pulses are recorded on the photo (see figure). Due to the shortness of the pulses and small size of the lamp, each pulse corresponds to a coloured dot on the photo. The colors of the dots are provided with the corresponding lettering: o — orange, b — blue, r —red, g — green, and y — yellow). The friction forces acting on the disk can be neglected.

i) Mark on the figure by numbers (1–4) the order of the pulses (dots). Motivate your answer. What can be said about the value of the exposure time?

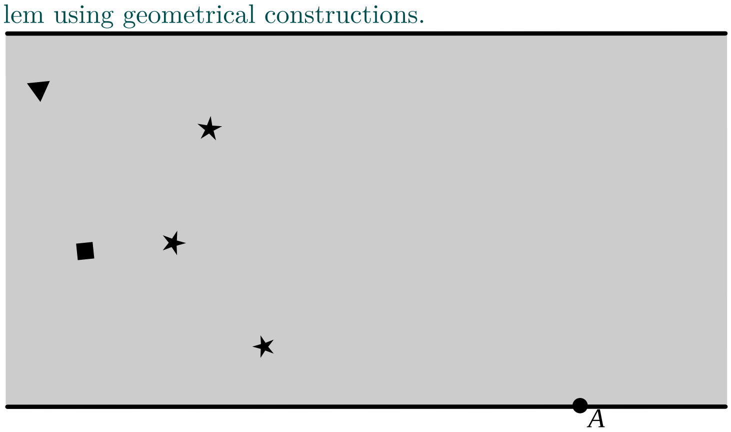
ii) Using the provided figure, find the radius of the disk R, the velocity of the center of the disk v and the angular velocity ω (it is known that $\omega < 30\,\text{rad/s}$). The scale of the figure is provided by the image of a line of length $l = 10\,\text{cm}$;



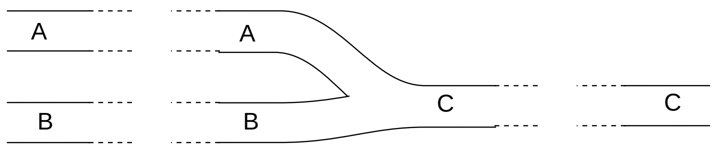
pr 55. The photo depicts a jet of water, together with background grid. The pitch of the grid equals to the diameter of the jet at the exit from the horizontal pipe. The water flow rate is constant in time, and if a vessel of volume $V = 150\,\text{cm}^3$ is used to collect the outflowing water, it is filled during the time period of $t = 5\,\text{min}$. Find the diameter of the jet at the exit of the pipe.



pr 56. On a wide river, two boats move with constant velocities. The velocity of the water in the river is constant across the whole area depicted in the figure, and parallel to the coastline. The figure is based on a photo which was taken from air, the camera being directed straight down. The positions of the boats are marked with a square and a triangle, and the positions of litter fallen from the boats — with pentagrams. One of the boats departed from the point A; it is known that the boats did meet with each other at a certain moment. From which coastal point did the other boat depart? Solve the problem using geometrical constructions.



pr 57. Let us consider the merger of two traffic lanes A and B into a single lane C, see figure. During a rush hour, all the lanes are filled with cars; the average distance between the cars can be assumed to be equal for all the lanes. The lengths of the lanes A, B, and C is respectively equal to $L_A = 1\,\text{km}$, $L_B = 3\,\text{km}$, and $L_C = 2\,\text{km}$. The average speed of cars on the lane A is $v_A = 3\,\text{km/h}$; and the travel time of a car on the lane B is $t_B = 36\,\text{min}$. How long will it take for a car to travel from the beginning of the lane A to the end of the lane C?



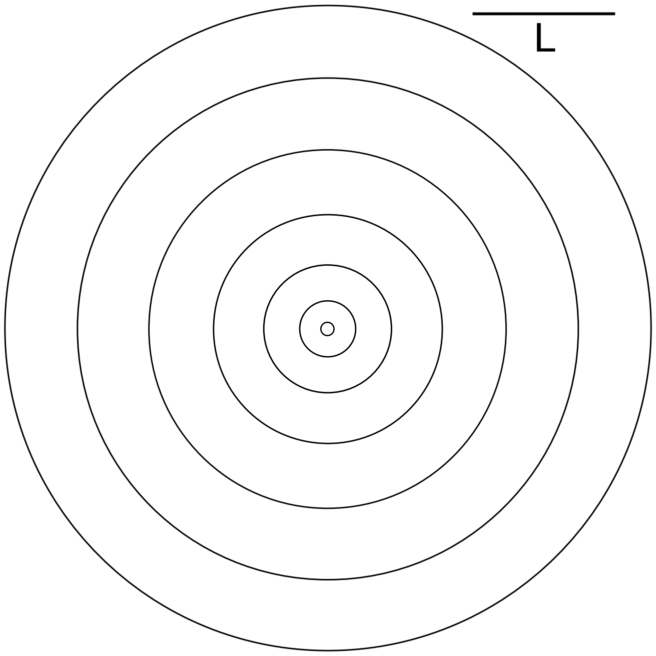
pr 58. On a windless rainy day, a standing man gets wet during $t = 2\,\text{min}$; if he runs with the speed $v_2 = 18\,\text{km/h}$, he gets wet during $t_2 = 0.5\,\text{min}$. How long will it take for him to get wet when walking with the speed $v_1 = 6\,\text{km/h}$? Assume that the body shape of a man can be approximated (a) with a vertical rectangular prism; (b) with a sphere. Here "getting

wet" is defined as receiving a certain amount of water.

pr 59. A photographer took a photo of a waterfall. Due to the reflected sunlight, the droplet were seen as speckles. Owing to the fast falling speed of the droplets, the speckles created bright stripes on the photo. When the camera was in a normal "landscape" position, the length of the stripes was $l_1 = 120$ pixels; when the camera was rotated around the optical axis of the lens by 180° into a "head-down" position, the length of the stripes was $l_2 = 200$ pixels. What was the length of the stripes when the camera was in a "portrait" position, i.e. rotated by 90°? Assume that the exposure time was equal in all three cases. If there are several possibilities, give all the possible answers.

Hint. The main components of the camera are the lens which creates an image on the sensor, and the shutter. The purpose of the shutter is to limit the time during which the sensor is exposed to the light to a short (and appropriate) period of time: normally, it covers the sensor, and the image created by the lens falls on the sensor only when the shutter is opened. The shutter is made of two curtains: at the beginning, the first curtain covers the sensor; when a photo is taken, it moves down with a certain speed v opening the sensor; once the sensor has been opened for the required period of time, the second curtain moves down with the same speed v, covering again the sensor. In order to achieve very short exposure times, both curtains move together, creating a narrow moving slit through which the light can pass through to reach the sensor.

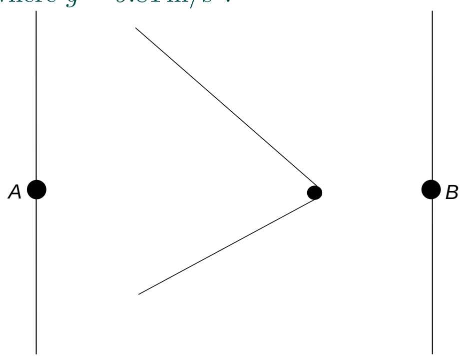
pr 60. If a stone is thrown into a pond, a circular wave is created which expands in time. The following figure depicts the propagation of such a wave: different circles correspond to the position of the wave crest at different moments of time; the underlying snapshots have been taken with a regular (but unknown) interval.



Note that the wave speed depends on the wavelength, and here, the effective wavelength grows in time: at the beginning, the wavelength is of the order of the circle radius, but later, it tends towards a value which is of the order of the water depth h. It turns out that at the beginning, the wavecrest moves with acceleration $a = g/\pi$, where g is the free fall acceleration; later, it tends towards the value $v_\infty = \sqrt{hg}$. Based on this knowledge, estimate the depth of the pond h assuming that it is constant everywhere; express your answer in terms of the length scale L provided in the figure. You can take measurements from the figure using a ruler.

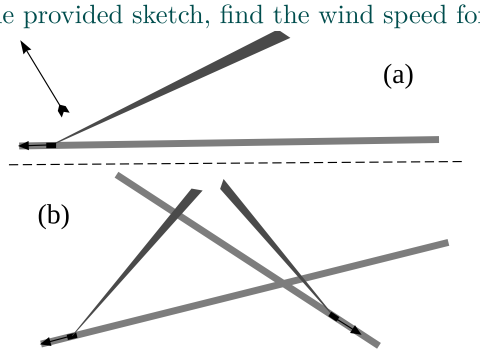
pr 61. A motorboat approaches a straight coastline perpendicularly, and at a distance L starts turning back by drawing a half-circle of radius R; later it departs perpendicularly to the coastline. The speed of the boat is constant and equal to v, the water wave speed can be assumed to be constant and equal to u (with $u < v$). How long will it take for the waves of the wake behind the boat to reach the coast (as measured from the moment when the boat starts turning)?

pr 62. On a wide river, a motorboat moves with a constant speed $v = 7\,\mathrm{m/s}$ from village A to village B over the river. When answering the following questions you may take measurements from the figure below depicting waves behind the boat. What is the water speed in the river, and what is the water depth h? Note: wave speed in shallow water[17] is $w = \sqrt{gh}$, where $g = 9.81\,\mathrm{m/s^2}$.



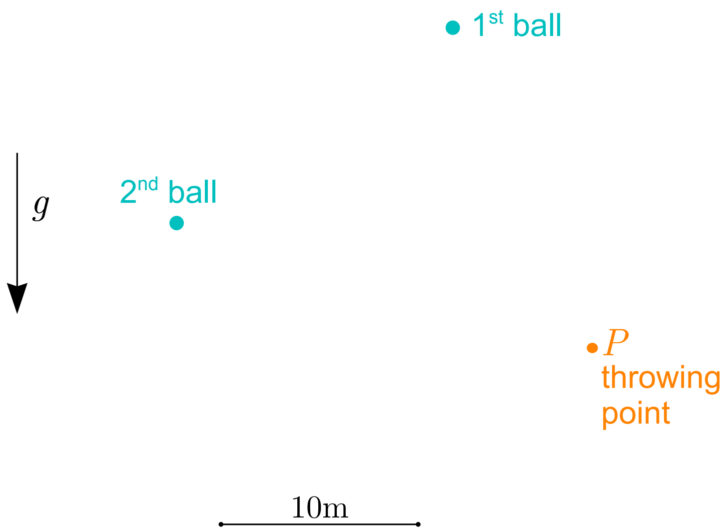
pr 63. Provided sketches (a) and (b) are made on the basis of satellite images, preserving proportions. They represent tractors, together with their smoke trails. The tractors were moving along the roads in the direction indicated by the arrows. The velocity of all the tractors was $v_0 = 30\,\mathrm{km/h}$. For sketch (a), the direction of wind is indicated by another arrow. When solving the problem, you may draw lines and measure distances using a ruler.

i) Using the provided sketch, find the wind speed for case (a).
ii) Using the provided sketch, find the wind speed for case (b).



[17] when water depth is noticeably smaller than the wavelength

pr 64. The following snapshot (a larger version is on an extra sheet) depicts two balls that were thrown simultaneously and with the same initial speed, but in different directions from point P. What was the initial speed? Use $g = 9.8\,\mathrm{m/s^2}$.



pr 65. A boat travelled from its home port to an island at the distance of $l = 4\,\mathrm{km}$ directly towards south. Its trajectory consisted of three straight segments the directions of which were not recorded. During each of the segments, the boat maintained a constant velocity; however, a different speed was maintained for diffe- rent segments. During the travel time, wind speed and direction was measured from the boat. The travel time on the first segment was $t_1 = 3\,\mathrm{min}$, the measured wind speed was $v_1 = 15\,\mathrm{m/s}$ and the wind blew directly from east. The travel time on the first segment was $t_2 = 1.5\,\mathrm{min}$, the measured wind speed was $v_2 = 10\,\mathrm{m/s}$ and the wind blew directly from southeast. The travel time on the first segment was $t_3 = 1.5\,\mathrm{min}$, the measured wind speed was $v_3 = 5\,\mathrm{m/s}$ and the wind blew directly from southwest. What was the wind speed? It is known that the wind speed and direction were constant during all the travel time.

pr 66. There is a long chute of constant slope angle along which balls can slide frictionlessly (the chute is narrow so that the motion of the balls is essentially one-dimensional). Let there be N identical perfectly elastic balls sliding on that chute. The total number of pair-wise collisions between the balls in the chute depends on their initial velocities and positions. What is the largest possible number of collisions? (If you don't know how will move two absolutely elastic balls after a collision, check the hints section.)

8 HINTS

1. In the water's frame of reference, it is clear that departing from the barge and returning to it took exactly the same amount of time.
2. In the frame of the red plane, the blue plane moves along a line s which forms an angle $\alpha = \arctan\frac{3}{4}$ with the horizontal dashed line in the figure. The distance of the red plane from this line is most conveniently found considering two similar right triangles, the larger of which is formed by the line s, and the two dashed lines in the figure.
3. Use the frame which moves with the speed $\frac{v}{2}$ to find the horizontal component of the intersection point's velocity. Now, in the lab frame, we know the direction of this velocity, as well as the horizontal projection; apply the idea 4.
4. Since the ascending speed is constant, it is sufficient to calculate it for a single position of the balloon — when its height is small and hence, the horizontal velocity is almost zero; apply the idea 5. For $t = 7\,\mathrm{min}$, the angular ascending speed is zero, hence the balloon needs to move along the line connecting the balloon and the observation point; apply the idea 4.
5. In the boards frame of reference, there is only horizontal force (the friction force) has a constant direction, antiparallel to the velocity.
6. According to the idea 7, we use the conveyor's frame, but as we are asked about the speed in lab frame, we need to switch back to the lab frame. In the conveyor's frame, the velocity vector becomes shorter while preserving the direction, i.e. can be represented as $\vec{w} = k\vec{w}_0$, where its initial value $\vec{w}_0 = \vec{v}_0 - \vec{u}_0$ and the factor k takes numerical values from 0 to 1. Hence, the velocity in the lab frame $\vec{v} = \vec{u}_0 + k\vec{w}_0$: this is a vector connecting the right angle of the right triangle defined by its catheti $\vec{u}_0$ and $\vec{v}_0$ with a point on the hypotenuse; the specific position of this point depends on the value of the factor k (which is a function of time).
7. Express the lateral displacement of the ball as the sum of two components: lateral displacement in the air's frame of reference (a trigonometrical task; this does not depend on t), and the lateral displacement of the moving frame.
8. Algebraic approach: take one of the axes (say x) to be perpendicular to the racket's plane and the other one (y) parallel to it. Absence of rotation means that the y-components of the ball's and racket's velocities are equal, $u_y = v_y$, and there is no parallel force acting on the ball, hence $v_y' = v_y$. Using idea 13 we find that $v_x' = -v_x + 2u_x$. Applying idea 15 to the vectors $\vec{v}$ and $\vec{v}'$ gives us an equation for finding u_x; apply Pythagoras' theorem obtain $|\vec{u}|$. To find angle β, express $\tan\beta = u_y/u_x$.

   Geometric approach: draw a right trapezoid as follows: we decompose $\vec{v}$ into parallel and perpendicular components, $\vec{v} = \vec{v}_x + \vec{v}_y$; let us mark points A, B and C so that $\vec{AB} = \vec{v}_x$ and $\vec{BC} = \vec{v}_y$ (then, $\vec{AC} = \vec{v}$). Next we mark points D, E and F so that $\vec{CD} = \vec{v}_y' = \vec{v}_y$, $\vec{DE} = -\vec{v}_x$, and $\vec{EF} = 2\vec{u}_x$; then, $\vec{CF} = \vec{v}_y' - \vec{v}_x + 2\vec{u}_x \equiv \vec{v}'$ and $\vec{AF} = 2\vec{v}_y + 2\vec{u}_x \equiv 2\vec{u}$. Due to the problem conditions, $\angle ACF = 90°$. Let us also mark point G as the centre of AF; then, GC is both the median of the right trapezoid $ABDF$ (and hence, parallel to AB and the x-axis), and the median of the triangle ACF. What is left to do, is expressing the hypotenuse of $\triangle ACF$ in terms of $v = |AC|$, and apply the idea 16.

9. Use the frame where the mirror is at rest: the source S rotates with angular velocity $-\omega$. Now go back to the lab frame and find the angular velocity of the image in that frame.

10. The area under the graph, from $t = 0$ until the given moment, gives the displacement, and regions below the t-axis make negative contributions. Hence, if a certain moment t corresponds to a maximal displacement then $v(t) = 0$ (otherwise, the displacement could be increased somewhat by making t slightly smaller or larger, depending on the sign of $v(t)$. (Alternatively, we can say that extrema correspond to zero derivative, and the derivative of Δx is v.) The same arguments lead us to the conclusion that for slightly smaller t-values we must have positive v, and for slightly larger t-values — negative v. Hence, the candidates are $t = 4.7\,\mathrm{s}$, $t = 7\,\mathrm{s}$, $t = 12.5\,\mathrm{s}$, and $t = 18.3\,\mathrm{s}$. Calculate the surface areas to see, which of them maximizes the displacement.

11. Let us divide the displacement into small pieces, $s = \sum \Delta s$, where $\Delta s = v\Delta t$. If the function $v(t)$ were known, the last formula would have been completed our task, because $\sum v(t)\Delta t$ is the sum of rectangles making up the area under the $v - t$-graph . However, the acceleration is given to us as a function of v, hence we need to substitute Δt with Δv. While trying to do that, we can introduce the acceleration (which is given to as a function of v):

$$\Delta t = \Delta v \cdot \frac{\Delta t}{\Delta v} = \frac{\Delta v}{\Delta v / \Delta t} = \frac{\Delta v}{a}.$$

This result serves us perfectly well:

$$s = \sum v\Delta t = \sum \frac{v}{a}\Delta v \to \int \frac{v}{a(v)} dv,$$

i.e. the displacement equals to the surface area under a graph which depicts $\frac{v}{a(v)}$ as a function of v.

12. Use the frame of one of the sliding balls — according to the acceleration addition rule, the other one moves with a constant horizontal acceleration. Apply the idea 2 to find the position where the distance is minimal. Express the answer in terms of the distance AB; apply the idea 10 for finding the distance AB.

13. If the axis x is parallel to the plane (and points downwards), the ball performs a free fall along x-axis, the acceleration being equal $g \sin\alpha$; if the axis y is perpendicular to the plane, the ball bounces along the y-axis up and down, with the free fall acceleration $g\cos\alpha$.

14. Use perpendicular coordinates so that the x-axis is along the contact line of the two surfaces and y-axis lays on the inclined surface; then, motion in the x-direction is independent of the motion in $y - z$-plane. Due to the idea 21, the speed remains constant throughout the transition from one plane to the other. Use fact 3 in conjunction with the provided graph to figure out the value of g_y, the projection of the free fall acceleration to the y-axis. Now, since we know the full free fall acceleration $g = 9.8\,\mathrm{m/s^2}$, the relationship $g_y = g\sin\alpha$ allows us to find the angle α.

15. According to the first method, we use the frame which co-rotates with the turtles, so that in the new frame, the turtles move radially towards the centre; when applying the rule of velocity addition, for each of the turtles we need to use the local velocity vector of the rotating frame at the location of the particular turtle. According to the second method, at each moment, let us project the velocities of two turtles on the straight line connecting them — that way we can find the rate of decrease in the distance between two turtles.

16. Write down the relationship between small increments: $dk = v \cdot dt/(L + ut)$; the answer is obtained upon integration of the left- and right-hand parts of this equality.

17. Apparently the boy will need to arrive to the coast OM at the right angle, c.f. idea 2. Now we can apply the idea 25 together with the fact 5 to the refraction of his trajectory at the coast OP to find the angle at which he needs to arrive to the coast OP (the answers are expressed straightforwardly in terms of this angle).

18. *First approach:* use the water's frame of reference, because then the swimming speed is independent of the swimming direction. In that frame, the boy moves on the coast with the speed equal to $v + w$. It is clear that the boy needs to start immediately swimming, i.e. it doesn't matter weather the point A is moving or motionless in the new frame. It is also clear that if we have found the fastest way of reaching the point C, the same trajectory would give us the fastest way of reaching any other upstream point on the same coast; in particular, we can take a point C' which moves together with the water (is motionless in the water's frame), and the optimal trajectory would still remain the same. With these modifications, we have a problem where we can apply the Fermat' principle; the resulting geometrical optics problem is essentially the problem of finding the angle of total internal reflection. For the *second approach*, once the front meets the point A, the front forms a cathetus of a right triangle APQ, where Q is the point where the front meets the riverbank, and P is the position of that Huygens source on the riverbank which creates the circular wave meeting the point A. Notice that the point P is the point where the boy needs to start swimming in the water's frame of reference, and is displaced by wT from the corresponding point in the lab frame.

19. Once you write down trajectory parametrically, $x = x(t)$ and $z = z(t)$, time t can be eliminated; as a result, you obtain an equation relating x, y, and the shooting angle α to each other, which we consider as an equation for finding angle α. The angle enters into this equation via two terms, one containing $\tan\alpha$, and the other — $\cos^{-2}\alpha$. In order to solve such equations, one possibility is to express all trigonometric functions via a single one. It is possible to express $\tan\alpha$ via $\cos\alpha$, but that involves a square root, which is inconvenient. Meanwhile, $\cos^{-2}\alpha$ can be nicely expressed in terms of $\tan\alpha$, and so, this is the way to go; as a result, you'll obtain a quadratic equation for $\tan\alpha$.

20. Use the fact that all vertical rays are reflected by the range boundary so that they will pass through the focus. It is enough to find the intersection point of two rays. Take one ray at $x = 0$, and the other one such that it will hit the range boundary at the level $z = 0$ (we know the tangent of the range boundary at that point due to the fact 8).

21. Step 1, proof by contradiction: if the trajectory doesn't touch neither of the edges, the throwing speed can obviously be reduced slightly while keeping the angle constant. If it touches only one edge, let it be the farther edge, the boy can step slightly forward so that now the trajectory doesn't touch neither of the edges. Step 2: from the law $v(z)^2 - 2gz =$ cosnt, we conclude that the speed at the height $z = c$ [denoted as $v(c)$] is a uniquely defined growing function of the speed at the ground level $v(0)$: if the speed $v(0)$ is minimal then $v(c)$ is also minimal. Step 3, brute force approach: find the minimal throwing speed from the right edge F of the roof by requiring that the other edge with coordinates $P = (a-c, \sqrt{b^2-(a-c)^2}$ belongs to the parabola found in problem 19. *Geometric approach (idea 30:* according to the idea 28 (keep in mind the facts 6 and 7), when throwing optimally from the point F to point P, the point P belongs to the envelope-parabola (which separates the region where the targets can be hit with the given speed, see idea 28), and F is its focus; let the respective minimal throwing speed be u [recall that this is the speed at the altitude of the point F, i.e. $u = v(c)$]. According to the idea 28, when throwing from F, any point of the envelope-parabola can be reached with the same speed u if we adjust the throwing angle accordingly; in order to reach the tip Q of the envelope-parabola we need to throw straight up. Now it would be a trivial task to find the throwing speed u from the energy conservation law if the height $h = |FQ|$ of the envelope-parabola were known. To obtain h, we use the property of a parabola (see fact 9): the sum of the distance of a point from the focus and the point's height is constant: $b + (a-b) = h + h$.

22. Due to the idea 28, together with facts 6 ,7, and 9, a vertical ray directed at the target is reflected by the projectile's trajectory to the focus, i.e. to the cannon. When making use of the idea 26 we see that this projectile's trajectory is also optimal for shooting the cannon's position from the location of the target; hence, the projectile's trajectory reflects a vertical ray directed to the cannon towards the target. If we combine these two observations we see that a vertical ray directed to the cannon is rotated after two reflections from the trajectory by 180°, which means that the reflecting surfaces must have been perpendicular to each other (showing this mathematically is left to the reader as a simple geometrical task).

For the alternative solution, we consider the motion of two projectiles of initial velocities $\vec{v}$ and $\vec{v}' = \vec{v} + \Delta\vec{v}$ as suggested above, i.e. with $|\Delta\vec{v}| \ll |\vec{v}|$, $\Delta\vec{v} \perp \vec{v}$ and $|\vec{v}| \approx |\vec{v}'|$. In the free-falling frame, they depart at a constant velocity $\Delta\vec{v}$, the relative velocity which was given at the beginning. Hence, at the destination, the displacement vector between the projectiles $\Delta\vec{r} = t\Delta\vec{v}$ (t being the flight time) which is perpendicular to the initial velocity $\vec{v}$. On the other hand, we can apply the fact 10 by considering the dependence of the x-coordinate of the projectile at the target's altitude as a function of the launching angle. Hence, we conclude that while the first projectile hits the target exactly, the other one must also come very close to it (at the target's level, $\vec{v}$ gives the optimal shooting angle, and hence, $\vec{v}'$ is nearly optimal): if one of them is at the target, the other one will be there very soon[18]. Recall that when the first projectile is at the target, the other one is displaced by $t\Delta\vec{v}$ from it, i.e. its current velocity $\vec{u} \parallel t\Delta\vec{v} \perp \vec{v}$.

23. Notice that all those points which have the same speed lie on the same distance from the instantaneous rotation center.

24. Equate the acceleration calculated in two inertial frames: in the lab frame, and in the frame of the wheel's centre.

25. a) First we use the idea 35: when applying it to the left rod we conclude that the joint's velocity is vertical, and when applying the idea to the right rod we conclude that the modulus is v. Now we can apply the fact 13 to calculate the horizontal projection of the joint's acceleration. In order to use the idea 34, we need to know the direction of the acceleration, as well. This knowledge is obtained if we switch to the frame moving up with constant speed v: the joint's speed is zero and hence, the centripetal acceleration is zero. b) Notice that in the frame moving up with constant speed v, the question b) is the same as mirror-reflected question a).

26. First method: find the instantaneous centre of rotation by figuring out the direction of velocities of for two points of the cylinder: first, point A where the rope meets the cylinder (notice that the velocity of A as a rope's point equals to the velocity of A as a cylinder's point, and apply the idea 35 to the rope), and point B where the cylinder meets the plate (what is the vertical component of the velocity of the cylinder's point B?).

Second method: make a drawing with two close positions of the cylinder and rope: let us mark on the left position of the rope a point P where the rope meets the cylinder, and on its right position — point P' which is at the same height as point P. Let us denote the point where the rope is fixed to the wall by Q. Then, the rope segment QP' consists of a straight segment QP'' and a curved segment $P''P'$. However, since the displacement of the cylinder PP' is small, the length of the curve $QP''P'$ has almost the same length as the straight line QP'. While the actual unwound length $|QP| - |QP''|$ is contributed by two rotations (rotation of the cylinder and rotation of the rope), the length difference $|QP| - |QP'|$ is contributed only by the cylinders rotation (the point P' is at the same relative position on the cylinder as the point P), and is, hence, equal to $\omega R\Delta t$. On the other hand, we can express this length trigonometrically in terms of the cylinder's displacement PP'.

27. Let us take y to be the coordinate of the upper end and x that of the lower end. Then the rod has length $l^2 = x^2 + y^2$; l is constant, so its derivative must be zero. Let us take the time derivative of the whole expression, using the chain rule of differentiation known from math-

[18] More precisely, the closest distance will be quadratically small; meanwhile, the displacement vector $t\Delta\vec{v}$ is linearly small, i.e. much larger.

ematics: $0 = x\dot{x} + y\dot{y} = xu + yv$ (a dot on top of a symbol means its time derivative). From that, we can express $u = -vy/x = -v\tan\alpha$.

28. Go to the frame which moves with speed u and where boundary between the queue and moving cars is stationary; equate the flux of cars (how many cars is passing in a unit time) in the region of queue to its value in the region of moving cars.

29. The intersection point P of two trails corresponds to the moment when the heads of the trains met: it was carried by wind to where it currently is. So, based on the speeds of trains we find their meeting point Q; since the speeds are equal, this is the middle point of the segment AB connecting the current positions of the train heads. The segment AB was covered during the given time interval at the speed $2v = 100\,\text{km/h}$; this value can be used as a scale for finding the wind speed based on the length of segment QP (you need to measure $|AB|$ and $|PQ|$).

30. First we conclude, based on the two collisions, that all the bodies move on the same plane, henceforth the (x, y)-plane. According to the idea 41, we plot the trajectories of the bodies in a 3D plot (as lines $x = x_a(t)$, $y = y_a(t)$; $x = x_b(t)$, $y = y_b(t)$; $x = x_c(t)$, $y = y_c(t)$. Collisions correspond to intersections of these lines, and intersection of two lines means that the two lines are coplanar (lie on a single plane).

31. In the free-falling frame, all the particles move with constant velocities; each particle had initial velocity equal to the wheel's velocity at the releasing point, i.e. tangential to the wheel and equal by modulus to ΩR. Hence the ensemble of particle expands as a circle, the radius of which can be calculated from the Pythagorean theorem. In the lab frame, the centre of the circle performs a free fall. A droplet reaching the point A corresponds to the expanding circle touching the ground.

32. Useful observations: each column of pixels is obtained very fast, essentially simultaneously; so, each column of pixels represents a vertical cut of the real object at the respective position for a certain moment of time. However, different columns correspond to different time. Each vertical curved shape on the photo is a sequence of vertical cuts and hence must correspond to the same blade of the propeller. At the upper half of the photo, the scanning ling moves towards the motion of blades, and none of the blades is missed: if we number the blades by 1, 2 and 3 then the blade sequence at the upper half must be 1, 2, 3, 1, 2, ... In lower half of the photo, the scanning line and blades move in the same direction, and the blades are faster, hence here the sequence is 3, 2, 1, 3 . . . . The scanning line moves apparently with a constant speed, hence the horizontal position on the photo can be used to measure the time.

33. i) While the grey combs moves by half of the teeth pitch, a dark stripe moves to where currently there is a white stripe, i.e. by half of the dark stripe distance. Hence, the stripe speed is as many times faster as many grey teeth can be counted per one stripe period.
ii) The product of two waves can be expanded as $a = \sin(\vec{k}_1\vec{r} - \omega t)\sin(\vec{k}_2\vec{r}) = \frac{1}{2}\{\cos[(\vec{k}_1 - \vec{k}_2)\vec{r} - \omega t] - \cos[(\vec{k}_1 + \vec{k}_2)\vec{r} - \omega t]\}$. The bands where $a \approx 1$ are where the "slow" sinusoid is close to one, $\cos[(\vec{k}_1 - \vec{k}_2)\vec{r} - \omega t] \approx 1$; this is a sinusoidal wave which moves with speed $u = \omega/|\vec{k}_1 - \vec{k}_2|$.

34. Let us consider the evolution of the vector $\vec{r}$ pointing from the dog to the fox; calculate the changing rates of the modulus $|\vec{r}|$, and of r_x, the projection of $\vec{r}$ to the x-axis (taken parallel to the fox's velocity). Is it possible to make such a linear combination of r_x and $|\vec{r}|$ that its time derivative would be zero? If yes, we would obtain a new conservation law which could be used to obtain immediately the answer.

35. Use the vector addition rule to draw a rectangle of velocities $\vec{u} + \vec{v} = \vec{w}$, where $\vec{w}$ is the velocity of the boy relative to the coast. Apply the idea 2: if we fix the position of the vector $\vec{u}$ and draw the possible triangles for different swimming directions, we'll see that the possible positions of the endpoint of $\vec{v}$ lie on a circle. Now it is not difficult to conclude that the optimal swimming corresponds to $\vec{v}$ being tangent to the circle.

36. Introduce coordinates x (the vertical position of the ring O) and y (the vertical position of the ring O'), with the origins at A and A', respectively. Also introduce the rope length L (to be eliminated later from the answer). Relate these quantities to each other via Pythagorean theorem (keep things squared to avoid square roots for easier manipulation) and apply the idea 38.

37. Apply idea 20: vertical motion is independent from the motion in the horizontal plane. The ball can escape the well if it hits the upper rim of the well. This will happen if the period of vertical motion relates to the time between two collisions with the well's walls for the horizontal motion as a rational number (ratio of two integers).

38. Use the idea 1: switch to the frame of the wedge; determine there the acceleration of the ball; apply the rule for the addition of accelerations to find the acceleration of the ball in the lab frame. Once knowing the acceleration and initial velocity, the trajectory can be also found easily.

39. Apply the fact 12 to calculate the acceleration — the time derivative of the velocity vector. The angular speed of the rotation of the velocity vector can be found using the idea 37.

40. Apply the idea 1: use the cone's (free-falling) frame where the table's corner moves upwards with the acceleration g.

41. Apply the idea 24 (work with polar coordinates r and φ). Express the squared curve length increment dl^2 via the squared increments $d\varphi^2$ and dr^2 (use the Pythagorean theorem); and relate dr^2 to dl^2 via the propagation speeds. Note that differential equations in the form $kdx = xdy$ (where k is a constant) can be solved by separating variables, i.e. bringing all the x-s and y-s to the respective sides of equality (here, $k\frac{dx}{x} = dy$), and integrating left-hand right-hand-sides of the equality (which leads here to

[19] It is nicer to keep $\ln(x/x_0)$ instead of $\ln(x) - \ln(x_0)$ as otherwise it would be difficult to check if the dimensionalities of the expressions are OK.

$k\ln(x/x_0) = y$, where x_0 is a constant emerging when taking the indefinite integrals[19].

42. Denoting the diagonal length via d, relate the distance of A_3 from the wall to d; express the time derivative of d in terms of v (c.f. idea 38 on Pg 12), and use this result to find the speeds of A_1 and A_2. Use the idea 1 (switch to the frame of the centre of the largest rhombus) to find the direction of the acceleration of B_2. Apply the idea 34 to deduce the modulus of the acceleration: switch to the frame where A_2 (or A_3) is at rest and the point B_3 moves along a circle, and determine the centripetal acceleration — the projection of the whole acceleration to the leg A_2B_3.

43. Apply the ideas 1 (use the frame of one of the boats), and the idea 2. You need the angle between the line AB and the relative velocity; note that the tangent of that angle can be expressed quite easily.

44. Apply the idea 33: you can find the direction of velocities for those two points of disc where the ropes are tangent to the disc. Indeed, keep in mind the idea 35 and notice that the upper ends of the ropes have zero velocities (where in contact, the rope and disc points have equal velocities).

45. Apply the idea 20: the motion along each of the boards are independent. Use the fact 4 (the free fall acceleration components will be $g\cos\alpha$ and $g\sin\alpha$); calculate the jumping periods for each of the motions.

46. Apply the idea 21 to conclude that the speed of the block remains constant. Express the angular speed ω of the unwound part of the rope in terms of its current length l; note that ω is also equal to the angular speed of the point P where the rope is tangent to the cylinder; relate this speed to the rate at which the rope is unwound, $\frac{dl}{dt}$. Apply the provided formula to conclude that $\frac{d(l^2)}{dt}$ remains constant during unwinding and winding. Don't forget that there is also a period when the rope is fully unwound and the block draws a semicircle.

47. Apply idea 35 to determine the projections of the box's velocity $\vec{v}$ to the directions of the ropes. Using the idea 16 one can conclude that the quadrilateral formed by the vector $\vec{v}$ as a diagonal and the projections of $\vec{v}$ as its two sides is a cyclic one whereas $\vec{v}$ is the diameter of its circumcircle. Apply the cosine theorem to determine the length of the other diagonal, and the sine theorem to determine $|\vec{v}|$, the diameter of the circumcircle.

48. First approach: apply the idea 1: use the inertial frame which moves with the initial velocity of the boy; while the acceleration of the boy is fixed by modulus, the direction can be adjusted as needed. Since in this frame, the initial velocity is zero, the optimizations task becomes quite trivial. Second approach: study the evolution of the boy's velocity vector $\vec{v}$ in (v_x, v_y)-plane: acceleration $\vec{a} = \frac{d\vec{v}}{dt}$ is constant by modulus, hence the endpoint of $\vec{v}$ moves with a constant speed $|\vec{a}|$ from its initial position $(0, v)$ to the final position $(v, 0)$.

49. There are at least three different solutions; the simplest one is based on the idea 30, and is quite similar to the geometric solution of the problem 21. All the solutions start in the same way as the ones of problem 30 by (a) showing that the optimal trajectory needs to touch the sphere at a certain point P before hitting the sphere at its top T, and (b) applying the idea 26 to consider throwing from T instead of throwing from an unknown point at the ground level.

The first method makes use of the geometric properties of the envelope parabola (see ideas 28 and 30, and facts 6, 7 and 9): $|TP| + |PA| = 2|TQ| + R$, where PA is a vertical line, A lies at the same height as the sphere's centre O, and Q is the tip of the envelope parabola. As the shooting speed is defined by $|TQ|$ (see the hints of problem 21 for more details), we only need calculate $|TP|$ and $|PA|$. Since TP is a line coming from the focus, it is reflected by the parabola at P to a vertical line; since parabola is tangent to the sphere, it is also reflected by the sphere to a vertical line which means that $\angle TPO = \angle OPA$. Since $TO \parallel PA$ we can conclude that $\angle TOP = \angle OPA$, and as $\triangle TOP$ is isosceles, $\angle OTP = \angle TPO$; together with equality $\angle TPO + \angle TOP + \angle OTP = 180°$ we conclude that $\angle OTP = 60°$, which gives us immediately $|TP| = R$ and $|PA| = \frac{R}{2}$.

The second method makes use of the expression for the envelope parabola (idea 28) with a focus at T: write down equation for finding the intersection points of the sphere and the parabola — this will be a biquadratic equation. According to the fact 7, in the optimal case the parabolic trajectory, envelope parabola, and sphere are tangent to each other at the same point; hence, at this optimal case there are exactly two symmetric solutions to the biquadratic equation (if the launching speed is smaller than optimal then there are 4 solutions, and if it is larger then there are no solutions). Apply idea 29 to find the speed.

The third solution makes use of the result of the problem 22 and idea 24: if we use $\varphi = \angle TOP$ as the parameter, we know that from the point T, the optimal shooting angle must be equal to $90° - \varphi$. Hence, we can write down the conditions that the trajectory goes through P and the final, and initial and final velocities are perpendicular (idea 15). This gives us three equations containing φ, touching time t, and initial speed v as unknowns. It appears that this system of equations simplifies nicely.

50. It can be seen that the red dots are repeated at the same places of the blue cycloid for each of its periods, and there are three dots per period; according to the idea 6, we'll make use of this fact (in generic case it considerably longer calculations would be needed) to conclude that the disk's rotation period is three times longer than the flashing period, i.e. 0.3 s. The distance travelled corresponds to the period length of ca 4 grid units; the value of a grid unit can be found by noticing that the cycloid height equals to $2a$.

51. i) v_{dist} is always negative, hence the overall, the distance decreased, hence the cars must have started from different cities. O could not have been the starting point of a car because in that case the final distance would have been the same or larger than at the beginning. Now we just need

to consider three possibilities, out of which two are easily excluded based on the behaviour for $0 < t < \frac{a}{v}$.
ii) Use the idea 18.
iii) and iv) Calculate relative velocity according to the vector subtraction rule and project it to the line connecting the cars for different positions on their path.

52. Approach the same way as in the case of problem 3: use the frame which rotates with the angular speed $\frac{\omega}{2}$ to obtain the angular speed of the line OP, P being the intersection point. Consider isosceles triangle QOP (where Q is the centre of the blue ring) to obtain the angular speed of the line QP.

53. Note that for a spike, such a point P appears sharp for which velocity is parallel to the spike. Apply the idea 33: the instantaneous rotation centre is the lowest point G of the wheel (as it is contact with the ground and has thus zero speed). Thus, GP must be perpendicular to the spike; recall now the idea 16.

54. i) Use the fact that orange dot is missing; the exposure time must be appropriate for capturing exactly four dots.
ii) Use the idea 1: in the frame of the disk's centre, the displacement vector $\vec{d}$ between neighbouring flashes has always the same modulus $d = 2R\sin(\omega\tau/2)$, and neighbouring displacement vectors are always rotated by the same angle $\omega\tau$. In the lab frame, additional constant displacement vector $\vec{v}\tau$ is to be added due to the translational motion of the frame: $\vec{d'} = \vec{d} + \vec{v}\tau$. Because of that, if we bring all the displacement vectors to such positions that their starting points coincide, the endpoints will lie on a circle of radius d. So, we redraw the displacement vectors $\vec{br}$, $\vec{rg}$, and $\vec{gy}$, and draw the circumcircle of the triangle formed by the endpoints of the vectors; from that figure we can measure both the rotation angle $\omega\tau$ (for finding ω), constant displacement $a = v\tau$ (for finding v), and the circle's radius $d = 2R\sin(\omega\tau/2)$ (for finding R).

55. For the parabolic shape $y = kx^2$ of the jet, the factor k can be determined from the figure; this relates the unknown grid unit d, the initial speed v of the jet, and $g = 9.81\,\mathrm{m/s^2}$ to each other. The idea 39 relates the flow rate $\frac{\pi}{4}d^2v$ to the vessel filling rate; we have two equations and two unknown parameters, so the system of equations can be solved.

56. Using the idea 1 we can conclude that all the litter must lie on the same line with that boat from which these fell: this allows us to conclude, which litter corresponds to which boat. Using the idea 40 we mark the point where the boats met — the intersection point of two trails. The distance of the boats from that point is proportional to the speed of the boats. The distance of A from the point where the trail of the boat intersects with the coastline gives the distance carried by the water flow (the frame displacement); from that moment when the boats met, the frame displacement was smaller, and can be found geometrically from similar triangles formed by the following lines: the trail, the line connecting the boat with its starting point, coastline, and a parallel line to the coast, drawn through the intersection point of the trails. Analogous construction of similar triangles for the other boat will complete the task.

57. Apply the idea 39: consider the number of cars passing a given point in unit time (the car frequency), and equate the sum of car frequencies of the two merging lanes with the frequency for the lane C; this will give you the speed on the lane C (the rest of the calculations are straightforward).

58. Apply the idea 39: introduce the density ρ of water in air; then, the water flux (mass per unit time) is given by $S\rho v$, where S is the cross-sectional area, and v is the speed of the man relative to the rain droplets. For a rectangular man we calculate the total flux in two parts — water falling on the horizontal surface due to the vertical velocity of the rain droplets, and water falling on the vertical surface due to the motion of the man. For a spherical man, we need to calculate the relative speed by using the triangle rule for adding the velocities. In either case we obtain a system of equations from which the surface areas and wind's speed can be eliminated.

59. The length of the trails is defined by the time interval during which the droplet's image remains within the gap between the curtains. This, in turn, is inversely proportional to that component of the image's relative velocity which is perpendicular to the curtain's edge. In one case, the velocity of the curtains $\vec{v}$ and the velocity of the droplet's image $\vec{u}$ are parallel, in the other case — antiparallel, and in the third case — perpendicular. In the antiparallel case there are two possibilities: we don't know which is faster, the curtain or the image. While formally we do have three unknown quantities, v, u, and the gap's width d, these enter the equations only in two combinations, v/d and u/d, i.e. we have essentially only two unknown parameters. Therefore, the expressions for l_1 and l_2 can be solved with respect to v/d and u/d, which are further used to calculate the value of l_3.

60. For small circles, the radii should form a sequence $r_n = \frac{g}{2\pi}n^2\tau^2$ (with $n = 1, 2, \ldots$); for larger rings we should have $r_{n+1} - r_n = \sqrt{hg}\tau$. By taking measurements from the figure, we can determine both $\frac{g}{2\pi}n^2\tau^2$ and $\sqrt{hg}\tau$ as a product of L and a certain numerical value; this gives us two equations from where we can eliminate τ and express h.

61. Apply the idea 27 and construct the wave front behind the boat. Note that locally, the front propagates perpendicularly to itself, so that the first to arrive to the coast is that part of the wave front which was initially (i.e. at the point of creation behind the boat) parallel to the coast. We can also conclude by using the Huygens principle that close to the boat, the wavefront forms angle $\arcsin\frac{u}{v}$ with the boat's trajectory.

62. Apply the idea 1: use the water's frame of reference. We assume that the water moves as a whole, across the whole depth, hence in the water's frame, the wave speed is the same for all the propagation directions. Thus we can conclude that in the water's frame, the boat's velocity is parallel to the bisector of the angle formed by the waves. We know that in lab frame, the boat moves parallel to the line AB, hence we can deduce the water speed from the triangle of velocities using the known value of the boat's speed. From the Huygens principle we know that the angle between the bisector and wave front is arcsin $\frac{v}{w}$, hence we can measure the angle to find w and calculate the depth $h = w^2/g$.

63. i) Use the idea 1: in the air's frame, the trail is parallel to the relative velocity of the tractor. Hence, $\vec{v}_{\text{tractor}} - \vec{v}_{\text{wind}}$ is parallel to the trail; these two vectors form a triangle which can be easily constructed (we know the directions of its two sides, and one length), from where the length of $\vec{v}_{\text{wind}}$ can be measured.
ii) Following the idea 40 we'd like to make use of the intersection point of the trails. However, the tractors didn't meet. So, we need to draw one more trail which would have been observed if the tractors were meeting at the road crossing (shift the left tractor together with its trail appropriately). Then, the wind speed can be related immediately to the distance of the trails intersection point from the road crossing (you just need to compare it with the distance of the tractors from the crossing).

64. The main idea is to consider motion in a free-falling frame. More specifically, in the free-falling frame, the balls move with constant velocities, hence are at the base vertices B and C of an isosceles triangle. In that frame, the throwing point is the top vertex A of the isosceles triangle. The current position of the point A in the lab frame can be found by constructing the isosceles triangle. The point A has fallen in the lab frame from point P with free fall acceleration, and the falling time can be found by measuring the falling distance $|AP|$. The velocities can be found by measuring the flight distance $|AB|$ (or $|AC|$) in the falling frame.

65. Consider the motion of the boat in the air's frame of reference. More specifically, based on wind data, calculate the displacement vector (i.e. the displacement in east-west and the displacement in north-south directions); knowing the displacement vector in lab frame, calculate the displacement vector of air, and based on that — its speed.

66. To begin with, you need to know that (as it follows from the energy and momentum conservation laws) if two absolutely elastic balls of equal mass collide centrally while moving along the same line, they will exchange their velocities: the ball A departs with the initial velocity of the ball B and vice versa.

Next, there are two main ideas which need to be applied: first, consider the motion of balls in a free-sliding frame where all the balls move with constant speeds. The second one is the idea 41: add a time axis and study the graphs. You'll notice that the $x-t$ graph consists of N intersecting lines with the intersection points corresponding to collisions. The number of intersections is found as the number of different possibilities of picking 2 lines out of the set of N lines.

9 ANSWERS

1. $v_r = 4\,\text{km/h}$, $v_b = 16\,\text{km/h}$.

2. 4 km.

3. $u = v/2\sqrt{1-(a/2r)^2}$.

4. Ascending velocity 4.85 m/s; $h = 2000\,\text{m}$; wind velocity $u = 2.8\,\text{m/s}$.

5. Straight line.

6. $2/\sqrt{5}$ m/s

7. $t = \frac{s}{u} + \frac{L}{v\cos\alpha} = 1.8\,\text{s}$

8. $u = v/2\cos\alpha$; $\beta = 180° - 2\alpha$

9. $v(t) \equiv 2\omega a$.

10. 18.75 m

11. 39 m

12. $t = \sqrt{(t_1^2 - t_2^2)/2}$

13. $8d\tan\alpha$

14. $\alpha = \arcsin 0.5 = 30°$

15. 6.7 s

16. $e^{100} - 1$ seconds

17. $x = \cos\alpha(l - h\tan\beta)$ and $t = \frac{h\cos\beta}{v} + \frac{l\sin\alpha}{u}$, where $\beta = \arcsin(v\sin\alpha/u)$, if $\tan\beta < l/h$; otherwise $x = 0$ and $t = \sqrt{h^2+l^2}/v$.

18. $x = a\left(\frac{w}{u\cos\alpha} - \tan\alpha\right)$, where $\alpha = \arcsin\left(\frac{u}{w+v}\right)$.

19. $z \le \frac{v_0^2}{2g} - \frac{gx^2}{2v_0^2}$

20.

21. $v_{min} = \sqrt{g(a+b+c)}$

22.

23. Concentric arcs of radii l_1 and l_2, where l_1 and l_2 are the distances of the instantaneous rotation centre from the top and bottom plates, respectively.

24. $r = 4R$

25. $a_1 = v_0^2/\sqrt{3}l$; $a_2 = v_0^2/\sqrt{3}l$

26. $v_0 = v/(1+\sin\alpha)$

27. $u = -v\tan\alpha$, $a = \frac{v^2}{l\cos^3\alpha}$.

28. $u = \frac{v}{v\tau/l - 1} \approx 3.4\,\text{m/s}$.

29. $u \approx 15\,\text{km/h} \approx 4.2\,\text{m/s}$, $\alpha \approx 27°$.

30. yes

31. $t = 2\sqrt{\frac{R}{g}\left(1 + \frac{R\Omega^2}{g}\right)}$ and $\alpha = \arctan(\Omega t)$.

32. counterclockwise; 3; 15 Hz

33. 7 cm/s; $\frac{1\,\text{cm/s}}{\sqrt{\alpha^{-2}+1/49}} \approx 5.7\,\text{cm/s}$.

34. $l/2$.

35. $L\sqrt{3}$.

36. $v_0 = v\left(\frac{1}{\cos\alpha} - 1\right)$, $a = \frac{v^2}{b}\tan^3\alpha$.

37. $nv_0\sqrt{2H/g} = mR\cos\alpha$ with integer n and m

38. $a = 2a_0\sin(\alpha/2)$

39. $a = v_1 v_2/l$

40. $v \ge \sqrt{r^2 g/2h}$

41. a logarithmic spiral $\sqrt{(\frac{v}{c})^2 - 1}\ln\frac{r}{r_0} = \varphi$.

42. $v_0/6$, $v_0/2$, $v_0\sqrt{5}/6$, $\sqrt{2}v_0^2/36l$

43. $l\sin\phi$, where $\tan\phi = |v_1\sin\beta - v_2\sin\alpha|/|v_2\cos\beta - v_1\cos\alpha|$

44. $\omega R/\cos(\alpha/2)$

45. $\sqrt{a\tan\alpha/b}$

46. $2\pi^2 kr(2k+1)/v$

47. $\sqrt{v_1^2+v_2^2-2v_1v_2\cos\alpha}/\sin\alpha$

48. $\sqrt{2}v/\mu g$; a parabola.

49. $\sqrt{4.5gR}$

50. $20\,\mathrm{cm/s}$.

51. i) from A and C to the capital; ii) $a(2-\sqrt{2})$; iii) for A-B: 0 until turning point, $-2v$ henceforth; for B-C: constantly $-v\sqrt{2}$; iv) speed drops briefly down to $-2v$.

52. $v_{\min} = v_{\max} = \omega r$.

53. A circle (ring) touching the ground and passing through the wheel's centre.

54. blue pulse was the first one; $300\,\mathrm{ms} < T < 500\,\mathrm{ms}$; $v \approx 65\,\mathrm{cm/s}$, $\omega \approx 23\,\mathrm{rad/s}$, $R \approx 5\,\mathrm{cm}$

55. We define $k \approx 0.014$ via the shape of the jet $y = kx^2$ using the units of the figure grid; then $d = \left(32V^2k/\pi^2t^2g\right) \approx 1\,\mathrm{mm}$.

56. Let a line s_1 connect the triangle (henceforth T) with two lower stars, and a line s_2 — the square (S) with the remaining star; draw a horizontal line s_3 through the intersection point of s_1 and s_2; mark the intersection point Q of the lines s_3 and TA; the starting point B of the other boat is the intersection point of the upper coast with the line SQ.

57. $35\,\mathrm{min}$

58. $1\,\mathrm{min}$; $\sqrt{1.5}\,\mathrm{min} \approx 73\,\mathrm{s}$.

59. 150 or 600 pixels.

60. $h \approx 3.2L$.

61. $\frac{R}{v}\arccos\frac{u}{v} + \frac{L}{u} - R\sqrt{u^{-2}-v^{-2}}$.

62. $v \approx 1.8\,\mathrm{m/s}$; $h \approx 2.0\,\mathrm{m}$

63. i) $v_{\mathrm{wind}} \approx 13\,\mathrm{km/h}$; ii) $v_{\mathrm{wind}} \approx 21\,\mathrm{km/h}$

64. $v \approx 20\,\mathrm{m/s}$

65. $\approx 12\,\mathrm{m/s}$

66. $\frac{N(N-1)}{2}$.