# PROBLEMS ON MECHANICS

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## INTRODUCTION

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This booklet is a sequel to a similar collection of problems on kinematics. Similarly to that collection the aim here is to present the most important ideas using which one can solve most (> 95%) of olympiad problems on mechanics. Usually a problem is stated first, and is followed by some relevant ideas and suggestions (letter 'K' in front of the number of an idea refers to the correspondingly numbered idea in the kinematics booklet). The answers to the problems are listed at the end of the booklet. They are preceded by quite detailed hints (no full solutions), but think carefully before reading the hints as a last resort!

The guiding principle of this booklet argues that almost all olympiad problems are "variations" on a specific set of topics — the solutions follow from corresponding solution ideas. Usually it is not very hard to recognize the right idea for a given problem, having studied enough solution ideas. Discovering all the necessary ideas during the actual solving would certainly show much more creativity and offer a greater joy, but the skill of conceiving ideas is unfortunately difficult (or even impracticable) to learn or teach. Moreover, it may take a long time to reach a new idea, and those relying on trying it during an olympiad would be in disadvantage in comparison to those who have mastered the ideas.

In science as a whole, solution ideas play a similar role as in olympiads: most scientific papers apply and combine known ideas for solving new (or worse, old) problems, at best developing and generalising the ideas. Genuinely new good ideas occur extremely rarely and many of them are later known as masterpieces of science. However, as the whole repertoire of scientific ideas encompasses immensely more than mere mechanics, it is not so easy to remember and utilise them in right places. The respective skill is highly valued; an especial achievement would be employing a well-known idea in an unconventional (unexpected, novel) situation.

In addition to ideas, the booklet also presents "facts" and "methods". The distinction is largely arbitrary, some facts could have been called methods and vice versa. In principle, an "idea" should have wider and/or more creative applications than a "fact"; a "method" is a universal and conventionalized "idea".

Several sources have been used for the problems: Estonian olympiads' regional and national rounds, journal "Kvant", Russian and Soviet Union's olympiads; some problems have been modified (either easier or tougher), some are "folklore" (origins unknown).

## STATICS

For problems on statics the solution is usually standard: we have to write down the condition of force balance for the $x$-, $y$- and (if necessary) $z$-components; often the condition of torque balance must be added. Usually the main ingenuity lies in

**IDEA 1:** choose optimal axes to zero as many projections of forces as possible. It is especially good to zero the projections of the forces *we do not know and are not interested in*,

for instance, the reaction force between two bodies or the tensile force in a string (or a rod). To zero as many forces as possible it is worthwhile to note that *a)* the axes may not be perpendicular; *b)* if the system consists of several bodies, then a different set of axes may be chosen for each body.

**IDEA 2:** for the torques equation it is wise to choose such a pivot point that zeroes as many moment arms as possible. Again it is especially beneficial to zero the torques of *"uninteresting" forces*.

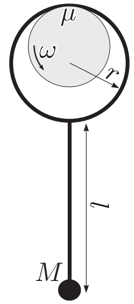
For example, if we choose the pivot to be at the contact point of two bodies, then the moment arms of the friction force between the bodies and of their reaction force are both zero.

**IDEA 3:** in case of a *two-dimensional system*, we can write two equations per body for the forces ($x$- and $y$-components) and one equation (per body) for the torques.

An equation for the torques can be written about any pivot point ("axis" of rotation). In principle, we could write several equations for several pivots at the same time, but together with the equations for the forces *the maximum number of linearly independent equations equals the number of degrees of freedom of the body* (three in the two-dimensional case, as the body can rotate in a plane and shift along the $x$- and $y$-axis). Accordingly, all is fine if we write one forces equation and two torques equations (or just three torques equations — as long as the pivots do not lie on a straight line); on the other hand, if we wrote two equations of both types, then one of the four equations would always be a redundant consequence of the three others and needless to write down.

So, an equation for the force balance may be replaced by an equation for the torque balance about an additional pivot. Such a substitution may turn out to be useful if the unwanted (uninteresting) forces are unparallel, because a choice of a projection axis can zero only one force in the balance of forces, while a choice of a pivot for the torques can zero two forces at once.

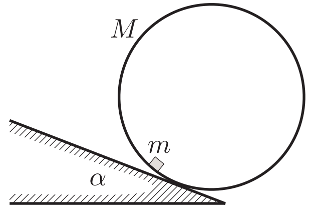
**PROB 1.** An end of a light wire rod is bent into a hoop of radius $r$. The straight part of the rod has length $l$; a ball of mass $M$ is attached to the other end of the rod. The pendulum thus formed is hung by the hoop onto a revolving shaft. The coefficient of friction between the shaft and the hoop is $\mu$. Find the equilibrium angle between the rod and the vertical.




Here we mainly need idea 2 with some simplification offered by

**FACT 1:** on an inclined surface, slipping will start when the slope angle $\alpha$ fulfills $\tan\alpha = \mu$.

**PROB 2.** On an incline with slope angle $\alpha$ there lies a cylinder with mass $M$, its axis being horizontal. A small block with mass $m$ is placed inside it. The coefficient of friction between the block and the cylinder is $\mu$; the incline is nonslippery. What is the maximum slope angle $\alpha$ for the cylinder to stay at rest? The block is much smaller than the radius of the cylinder.

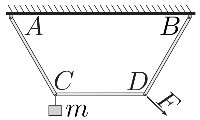



Here we can again use fact 1 and idea 2 if we add

**IDEA 4:** sometimes it is useful to consider a system of two (or more) bodies as one whole and write the equations for the forces and/or the torques for the whole system.

Further, the net force (or torque) is the sum of forces (torques) acting on the constituents (the effort is eased as the internal forces are needless — they cancel each other out). In our case, it is useful to assemble such a whole system from the cylinder and the block.

**PROB 3.** Three identical rods are connected by hinges to each other, the outmost ones are hinged to a ceiling at points $A$ and $B$. The distance between these points is twice the length of a rod. A weight of mass $m$ is hanged onto hinge $C$. At least how strong a force onto hinge $D$ is necessary to keep the system stationary with the rod $CD$ horizontal?




Again we can use idea 2. The work is also aided by

**FACT 2:** if forces are applied only to two points of a rod and the fixture of the rod is not rigid (the rod rests freely on its supports or is attached to a string or a hinge), then the tension force in the rod is directed along the rod.

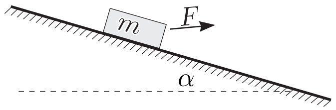
Indeed, the net external force $\vec{F}$ onto either point of application of the forces must point along the rod, as its torque with respect to the other point of application must be zero. In addition to the external forces, the point is acted on by tension force $\vec{T}$ that must compensate the rest of the forces, so $\vec{F} = -\vec{T}$.

Some ideas are very universal, especially the mathematical ones.

**IDEA K5:** some extrema are easier to find without using derivatives,

for example, the shortest path from a point to a plane is perpendicular to it.

**PROB 4.** What is the minimum force needed to dislodge a block of mass $m$ resting on an inclined plane of slope angle $\alpha$, if the coefficient of friction is $\mu$? Investigate the cases when *a)* $\alpha = 0$; *b)* $0 < \alpha < \arctan \mu$.



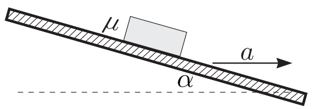

**IDEA 5:** force balance can sometimes be resolved vectorially without projecting anything onto axes.

Fact 1, or rather its following generalisation, turns out to be of use:

**FACT 3:** if a body is on the verge of slipping (or already slipping), then the sum of the friction force and the reaction force is angled by $\arctan \mu$ from the surface normal.

This fact is also beneficial in the next problem.

**PROB 5.** A block rests on an inclined surface with slope angle $\alpha$. The surface moves with a horizontal acceleration $a$ which lies in the same vertical plane as a normal vector to the surface. Determine the values of the coefficient of friction $\mu$ that allow the block to remain still.



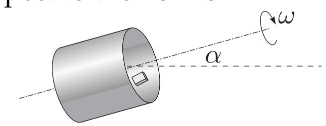

Here we are helped by the very universal

**IDEA 6:** many problems become very easy in a non-inertial translationally moving reference frame.

To clarify: in a translationally moving reference frame we can re-establish Newton's laws by imagining that every body with mass $m$ is additionally acted on by an inertial force $-m\vec{a}$ where $\vec{a}$ is the acceleration of the frame of reference. Note that that the fictitious force is totally analogous to the gravitational force and (as an aside) their equivalence is the cornerstone of the theory of general relativity (more specifically, it assumes the inertial and gravitational forces to be indistinguishable in any local measurement).

**IDEA 7:** The net of the inertial and gravitational forces is usable as an effective gravitational force.

**PROB 6.** A cylinder with radius $R$ spins around its axis with an angular speed $\omega$. On its inner surface there lies a small block; the coefficient of friction between the block and the inner surface of the cylinder is $\mu$. Find the values of $\omega$ for which the block does not slip (stays still with respect to the cylinder). Consider the cases where *(a)* the axis of the cylinder is horizontal; *(b)* the axis is inclined by angle $\alpha$ with respect to the horizon.

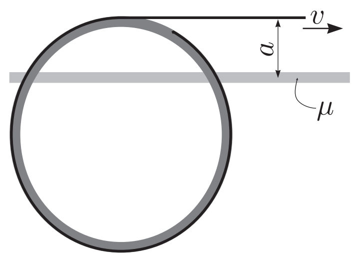



**IDEA 8:** a rotating frame of reference may be used by adding a centrifugal force $m\omega^2\vec{R}$ (with $\omega$ being the angular speed of the frame and $R$ being a vector drawn from the axis of rotation to the point in question) and Coriolis force. The latter is unimportant *(a)* for a body standing still or moving in parallel to the axis of rotation in a rotating frame of reference (in this case the Coriolis force is zero); *(b)* for energy conservation (in this case the Coriolis force is perpendicular to the velocity and, thus, does not change the energy).

Warning: *in this idea, the axis of rotation must be actual, not instantaneous.* For the last problem, recall idea K5 and fact 3; for part *(b)*, add

**IDEA 9:** in case of three-dimensional geometry, consider two-dimensional sections. It is especially good if all interesting objects (for example, force vectors) lie on one section. The orientation and location of the sections may change in time.

**PROB 7.** A hollow cylinder with mass $m$ and radius $R$ stands on a horizontal surface with its smooth flat end in contact the surface everywhere. A thread has been wound around it and its free end is pulled with velocity $v$ in parallel to the thread. Find the speed of the cylinder. Consider two cases: *(a)* the coefficient of friction between the surface and the cylinder is zero everywhere except for a thin straight band (much thinner than the radius of the cylinder) with a coefficient of friction of $\mu$, the band is parallel to the thread and its distance to the thread $a < 2R$ (the figure shows a top-down view); *(b)* the coefficient of friction is $\mu$ everywhere. *Hint:* any planar motion of a rigid body can be viewed as rotation around an instant centre of rotation, i.e. the velocity vector of any point of the body is the same as if the instant centre were the real axis of rotation.




This is quite a hard problem. It is useful to note

**IDEA 10:** if a body has to move with a constant velocity, then the problem is about statics.

Also remember ideas 1 and 2. The latter can be replaced with its consequence,

**FACT 4:** if a body in equilibrium is acted on by three forces at three separate points, then their lines of action intersect at one point. If there are only two points of action, then the corresponding lines coincide.

Another useful fact is

**FACT 5:** the friction force acting on a given point is always antiparallel to the velocity of the point in the frame of reference of the body causing the friction.

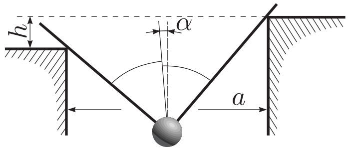
From time to time some mathematical tricks are also of use; here it is the property of inscribed angles (Thales' theorem),

**FACT 6:** a right angle is subtended by a semicircle (in general: an inscribed angle in radians equals half of the ratio between its arc-length and radius).

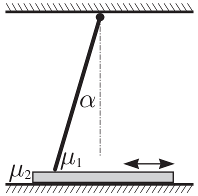
The property of inscribed angles is also useful in the next problem, if we add (somewhat trivial)

**IDEA 11:** in stable equilibrium the potential energy of a body is minimum.

**PROB 8.** A light wire is bent into a right angle and a heavy ball is attached to the bend. The wire is placed onto supports with height difference $h$ and horizontal distance $a$. Find the position of the wire in its equilibrium. Express the position as the angle between the bisector of the right angle and the vertical. Neglect any friction between the wire and the supports; the supports have little grooves keeping all motion in the plane of the wire and the figure.



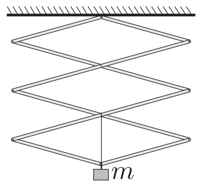

**PROB 9.** A rod with length $l$ is hinged to a ceiling with height $h < l$. Underneath, a board is being dragged on the floor. The rod is meant to block the movement the board in one direction while allowing it move in the opposite direction. What condition should be fulfilled for it to do its job? The coefficient of friction is $\mu_1$ between the board and the rod, and $\mu_2$ between the board and the floor.




Let's remember fact 3: if the relative sliding between two bodies has a known direction, then the direction of the sum of the friction and reaction force vectors is always uniquely determined by the coefficient of friction. If a force makes one of the bodies move in such a way that the reaction force grows, then they jam: the larger the forces we try to drag the bodies with, the larger friction and reaction forces restrain them.

**IDEA 12:** Friction can block movement. In such a case, all forces become negligible except for the friction force, reaction force and the externally applied force that tries to make the system move, because gravitational (and such) forces are fixed, but the said forces become the larger the harder we push or pull.

**PROB 10.** Four long and four half as long rods are hinged to each other forming three identical rhombi. One end of the contraption is hinged to a ceiling, the other one is attached to a weight of mass $m$. The hinge next to the weight is connected to the hinge above by a string. Find the tension force in the string.



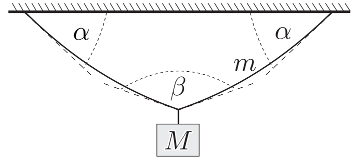

This problem is the easiest to solve using *the method of virtual displacement*.

**METHOD 1:** Imagine that we are able to change the length of the string or rod the tension in which is searched for by an infinitesimal amount $\Delta x$. Equating the work $T\Delta x$ by the change $\Delta\Pi$ of the potential energy, we get $T = \Delta\Pi/\Delta x$.

*Generalisation:* if some additional external forces $\vec{F}_i$ $(i = 1,2,\ldots)$ act on the system with the displacements of their points of action being $\delta\vec{x}_i$, while the interesting string or rod undergoes a virtual lengthening of $\Delta x$, then $T = (\Delta\Pi - \sum_i \delta\vec{x}_i \cdot \vec{F}_i)/\Delta x$.

The method can also be used for finding some other forces than tension (for example, in problems about pulleys): by imaginarily shifting the point of action of the unknown force one can find the projection of this force onto the direction of the virtual displacement.

**PROB 11.** A rope with mass $m$ is hung from the ceiling by its both ends and a weight with mass $M$ is attached to its centre. The tangent to the rope at its either end forms angle $\alpha$ with the ceiling. What is the angle $\beta$ between the tangents to the rope at the weight?




**FACT 7:** The tension in a freely hanging string is directed along the tangent to the string.

In addition, we can employ

**IDEA 13:** consider a piece of string separately and think about the componentwise balance of forces acting onto it.

In fact, here we do not need the idea as a whole, but, rather, its consequence,

**FACT 8:** the horizontal component of the tension in a massive string is constant.

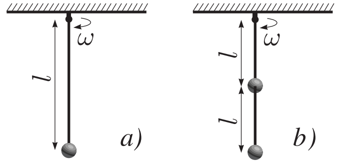
In problems about ropes one may sometimes use

**IDEA 14:** If the weight of a hanging part of a rope is much less than its tension, then the curvature of the rope is small and its horizontal mass distribution can quite accurately be regarded as constant.

This allows us to write down the condition of torque balance for the hanging portion of the rope (as we know the horizontal coordinate of its centre of mass). The next problem illustrates that approach.

**PROB 12.** A boy is dragging a rope with length $L = 50\,\mathrm{m}$ along a horizontal ground with a coefficient of friction of $\mu = 0.6$, holding an end of the rope at height $H = 1\,\mathrm{m}$ from the ground. What is the length $l$ of the part of the rope not touching the ground?

**PROB 13.** A light rod with length $l$ is hinged in such a way that the hinge folds in one plane only. The hinge is spun with angular speed $\omega$ around a vertical axis. A small ball is fixed to the other end of the rod. *(a)* Find the angular speeds for which the vertical orientation is stable. *(b)* The ball is now attached to another hinge and, in turn, to another identical rod; the upper hinge is spun in the same way. What is now the condition of stability for the vertical orientation?




For answering about the stability of an equilibrium, usually the following fact works best.

**IDEA 15:** Presume that the system deviates a little from the equilibrium, either by a small displacement $\Delta x$ or by a small angle $\Delta\varphi$, and find the direction of the appearing force or torque — whether it is towards the equilibrium or away from it. NB! compute approximately: in almost all cases, an approximation linear in the deviation is enough.

Incidentally use all formulae of approximate calculation known from mathematics ($\sin\varphi \approx \varphi$ and others);

**IDEA 16:** $f(x+\Delta x) \approx f(x) + f'(x)\Delta x$ $[+f''(x)\frac{\Delta x^2}{2}]$; $(x+\Delta x)(y+\Delta y) \approx xy + x\Delta y + y\Delta x$ etc (consider them wherever initial data suggest some parameter to be small).

The case *(b)* is substantially more difficult as the system has two degrees of freedom (for example, the deviation angles $\Delta\varphi_1$ and $\Delta\varphi_2$ of the rods). Although idea 15 is generalisable for more than one degrees of freedom, apparently it is easier to start from idea 11.

**IDEA 17:** The equilibrium $x = y = 0$ of a system having two degrees of freedom is stable if (and only if) the potential energy $\Pi(x,y)$, when viewed as a one-variable function $\Pi(x,kx)$, has a minimum for all real constants $k$.

**PROB 14.** If a beam with square cross-section and very low density is placed in water, it will turn one pair of its long opposite faces horizontal. This orientation, however, becomes unstable as we increase its density. Find the critical density when this transition occurs. The density of water is $\rho_v = 1000\,\mathrm{kg/m^3}$.

**IDEA 18:** The torque acting on a body placed into a liquid is equal to torque from buoyancy, if we take the latter force to be acting on the centre of the mass of the displaced liquid.

Indeed, consider a body with density of the liquid and shape identical to the part of the given body that is immersed in the liquid. Of course it must be in equilibrium when placed in water: whatever point we choose to measure torques from, the sum of moments from pressure forces is always equal to the opposite value of torque from gravity. When calculating the moments from buoyancy in this question, it is useful to keep in mind that we can give negative mass to bits of some body: if two bits overlap that have the same density with different signs, they add up to zero density. The last suggestion can be formulated in a more general way:

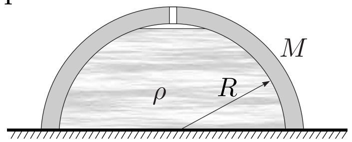
**IDEA 19:** In order to achieve a more symmetric configuration or to make the situation simpler in some other way, it is sometimes useful to represent a region with zero value of some quantity as a superposition of two regions with opposite signs of the same quantity.

This quantity can be mass density (like in this case), charge or current density, some force field etc. Often this trick can be combined with

**IDEA 20:** Make the problem as symmetric as possible.

This goal can be reached by applying idea 19, but also by using appropriate reference frames, dividing the process of solving into several phases (where some phases use symmetric geometry), etc.

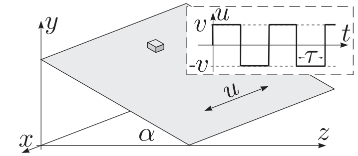
**PROB 15.** A hemispherical container is placed upside down on a smooth horizontal surface. Through a small hole at the bottom of the container, water is then poured in. Exactly when the container gets full, water starts leaking from between the table and the edge of the container. Find the mass of the container if water has density $\rho$ and radius of the hemisphere is $R$.




**IDEA 21:** If water starts flowing out from under an upside down container, normal force must have vanished between the table and the edge of the container. Therefore force acting on the system container+liquid from the table is equal solely to force from hydrostatic pressure.

The latter is given by $pS$, where $p$ is pressure of the liquid near the tabletop and $S$ is area of the container's open side.

**PROB 16.** A block is situated on a slope with angle $\alpha$, the coefficient of friction between them is $\mu > \tan\alpha$. The slope is rapidly driven back and forth in a way that its velocity vector $\vec{u}$ is parallel to both the slope and the horizontal and has constant modulus $v$; the direction of $\vec{u}$ reverses abruptly after each time interval $\tau$. What will be the average velocity $w$ of the block's motion? Assume that $g\tau \ll v$.

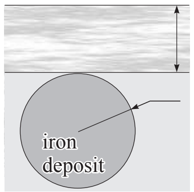



**IDEA 22:** If the system changes at high frequency, then it is often pratical to use time-averaged values $\langle X\rangle$ instead of detailed calculations. In more complicated situations a high-frequency component $\tilde{X}$ might have to be included (so that $X = \langle X\rangle + \tilde{X}$).

**METHOD 2:** (*perturbation method*) If the impact of some force on a body's motion can be assumed to be small, then solve the problem in two (or more) phases: first find motion of the body in the absence of that force (so-called zeroth approximation); then pretend that the body is moving just as found in the first phase, but there is this small force acting on it. Look what correction (so-called first correction) has to be made to the zeroth approximation due to that force.

In this particular case, the choice of zeroth approximation needs some explanation. The condition $g\tau \ll v$ implies that within one period, the block's velocity cannot change much. Therefore if the block is initially slipping downwards at some velocity $w$ and we investigate a short enough time interval, then we can take the block's velocity to be constant in zeroth approximation, so that it is moving in a straight line. We can then move on to phase two and find the average value of frictional force, based on the motion obtained in phase one.

**PROB 17.** Let us investigate the extent to which an iron deposit can influence water level. Consider an iron deposit at the bottom of the ocean at depth $h = 2\,\mathrm{km}$. To simplify our analysis, let us assume that it is a spherical volume with radius 1 km with density greater from the surrounding rock by $\Delta\rho = 1000\,\mathrm{kg/m^3}$. Presume that this sphere touches the bottom of the ocean with its top, i.e. that its centre is situated at depth $r+h$. By how much is the water level directly above the iron deposit different from the average water level?



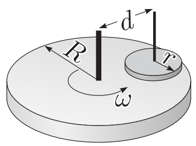

**IDEA 23:** The surface of a liquid in equilibrium takes an equipotential shape, i.e. energies of its constituent particles are the same at every point of the surface.

If this was not the case, the potential energy of the liquid could be decreased by allowing some particles on the surface to flow along the surface to where their potential energy is smaller.

**IDEA 24:** Gravitational potentials can be calculated exactly in the same way as electrostatic potentials.

The principle of superposition still holds and a sphere's potential only has a different factor: instead of $Q/4\pi\varepsilon_0 r$ in electrostatics the gravitational potential of a sphere with respect to infinity is $\varphi = -GM/r$; the minus sign comes from the fact that masses with the same sign ["+"] attract.

**PROB 18.** A horizontal platform rotates around a vertical axis at angular velocity $\omega$. A disk with radius $R$ can freely rotate and move up and down along a slippery vertical axle situated at distance $d > R$ from the platform's axis. The disk is pressed against the rotating platform due to gravity, the coefficient of friction between them is $\mu$. Find the angular velocity acquired by the disk. Assume that pressure is distributed evenly over the entire base of the disk.




**IDEA 25:** If we transform into a rotating frame of reference, then we can add angular velocities about instantaneous axes of rotation in the same way as we usually add velocities.

Thus $\vec{\omega}_3 = \vec{\omega}_1 + \vec{\omega}_2$, where $\vec{\omega}_1$ is angular velocity of the reference frame, $\vec{\omega}_2$ angular velocity of the body in the rotating frame of reference and $\vec{\omega}_3$ that in the stationary frame. In this question, we can use fact 5, ideas 2, 8, 10 and also

**IDEA K5:** Arbitrary motion of a rigid body can be considered as rotation about an instantaneous centre of rotation (in terms of velocity vectors of the body).

**METHOD 3:** (differential calculus) Divide the object into infinitesimally small bits or the process into infinitesimally short periods (if necessary, combine this with idea 16).

Within an infinitesimal bit (period), quantities changing in space (time) can be taken constant (in our case, that quantity is the direction of frictional force vector). If necessary (see the next question), these quantities may be summed over all bits — this is called integration.

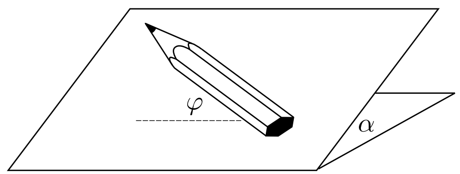
**PROB 19.** A waxing machine consists of a heavy disk with mass $M$ densely covered with short bristles on one side, so that if it lies on the floor, then its weight is evenly distributed over a circular area with radius $R$. An electrical motor makes the disk rotate at angular velocity $\omega$, the user compensates for the torque from frictional forces by a long handle. The same handle can be used to push the machine back and forth along the floor. With what force does the machine have to be pushed to make it move at velocity $v$? Assume that angular velocity of the disk is large, $\omega R \gg v$, and that the force needed to compensate for the torque can be neglected. The coefficient of friction between the bristles and the floor is $\mu$.

Here we need fact 5, ideas K5 and 19 and additionally

**IDEA 26:** Try to determine the region of space where forces (or torques etc) cancel at pairs of points.

These pairs of points are often symmetrically located. Idea 20 is relevant as well.

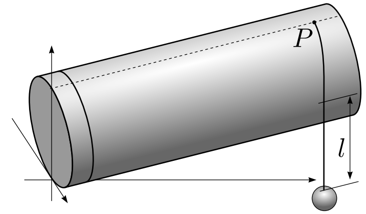
**PROB 20.** A hexagonal pencil lies on a slope with inclination angle $\alpha$; the angle between the pencil's axis and the line of intersection of the slope and the horizontal is $\varphi$. Under what condition will the pencil not roll down?




**IDEA 27:** When solving three-dimensional problems, sometimes calculating coordinates in appropriately chosen axes and applying formulae of spatial rotations can be of use.

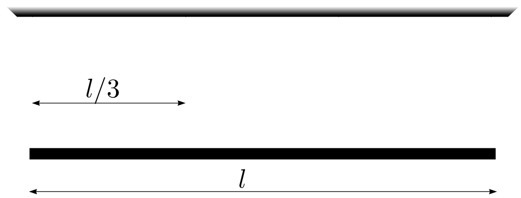
What (which vector) could be expressed in terms of its components in our case? The only promising option is the small shift vector of centre of mass when its starts to move; ultimately we are only interested in its vertical component.

**PROB 21.** A slippery cylinder with radius $R$ has been tilted to make an angle $\alpha$ between its axis and the horizontal. A string with length $L$ has been attached to the highest point $P$ of some cross-section of the cylinder, the other end of it is tied to a weight with mass $m$. The string takes its equilibrium position, how long ($l$) is the part not touching the cylinder? The weight is shifted from its equilibrium position in such a way that the shift vector is parallel to the vertical plane including the cylinder's axis; what is the period of small oscillations?




**IDEA 28:** Unfolding a three-dimensional object and looking at its surfaces in the same plane can assist in solving problems, among other things it helps to find shortest distances.

**PROB 22.** A uniform bar with mass $m$ and length $l$ hangs on four identical light wires. The wires have been attached to the bar at distances $\frac{l}{3}$ from one another and are vertical, whereas the bar is horizontal. Initially, tensions are the same in all wires, $T_0 = mg/4$. Find tensions after one of the outermost wires has been cut.




**IDEA 29:** If more fixing elements (rods, strings, etc) than the necessary minimum have been used to keep a body in static equilibrium (i.e. more than the number of degrees of freedom) and fixing elements are absolutely rigid, then tensions in the elements cannot be determined. In order to

make it possible, the elements have to be considered elastic (able to deform).

Let us note that this statement is in accordance with idea 3 that gives the number of available equations (there can be no more unknowns than equations). In this particular case, we are dealing with effectively one-dimensional geometry with no horizontal forces, but the body could rotate (in absence of the wires). Thus we have two degrees of freedom, corresponding to vertical and rotational motion. Since the wires are identical, they must have the same stiffness as well; the word "wire" hints at large stiffness, i.e. deformations (and the inclination angle of the bar) are small.

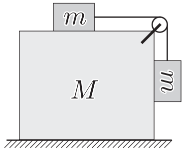
## DYNAMICS

A large proportion of dynamics problems consist of finding the acceleration of some system or forces acting between some bodies. There are several possible approaches for solving these questions, here we consider three of them.

**METHOD 4:** For each body, we find all the forces acting on it, including normal forces and frictional forces, and write out Newton's second law in terms of components (i.e. by projecting the equation on $x$, $y$, and possibly $z$-axes).

We need the same number of equations as we have unknowns; following idea 1 can help to reduce that number.

**PROB 23.** A block with mass $M$ lies on a slippery horizontal surface. On top of it there is another block with mass $m$ which in turn is attached to an identical block by a string. The string has been pulled across a pulley situated at the corner of the big block and the second small block is hanging vertically. Initially, the system is held at rest. Find the acceleration of the big block immediately after the system is released. You may neglect friction, as well as masses of the string and the pulley.




This question can be successfully solved using method 4, but we need two more ideas.

**IDEA 30:** If a body is initially at rest, then its shift vector is parallel to the force acting on it (and its acceleration) right after the start of its motion.

**IDEA 31:** If bodies are connected by a rope or a rod or perhaps a pulley or one is supported by the other, then there is an arithmetic relation between the bodies' shifts (and velocities, accelerations) that describes the fact that length of the string (rod, etc.) is constant.

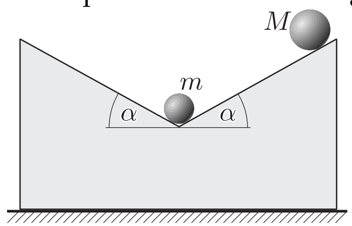
If bodies start at rest or if motion is along a straight line, then the same relation holds between accelerations, since the relation for shifts can be differentiated w.r.t. time. This relation is usually relatively simple, but in some problems it is easy to make a mistake.

**METHOD 5:** Otherwise the same as method 4, but motion is investigated in a non-inertial frame of reference (see idea 6) where one of the bodies is at rest.

Method 5 is useful in many questions concerning wedges, where it can be difficult to write out the condition for an object to stay on the wedge in the laboratory frame. Applying idea 31 is also often easier in the wedge's frame of reference than in the laboratory frame. Since the body defining the reference frame is at rest, we can write out the condition(s) of equilibrium for it.

**FACT 9:** If the frame of reference of an accelerating body is used (method 5), then in the new frame the forces acting on this body add up to zero.

**PROB 24.** A wedge has been made out of a very light and slippery material. Its upper surface consists of two slopes making an angle $\alpha$ with the horizontal and inclined towards one another. The block is situated on a horizontal plane; a ball with mass $m$ lies at the bottom of the hole on its upper surface. Another ball with mass $M$ is placed higher than the first ball and the system is released. On what condition will the small ball with mass $m$ start slipping upwards along the slope? Friction can be neglected.



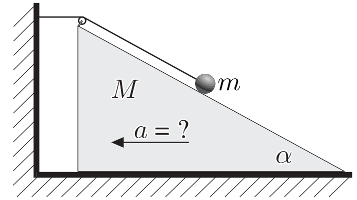

The final method is based on using *generalised coordinates* and originates from theoretical mechanics. There its description requires relatively complicated mathematical apparatus, but in most problems it can be used in a much simpler form.

**METHOD 6:** Let us call $\xi$ a generalised coordinate if the entire state of a system can be described by this single number. Say we need to find the acceleration $\ddot{\xi}$ of coordinate $\xi$. If we can express the potential energy $\Pi$ of the system as a function $\Pi(\xi)$ of $\xi$ and the kinetic energy in the form $K = \mathcal{M}\dot{\xi}^2/2$ where coefficient $\mathcal{M}$ is a combination of masses of the bodies (and perhaps of moments of inertia), then

$$\ddot{\xi} = -\Pi'(\xi)/\mathcal{M}.$$

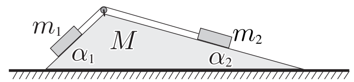
Here, a dot denotes differentiation w.r.t. time and dash w.r.t. coordinate $\xi$. Indeed, due to conservation of energy $\Pi(\xi) + \mathcal{M}\dot{\xi}^2/2$ =Const. Differentiating that w.r.t. time and using the chain rule, we obtain $\Pi'(\xi)\dot{\xi} + \mathcal{M}\dot{\xi}\ddot{\xi} = 0$. We reach the aforementioned formula after dividing through by $\dot{\xi}$.

**PROB 25.** A small block with mass $m$ lies on a wedge with angle $\alpha$ and mass $M$. The block is attached to a rope pulled over a pulley attached to the tip of the wedge and fixed to a horizontal wall (see the figure). Find the acceleration of the wedge. All surfaces are slippery (there is no friction).




Full solution of this problem is given in the hints' section to illustrate method 6

**PROB 26.** A wedge with mass $M$ and acute angles $\alpha_1$ and $\alpha_2$ lies on a horizontal surface. A string has been drawn across a pulley situated at the top of the wedge, its ends are tied to blocks with masses $m_1$ and $m_2$. What will be the acceleration of the wedge? There is no friction anywhere.




It may seem that there is more than one degree of freedom in this question: the wedge can move and the string can shift w.r.t. the wedge. However, we are saved by

**IDEA 32:** If $x$-components of the sum of external forces and of centre of mass velocity are both zero, then the $x$-coordinate of the centre of mass remains constant.

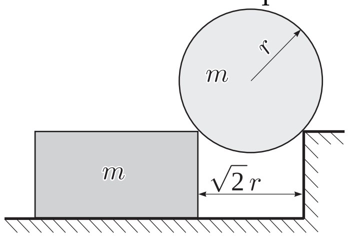
We can use this circumstance to reduce the effective number of degrees of freedom. In our particular case, the system consists of two components and thus the shift of component can be expressed by that of the other.

**IDEA 33:** The $x$-coordinate of the centre of mass of a system of bodies is

$$X_C = \sum x_i m_i / \sum m_i,$$

where $m_i$ denotes mass of the $i$-th component and $x_i$ the coordinate of its centre of mass. The formula can be rewritten in integral form, $X_C = \int x dm / \int dm$, where $dm = \rho(x,y,z)dV$ is differential of mass.

**PROB 27.** Two slippery horizontal surfaces form a step. A block with the same height as the step is pushed near the step, and a cylinder with radius $r$ is placed on the gap. Both the cylinder and the block have mass $m$. Find the normal force $N$ between the cylinder and the step at the moment when distance between the block and the step is $\sqrt{2}r$. Initially, the block and the step were very close together and all bodies were at rest. Friction is zero everywhere. Will the cylinder first separate from the block or the step?




It is easy to end up with very complicated expressions when solving this problem, this may lead to mistakes. Therefore it is wise to plan the solution carefully before writing down any equations.

**IDEA 34:** Newton's laws are mostly used to find acceleration from force, but sometimes it is clever to find force from acceleration.

But how to find acceleration(s) in that case? It is entirely possible if we use method 6, but this path leads to long expressions. A tactical suggestion: if you see that the solution is getting very complicated technically, take a break and think if there is an easier way. There is a "coincidence" in this particular problem: straight lines drawn from the sphere's centre to points of touching are perpendicular; can this perhaps help? It turns out that it does.

**IDEA 35:** Pay attention to special cases and use simplifications that they give rise to!

Let us remind what we learned in kinematics:

**IDEA K29:** In case of motion along a curve, the radial component (perpendicular to the trajectory) of a point's acceleration $v^2/R$ is determined by velocity $v$ and radius of curvature $R$; the component along the trajectory is linear acceleration (equal to $\varepsilon R$ in case of rotational motion, $\varepsilon$ is angular acceleration).

The centre of mass of the cylinder undergoes rotational motion, method 6 is necessary to find angular acceleration — but we hoped to refrain from using it. An improvement on idea 1 helps us out:

**IDEA 36:** Project Newton's 2nd law on the axis perpendicular to an unwanted vector, e.g. an unknown force or the tangential component of acceleration.

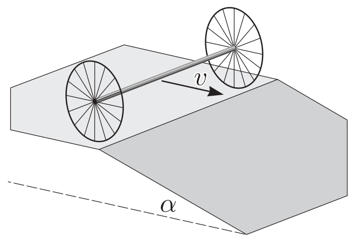
We can easily find the cylinder's velocity (and thus the radial component of acceleration) if we use

**IDEA 37:** If energy is conserved (or its change can be calculated from work done etc), write it out immediately. Energy is conserved if there is no dissipation (friction, inelastic collisions etc) and external forces acting on the system are static (e.g. a stationary inclined plane); forces changing in time (force acting on a moving point, moving inclined plane) change energy as well. Idea 31 helps to write out conservation of energy (relation between bodies' velocities!). To answer the second question, we need

**IDEA 38:** Normal force vanishes at the moment when a body detaches from a surface.

Also, review idea 31 for horizontal components of accelerations.

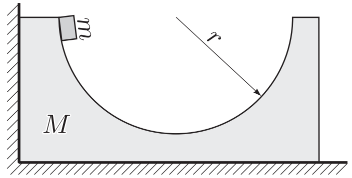
**PROB 28.** Light wheels with radius $R$ are attached to a heavy axle. The system rolls along a horizontal surface which suddenly turns into a slope with angle $\alpha$. For which angles $\alpha$ will the wheels move without lifting off, i.e. touch the surface at all times? Mass of the wheels can be neglected. The axle is parallel to the boundary between horizontal and sloped surfaces and has velocity $v$.




**IDEA 39:** To answer the question whether a body lifts off, we have to find the point on the non-lifting-off trajectory with smallest normal force.

If normal force has to be negative at that point, then the body lifts off; the critical value is zero — compare with idea 38). Also, review ideas 1, 37 and K29.

**PROB 29.** A block with mass $M$ lies on a horizontal slippery surface and also touches a vertical wall. In the upper surface of the block, there is a cavity with the shape of a half-cylinder with radius $r$. A small pellet with mass $m$ is released at the upper edge of the cavity, on the side closer to the wall. What is the maximum velocity of the block during its subsequent motion? Friction can be neglected.



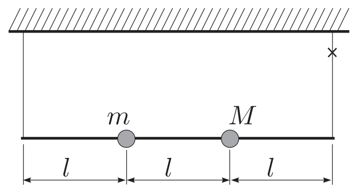

**IDEA 40:** Conservation law can hold only during some period of time.

**IDEA 41:** Momentum is conserved if the sum of external forces is zero; sometimes momentum is conserved only along one axis.

You will also need idea 37.

**IDEA 42:** Velocity is maximal (or minimal) when acceleration (and net force) is zero (since $0 = \frac{dv}{dt} = a$); shift is extremal when velocity is zero. Possible other pairs: electrical charge (capacitor's voltage)-current, current-inductive emf, etc.

**PROB 30.** A light rod with length $3l$ is attached to the ceiling by two strings with equal lengths. Two balls with masses $m$ and $M$ are fixed to the rod, the distance between them and their distances from the ends of the rod are all equal to $l$. Find the tension in the second string right after the first has been cut.




There are several good solutions for this problem, all of which share applying idea 34 and the need to find the angular acceleration of the rod. Firstly, angular acceleration of the rod can be found from method 6 by choosing angle of rotation $\varphi$ to be the generalised coordinate.

Secondly, we may use Newton's 2nd law for rotational motion: we find the torque on the rod about the point of attachment of the second string and equate it to $I\varepsilon$ with angular acceleration $\varepsilon$ and moment of inertia $I = ml^2 + 4Ml^2$. More generally,

**IDEA 43:** When a body is rotating around the axis $s$, the net torque it experiences is $M = I\varepsilon$ (not to be confused with the body's mass), where $I$ is its moment of inertia with respect to the axis $s$, $I = \sum m_i r_i^2 = \int r^2 \cdot dm = \int r^2 \rho \cdot dV$ and $r_i$ is the distance of $i$-th particle from the axis $s$ (the sum is evaluated over all particles of the body). Kinetic energy is $K = \frac{1}{2} I\omega^2$.

Once the angular acceleration is found, in order to apply the idea 34 it may be helpful to use

**IDEA 44:** The more general and sometimes indispensable form of Newton's 2nd law is $\vec{F} = \frac{d\vec{P}}{dt}$, where $\vec{P}$ is the net momentum of the system and $\vec{F}$ is the sum of external forces acting on the system. An analogous formula is $\vec{M} = \frac{d\vec{L}}{dt}$, where $\vec{L}$ is the net angular momentum of the system (with respect to a given point) and $\vec{M}$ is the sum of external torques.

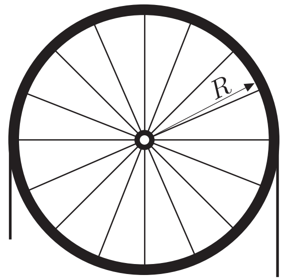
In our case this last method is fruitful when applied both to forces and torques.

Another solution method is to consider the rod and the balls as three different (interacting) bodies. Then the balls' accelerations can be found as per idea 31; one can also employ

**IDEA 45:** Net force and torque acting on very light bodies (compared to other bodies) are zero.

Clearly if this were not true, a non-zero force would generate an infinite acceleration for a massless body.

**PROB 31.** An inextensible rough thread with mass per unit length $\rho$ and length $L$ is thrown over a pulley such that the length of one hanging end is $l$. The pulley is comprised of a hoop of mass $m$ and radius $R$ attached to a horizontal axle by light spokes. The initially motionless system is let go. Find the force on the axle immediately after the motion begins. The friction between the pulley and the axle is negligible.




Why not proceed as follows: to find the force, we will use idea 34; the acceleration of the system will be found using Method 6. To apply idea 34 most handily, let us employ

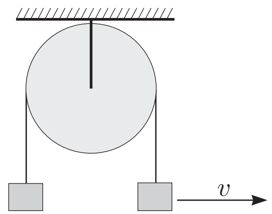
**IDEA 46:** Newton's 2nd law can be written as $\vec{F} = M\vec{a}_C$, where $\vec{a}_C$ is the acceleration of the centre of mass.

This idea is best utilised when a part of the system's mass is motionless and only a relatively small mass is moved about (just like in this case: the only difference after a small period of time is that a short length of thread is "lost" at one end and "gained" at the other end). Obviously idea 32 will be useful here, and idea 19 will save you some effort. Bear in mind that in this case we are not interested in the centre of mass coordinate per se, but only in its change as a function of time; therefore in the expression for this coordinate we can omit the terms that are independent of time: their time derivatives will vanish. The time-dependent part of the centre of mass coordinate should be expressed using the same coordinate that we will use with Method 6 (since Method 6 will produce its second derivative with respect to time). A technical bit of advice may help: a vector is specified by *(a)* its magnitude and direction; *(b)* its projections onto coordinate axes in a given coordinate system;

**IDEA 47:** sometimes it is easier to compute the components of a vector, even if we are interested in its magnitude only.

Above all, this applies when the direction of the vector is neither known nor apparent. In this instance, we should find $F_x$ and $F_y$ in a suitable coordinate system.

**PROB 32.** A thread is thrown over a pulley. At its both ends there are two blocks with equal masses. Initially the two blocks are at the same height. One of them is instantaneously given a small horizontal velocity $v$. Which of the two blocks will reach higher during the subsequent motion? The pulley's mass is negligible.




This problem is really tough, because the key to the solution is a very specific and rarely used

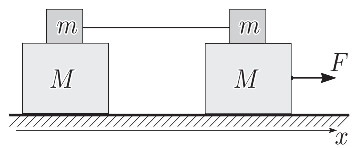
**IDEA 48:** If the centre of mass of a system cannot move, then the net force acting on it is zero.

Here the centre of mass can move about a little bit, but in the longer term (averaged over one period of the pendulum-like motion of the kicked block — cf. idea 22) it is motionless: the blocks have the same mass and if one of them rises, then in the expression for the centre of mass this will be compensated by the descent of the other block. This is also true for the horizontal coordinate of the centre of mass, but it is enough to consider the vertical coordinate only to solve the problem. Let us also bring up the rather obvious

**FACT 10:** the tension in a weightless thread thrown over a weightless pulley or pulled along a frictionless surface is the same everywhere.

The solution algorithm is then as follows: we write down Newton's 2nd law for *(a)* the system made out of two blocks and *(b)* one block; we average both equations and use the equality apparent from *(a)* to find the average tension in the thread, which we then substitute into equation *(b)*. Based on idea 22, we partition the tension in the thread into the average and the high-frequency component and use idea 16.

**PROB 33.** A system of blocks sits on a smooth surface, as shown in the figure. The coefficient of friction between the blocks is $\mu$, while that between the blocks and the surface is $\mu = 0$.




The bottom right block is being pulled by a force $F$. Find the accelerations of all blocks.

**IDEA 49:** When bodies are connected by frictional forces, then to answer some questions fully one needs to consider all possible combinations of there being relative slipping between all possible touching surfaces.

For example, if we are to assume that there is no slipping between two touching bodies, then they could be treated as a whole. Then one should find the frictional force $F_h$ between the bodies and determine when the assumption holds, or when is $F_h$ less that the maximum static friction force $\mu N$.

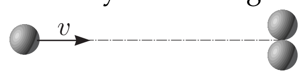
**PROB 34.** A billiard ball hits another stationary billiard ball. At which collection of points could the stationary ball be positioned such that it would be possible to achieve the situation where both balls will fall into two (different) pockets on the table? The collisions are perfectly elastic, the balls are perfectly slippery (hence the rotation of the balls is negligible).

**IDEA 50:** If an absolutely elastic ball hits another motionless identical ball and the rotation (rolling) of the balls can be ignored, then upon impact there will be a right angle between the velocity vectors of the two balls.

To prove this, note that the three velocity vectors (velocity before and the two velocities after the impact) form a triangle because of the momentum conservation law. The conservation of energy means that the sides of the triangle satisfy Pythagore's theorem. A special case of this result is (see the problem after next)

**FACT 11:** When an elastic ball undergoes a central collision with another identical stationary ball, then the first ball stops and the second gains the velocity of the first ball.

**PROB 35.** An absolutely elastic and slippery billiard ball is moving with velocity $v$ toward two motionless identical balls. The motionless balls are touching and their centres lie on a straight line that is perpendicular to the incoming ball's velocity vector. The moving ball is directed exactly toward the touching point of the two balls. Which velocity will the incoming ball have after the collisions? Consider two scenarios: *(a)* the incoming ball hits exactly in the middle between the balls; *(b)* its trajectory is a little bit off and it hits one of the stationary balls marginally earlier.




To answer the first question, it is necessary to use

**IDEA 51:** collisions (and other many-body interactions, like the motion of balls connected by threads or springs) are easier to treated in the centre of mass system, because in that system the momentum conservation is the easiest to write down (the net momentum is zero).

Also, do not forget idea 37! For the second question, let us use

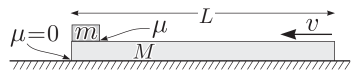
**IDEA 52:** if a force acting on a body during a known time does not change direction, then the transferred momentum has the same direction as the force.

**PROB 36.** $n$ absolutely elastic beads are sliding along the frictionless wire. What is the maximum possible number of collisions? The sizes of the beads are negligible, and so is the probability that more than two beads will collide at the same time.

**IDEA 53:** Representing the process visually, e.g. with a graph, tends to be great help.

Here is an auxiliary question: what would the elastic collision of two balls on an $x-t$ diagram look like?

**PROB 37.** A plank of length $L$ and mass $M$ is lying on a smooth horisontal surface; on its one end lies a small block of mass $m$. The coefficient of friction between the block and the plank is $\mu$. What is the minimal velocity $v$ that needs to be imparted to the plank with a quick shove such that during the subsequent motion the block would slide the whole length of the board and then would fall off the plank? The size of the block is negligible.

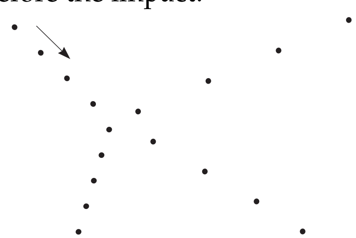



This problem has two more or less equivalent solutions. First, we could solve it using idea 6. Second, we could use ideas 37 and 51, further employing

**IDEA 54:** if a body slides along a level surface, then the energy that gets converted to heat is equal to the product of the friction force and the length of the sliding track.

Indeed, the friction force has a constant magnitude and, as seen in the reference frame of the support, it is always parallel to displacement.

**PROB 38.** The given figure has been produced off a stroboscopic photograph and it depicts the collision of two balls of equal diameters but different masses. The arrow notes the direction of motion of one of the balls before the impact. Find the ratio of the masses of the two balls and show what the direction of motion for the second ball was before the impact.



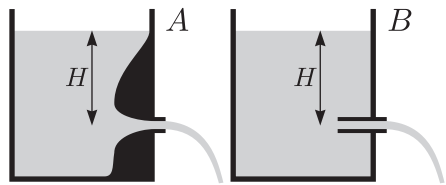
**IDEA 55:** sometimes it is beneficial to treat momenta as vectors, treating their vectorial sums and differences using triangle or parallelogram rules (this is also true of other vectorial quantities: displacements, velocities, accelerations, forces etc.)

To be more specific: when two bodies interact, the vector of the impulse is equal to the vectorial difference of their two momenta. Cf. idea 5.

**FACT 12:** In a stroboscopic photograph, the vector from one position of the body to the next is proportional to its velocity (vector).

**FACT 13:** (Newton's 3rd law) if two bodies have interacted, the changes of momenta of the two bodies are equal and opposite.

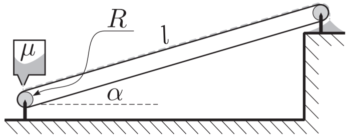
**PROB 39.** There are two barrels ($A$ and $B$) whose taps have different design, see figure. The tap is opened, the height of the water surface from the tap is $H$. What velocity does the water stream leave the barrels with?




**IDEA 56:** If it seems that it is possible to solve a problem using both energy and momentum conservation, then at least one of these is not actually conserved!

It could not be otherwise: the answers are, after all, different. It pays to be attentive here. While designing the tap $A$, there was a clear attempt to preserve the laminarity of the flow: energy is conserved. However, if, motivated by method 3, we were to write down the momentum given to the stream by the air pressure during an infinitesimal time $dt$ — $pSdt$ (where $S$ is the tap's area of cross-section), we would see that, owing to the flow of water, $p \neq \rho g$ (cf. dynamical pressure, Bernoulli's law!). On the other hand, for tap $B$ the laminar flow is not preserved; there will be eddies and loss of energy. We could nonetheless work with momentum: we write the expression for the pressure exerted on the liquid by the walls of the barrel (generally the pressures exerted by the left and the right hand side walls of the barrel cancel each other out, but there remains an uncompensated pressure $p = \rho g H$ exerted to the left of the cross-section of the tap $S$).

**PROB 40.** Sand is transported to the construction site using a conveyor belt. The length of the belt is $l$, the angle with respect to the horizontal is $\alpha$; the belt is driven by the lower pulley with radius $R$, powered externally. The sand is put onto the belt at a constant rate $\mu$ (kg/s). What is the minimal required torque needed to transport the sand? What is the velocity of the belt at that torque? The coefficient of friction is large enough for the sand grains to stop moving immediately after hitting the belt; take the initial velocity of the sand grains to be zero.




**FACT 14:** To make anything move — bodies or a flow (e.g. of sand) — force needs to be exerted.

For this problem, idea 56 and methode 3 will come in handy in addition to

**IDEA 57:** (the condition for continuity) for a stationary flow the flux of matter (the quantity of stuff crossing the cross-section of the flow per unite time) is constant and is independent of the cross-section: $\sigma v =$ Const [$\sigma(x)$ is the matter density per unit distance and $v(x)$ — the velocity of the flow].

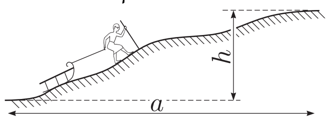
For a flow of incompressible (constant density) liquid in a pipe, such a density is $\sigma = \rho S$ and therefore $vS =$ Const. For a region of space where the flow is discharged — a sink — the mass increases: $\frac{dm}{dt} = \sigma v$ — this equation, too, could be called the condition for continuity.

**PROB 41.** A ductile blob of clay falls against the floor from the height $h$ and starts sliding. What is the velocity of the blob at the very beginning of sliding if the coefficient of friction between the floor and the blob is $\mu$? The initial horizontal velocity of the blob was $u$.

**IDEA 58:** If during an impact against a hard wall there is always sliding, then the ratio of the impulses imparted along and perpendicular to the wall is $\mu$.

Indeed, $\Delta p_\perp = \int N(t)dt$ (integrated over the duration of the impact) and $\Delta p_\parallel = \int \mu N(t)dt = \mu \int N(t)dt$.

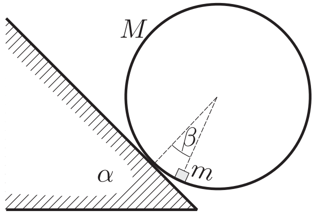
**PROB 42.** A boy is dragging a sled by the rope behind him as he slowly ascends a hill. What is the work that the boy does to transport the sled to the tip of the hill if its height is $h$ and the horizontal distance from the foot of the hill to its tip is $a$? Assume that the rope is always parallel to the tangent of the hill's slope, and that the coefficient of friction between the sled and the snow is $\mu$.




**FACT 15:** if the exact shape of a certain surface or a time dependence is not given, then you have to deal with the general case: prove that the proposition is true for an arbitrary shape.

Clearly, to apply the fact 15, one will need idea 3.

**PROB 43.** An empty cylinder with mass $M$ is rolling without slipping along a slanted surface, whose angle of inclination is $\alpha = 45°$. On its inner surface can slide freely a small block of mass $m = M/2$. What is the angle $\beta$ between the normal to the slanted surface and the straight line segment connecting the centre of the cylinder and the block?




Clearly the simplest solution is based on idea 6, but one needs to calculate the kinetic energy of a rolling cylinder.

**IDEA 59:** $K = K_c + M_\Sigma v_c^2/2$, where $K_c$ is the kinetic energy as seen in the centre of mass frame and $M_\Sigma$ — is the net mass of the system. Analogously: $\vec{P} = M_\Sigma \vec{v}_c$ (since $\vec{P}_c \equiv 0$) and the angular momentum $\vec{L} = L_c + \vec{r}_c \times \vec{P}$. Parallel-axis (Steiner) theorem holds: $I = I_0 + M_\Sigma a^2$, where $I$ is the moment of inertia with respect to an axis $s$ and $I_0$ — that with respect to an axis through the centre of mass (parallel to $s$) while $a$ is the distance between these two axes.

We will have to compute angular momentum already in the next problem, so let us clarify things a little.

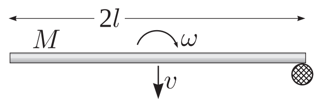
**IDEA 60:** Angular momentum is additive. Dividing the system into point-like masses, $\vec{L} = \sum \vec{L}_i$, where for $i$-th point-like mass $\vec{L}_i = \vec{r}_i \times \vec{p}_i$ (generally) or $L_i = h_i p_i = r_i p_{ti}$ (motion in a plane), $h_i = r_i \sin\alpha_i$ is the lever arm and $p_{ti} = p_i \sin\alpha$ — is the tangential component of the momentum). *Kinetic energy, momentum etc. are also additive.*

If in a three-dimensional space the angular momentum is a vector, for a motion in a plane this vector is perpendicular to the plane and is therefore effectively a scalar (and thus one can abandon cross products). It is often handy to combine ideas 59 and 60: we do not divide the system into particles but, instead, into rigid bodies ($L = \sum L_i$), we compute the moment of inertia $L_i$ of each body according to idea 59: the moment of inertia of the centre of mass plus the moment of inertia as measured in the centre of mass frame.

**IDEA 61:** Here are moments of inertia for a few bodies, with respect to the centre of mass. A rod of length of $l$: $\frac{1}{12}Ml^2$, solid sphere: $\frac{2}{5}MR^2$, spherical shell: $\frac{2}{3}MR^2$, cylinder: $\frac{1}{2}MR^2$, square with side length $a$, axis perpendicular to its plane: $\frac{1}{6}Ma^2$.

If the the rotation axis does not go through the centre of mass, then one can *(a)* find the moment of inertia with respect to the axis of interest using the parallel-axis (Steiner) theorem; *(b)* apply idea 59 to calculate kinetic energy or angular momentum (in which case it is only enough to know the moment of inertia with respect to the centre of mass).

**PROB 44.** A rod of mass $M$ and length $2l$ is sliding on ice. The velocity of the centre of mass of the rod is $v$, the rod's angular velocity is $\omega$. At the instant when the centre of mass velocity is perpendicular to the rod itself, it hits a motionless post with an end. What is the velocity of the centre of mass of the rod after the impact if *(a)* the impact is perfectly inelastic (the end that hits the post stops moving); *(b)* the impact is perfectly elastic.




In case of an absolutely elastic collision one equation follows from energy conservation; if the collision is inelastic, then another condition arises: that of a motionless end of the rod. Still, we have two variables. The second equation arises from

**IDEA 62:** if a body collides with something, then its angular momentum is conserved with respect to the point of impact.

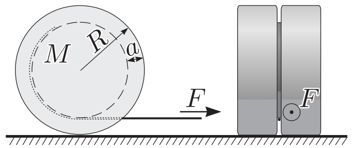
Indeed, during the impact the body's motion is affected by the normal and frictional forces, but both are applied through the point of impact: their lever arm is zero. If a body is moving in a gravitational or similar field, then in the longer term the angular momentum with respect to the point of impact may begin to change, but immediately before and after the collision it is nonetheless the same (gravity is not too strong as opposed to the normal forces that are strong yet short-lived; even though gravity's lever arm is non-zero, it cannot change the angular momentum in an instant).

**PROB 45.** If one hits something rigid — e.g. a lamppost — with a bat, the hand holding the bat may get stung (hurt) as long as the impact misses the so-called centre of percussion of the bat (and hits either below or above such a centre). Determine the position of the centre of percussion for a bat of uniform density. You may assume that during an impact the bat is rotating around its holding hand.

**METHOD 7:** Convert a real-life problem into the formal language of physics and math — in other words, create a model.

Phrased like that, it may seem that the method is rather pointless. However, converting and interpreting real-life scenarios — *modelling* the problem — is one of the most challenging and interesting aspects of physics. It is interesting because it supplies more creative freedom than solving an existing model using well-established ideas. Still, this freedom has limits: the model has to describe the reality as best as possible, the approximations have to make sense and it is desirable that the model were solvable either mentally or with aid of a computer. For a given problem, there is not much freedom left and the business is simplified: there clear hints as to sensible assumptions. Let us begin translating: "A rigid rod of length $l$ and uniform density is rotating around one end with the angular velocity $\omega$, the rotation axis is perpendicular to the rod. At a distance $x$ from the axis there is a motionless post that is parallel to the axis of rotation. The rod hits the post." Now we encounter the first obstacle: is the impact elastic or inelastic? This is not brought up in the text of the problem. Let us leave it for now: maybe we can get somewhere even without the corresponding assumption (it turns out that this is the case). Now we encounter the central question: what does it mean for the hand "not to get stung"? We know it hurts when something hits our hand — if this something gets an impulse from the hand during a short period of time (the impact), as this implies a large force. The hand is stationary, so the hand-held end of the bat should come to halt without receiving any impulse from the hand. Thus our interpretation of the problem is complete: "Following the impact, the rotation is reversed, $0 \geq \omega' \geq -\omega$; during the impact the axis of rotation imparts no impulse on the rod. Find $x$." The penultimate sentence hints at the usage of idea 62.

**PROB 46.** A massive cylinder of radius $R$ and mass $M$ is lying on the floor. A narrow groove of depth $a$ has been chiselled along the circumference of the cylinder. A thread has been wrapped around the groove and is now being pulled by its free end, held horizontally, with a force $F$. The cylinder is positioned such that the thread is being freed from below the cylinder. With what acceleration will the cylinder start moving? The friction between the floor and the cylinder is large enough for there to be no slipping.




There are multiple ways to tackle this problem, but let us use the following idea.

**IDEA 63:** The relation $I\varepsilon = M$ is clearly valid only if the centre of rotation is motionless; however, it turns out that it also holds when the instantaneous axis of rotation is moving translationally such that the distance of the body's centre of mass from the axis does not change (eg when rolling a cylindrical or spherical object).

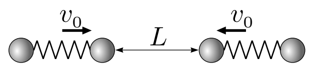
To prove this idea, recall idea 6: kinetic energy appears when work is done, $K = \frac{1}{2}I\omega^2 = M\varphi$ ($\varphi$ is the angle of rotation of the body, $\omega = d\varphi/dt$). If the moment of inertia with respect to the instantaneous axis of rotation $I$ does not depend on time, then $dK/dt = \frac{1}{2}Id\omega^2/dt = I\omega\varepsilon = dM\varphi/dt = M\omega$, which gives $I\varepsilon = M$.

**PROB 47.** A ball is rolling along a horizontal floor in the region $x < 0$ with velocity $\vec{v}_0 = (v_{x0}, v_{y0})$. In the region $x > 0$ there is a conveyor belt that moves with velocity $\vec{u} = (0, u)$ (parallel to its edge $x = 0$). Find the velocity of the ball $\vec{v} = (v_x, v_y)$ with respect to the belt after it has rolled onto the belt. The surface of the conveyor belt is rough (the ball does not slip) and is level with the floor.

**IDEA 64:** For cylindrical or spherical bodies rolling or slipping on a horizontal surface, the angular momentum is conserved with respect to an arbitrary axis lying in the plane of the surface.

Indeed, the points where the normal force and the gravity are applied are on the same straight line with the forces themselves and their sum is zero, meaning that their net torque is also zero; the force of friction is lying in the plane of the surface, and so its lever arm with respect to an axis in the same plane is zero.

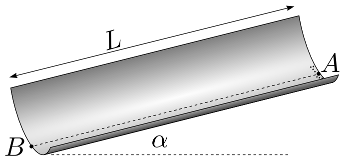
**PROB 48.** A "spring-dumbbell" comprises two balls of mass $m$ that are connect with a spring of stiffness $k$. Two such dumbbells are sliding toward one another, the velocity of either is $v_0$. At some point the distance between them is $L$ (see fig.). After which time is the distance between them equal to $L$ again? The collisions are perfectly elastic.




**IDEA 65:** If a system consisting of elastic bodies, connected by springs, threads etc., interacts with other bodies, then the duration of impact of the elastic bodies is significantly smaller than the characteristic times of other processes. The whole process can then be divided into simpler stages: an almost instantaneous collision of elastic bodies (that could be considered free, as e.g. the spring exerts an insignificant force compared to that exerted in an elastic collision) and the subsequent (or precedent, or in between the collisions) slow process: the oscillations of the spring etc.

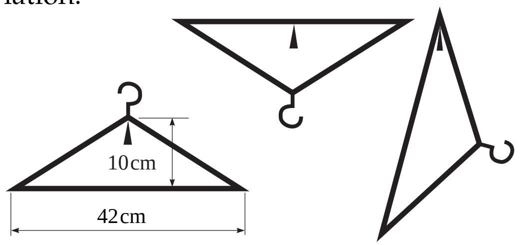
*Note:* this is a rather general idea, division into simpler steps can be useful if rapid (almost instantaneous) processes can occur in a dynamical system; see next problem for an example (also recall idea 51)

**PROB 49.** Small grains of sand are sliding without friction along a cylindrical trough of radius $R$ (see fig.). The inclination angle of the trough is $\alpha$. All grains have initial velocity zero and start near point $A$ (but not necessarily at the point $A$ itself). What should be the length of the trough such that all grains would exit it at the point $B$?




**IDEA 66:** If the motion of a spread collection of particles could be divided into oscillation in a known direction and an oscillation-free motion (so motion perpendicular to the oscillation), then the particles are focussed at certain points: where the oscillation phase of all particles is either zero or is an integer multiple of $2\pi$.

**PROB 50.** A coat hanger made of wire with a non-uniform density distribution is oscillating with a small amplitude in the plane of the figure. In the first two cases the longer side of the triangle is horizontal. In all three cases the periods of oscillation are equal. Find the position of the centre of mass and the period of oscillation.




*Background info:* A finite-size rigid body that oscillates around a fixed axis is known as the physical pendulum. Its frequency of small oscillations is easy to derive from the relation $I\ddot{\varphi} = -mgl\varphi$, where $I$ is the moment of inertia with respect to the axis of oscillation and $l$ is the distance of the centre of mass from that axis: $\omega^{-2} = I/mgl = I_0/mgl + l/g$ (here we employ the parallel-axis/Steiner theorem, see idea 59). The *reduced length* of the physical pendulum is the distance $\tilde{l} = l + I_0/ml$ such that the frequency of oscillation of a mathematical pendulum of that length is the same as for the given physical pendulum.

**IDEA 67:** If we draw a straight line of length $\tilde{l}$ such that it passes through the centre of mass and one of its ends is by the axis of rotation, then if we move the rotation axis to the other end of the segment (and let the body reach a stable equilibrium), then the new frequency of oscillation is the same as before. Conclusion: the set of points where the axis of rotation could be placed without changing the frequency of oscillation, consists of two concentric circles around the centre of mass.

Proof: the formula above could be rewritten as a quadratic equation to find the length $l$ corresponding to the given frequency $\omega$ (i.e. to the given reduced length $\tilde{l} = g/\omega^2$): $l^2 - l\tilde{l} + I_0/m = 0$. According to Vieta's formulae, the solutions $l_1$ and $l_2$ satisfy $l_1 + l_2 = l$, so that $l_1$ and $l_2 = \tilde{l} - l_1$ result in the same frequency of oscillations.

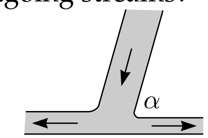
**PROB 51.** A metallic sphere of radius 2 mm and density $\rho = 3000\,\mathrm{kg/m^3}$ is moving in water, falling freely with the acceleration $a_0 = 0{,}57\mathrm{g}$. The water density is $\rho_0 = 1000\,\mathrm{kg/m^3}$. With what acceleration would a spherical bubble of radius 1 mm rise in the water? Consider the flow to be laminar in both cases; neglect friction.

**IDEA 68:** If a body moves in a liquid, the fluid will also move. (A) If the flow is laminar (no eddies), only the liquid adjacent to the body will move; (B) is the flow is turbulent, there will be a turbulent 'tail' behind the body. In either case the characteristic velocity of the moving liquid is the same as the velocity of the body.

Using method 6 we find that in the case (A) the kinetic energy of the system $K = \frac{1}{2}v^2(m + \alpha\rho_0 V)$, where the constant $\alpha$ is a number that characterizes the geometry of the body that correspond to the extent of the region of the liquid that will move (compared to the volume of the body itself). If a body is acted on by a force $F$, then the power produced by this force is $P = Fv = \frac{dK}{dt} = va(m + \alpha\rho_0 V)$. Thus $F = a(m + \alpha\rho_0 V)$: the effective mass of the body increases by $\alpha\rho_0 V$. In the problem above, the constant $\alpha$ for the spherical body can be found using the conditions given in the first half of the problem.

In case (B), if we assume that the velocity of the body is constant, we find $K = \frac{1}{2}v^2\rho_0(\alpha S v t)$, where $S$ is the cross-sectional area of the body and $\alpha S$ is the cross-sectional area of the turbulent 'tail'. This $\alpha$, again, characterizes the body. From here, it is easy to find $Fv = \frac{dK}{dt} = \frac{\alpha}{2}v^3\rho_0 S$, which gives $F = \frac{\alpha}{2}v^2\rho_0 S$.

**PROB 52.** A stream of water falls against a trough's bottom with velocity $v$ and splits into smaller streams going to the left and to the right. Find the velocities of both streams if the incoming stream was inclined at an angle $\alpha$ to the trough (and the resultant streams). What is the ratio of amounts of water carried per unit time in the two outgoing streams?




This is a rather hard problem. Let us first state a few ideas and facts.

**IDEA 69:** For liquid flow, Bernoulli's (i.e. energy conservation) law is often helpful: $p + \rho g h + \frac{1}{2}\rho v^2 =$ Const, where $p$ is the static pressure, $h$ is the height of the considered point and $v$ is the velocity of the flow at that point.

**FACT 16:** Inside the liquid close to its free surface the static pressure is equal to the external pressure.

To solve the second half of the problem, the following is needed:

**IDEA 70:** Idea 44 can be generalized in a way that would hold for open systems (certain amounts of matter enter and leave the system): $\vec{F} = \frac{d\vec{P}}{dt} + \vec{\Phi}_{P\mathrm{in}} - \vec{\Phi}_{P\mathrm{out}}$, where $\vec{\Phi}_{P\mathrm{in}}$ and $\vec{\Phi}_{P\mathrm{out}}$ are the entering and the outgoing fluxes of momentum (in other words, the net momentum of the matter entering and leaving the system, respectively).

The momentum flux of the flowing liquid could be calculated as the product of momentum volume density $\rho\vec{v}$ with the flow rate (volume of liquid entering/leaving the system per unit time).

What is the open system we should be considering in this case? Clearly, a system that would allow relating the incoming flow rate $\mu$ (kg/s) to the outgoing fluxes ($\mu_l$ ja $\mu_r$) using the formula above: a small imaginary region of space that would include the region where the stream splits into two.

**FACT 17:** If we can ignore viscosity, the component of the force exerted by the stream bed (including the 'walls' limiting the flow) on the flow that is parallel to these walls is zero.

**PROB 53.** Find the velocity of propagation of small waves in shallow water. The water is considered shallow if the wavelength is considerably larger than the depth of the water $H$. Thanks to this we can assume that along a vertical cross-section the horizontal velocity of all particles $v_h$ is the same and that the horizontal velocity of water particles is significantly smaller than the vertical velocity.

The smallness of the waves means that their height is significantly smaller than the depth of the water. This allows us to assume that the horizontal velocity of the water particles is significantly smaller than the wave velocity, $u$.

**IDEA 71:** A standard method for finding the velocity of propagation (or another characteristic) of a wave (or another structure with persistent shape) is to choose a reference system where the wave is at rest. In this frame, (a) continuity (idea 57) and (b) energy conservation (e.g. in the form of Bernoulli's law) hold. In certain cases energy conservation law can be replaced by the balance of forces.

(An alternative approach is to linearise and solve a system of coupled partial differential equations.)

**PROB 54.** A small sphere with mass $m = 1\,\mathrm{g}$ is moving along a smooth surface, sliding back and forth and colliding elastically with a wall and a block. The mass of the rectangular block is $M = 1\,\mathrm{kg}$, the initial velocity of the sphere is $v_0 = 10\,\mathrm{m/s}$. What is the velocity of the sphere at the instant when the distance between the sphere and the wall has doubled as compared with the initial distance? By how many times will the average force (averaged over time) exerted by the sphere on the wall have changed?

**IDEA 72:** If a similar oscillatory motion takes place, for which the parameters of the system change slowly (compared to the period of oscillation), then the so-called adiabatic invariant $I$ is conserved: it is the area enclosed by the closed contour traced by the trajectory of the system on the so-called phase diagram (where the coordinates are the spatial coordinate $x$ and momentum $p_x$).

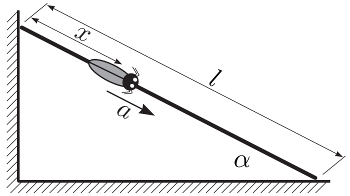
Let us be more precise here. The closed contour is produced as a parametric curve (the so-called phase trajectory) $x(t)$, $p_x(t)$ if we trace the motion of the system during one full period $T$. The phase trajectory is normally drawn with an arrow that indicated the direction of motion. The adiabatic invariant is not exactly and perfectly conserved, but the precision with which it is conserved grows if the ratio $\tau/T$ grows, where $\tau$ is the characteristic time of change of the system's parameters.

Adiabatic invariant plays an instrumental role in physics: from the adiabatic law in gases (compare the result of the previous problem with the adiabatic expansion law for an ideal gas with one degree of freedom!) and is applicable even in quantum mechanics (the number of quanta in the system — e.g. photons — is conserved if the parameters of the system are varied slowly).

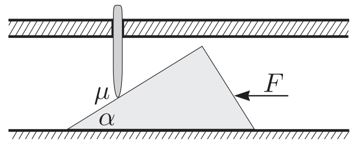
## REVISION PROBLEMS

**PROB 55.** A straight homogeneous rod is being externally supported against a vertical wall such that the angle between the wall and the rod is $\alpha < 90°$. For which values of $\alpha$ can the rod remain stationary when thus supported? Consider two scenarios: a) the wall is slippery and the floor is rough with the friction coefficient $\mu$ ; b) the floor is slippery and the wall is rough with the friction coefficient $\mu$.

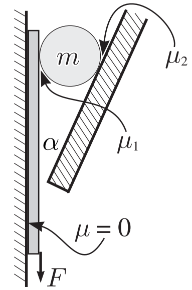
**PROB 56.** A light stick rests with one end against a vertical wall and another on a horizontal floor. A bug wants to crawl down the stick, from top to bottom. How should the bug's acceleration depend on its distance from the top endpoint of the stick? The bug's mass is $m$, the length of the stick is $l$, the angle between the floor and the stick is $\alpha$ and the stick's mass is negligible; both the floor and the wall are slippery ($\mu = 0$). How long will it take the bug to reach the bottom of the stick having started at the top (from rest)?



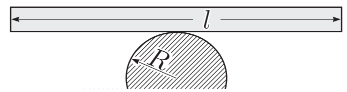

**PROB 57.** A wedge with the angle $\alpha$ at the tip is lying on the horizontal floor. There is a hole with smooth walls in the ceiling. A rod has been inserted snugly into that hole, and it can move up and down without friction, while its axis is fixed to be vertical. The rod is supported against the wedge; the only point with friction is the contact point of the wedge and the rod: the friction coefficient there is $\mu$. For which values of $\mu$ is it possible to push the wedge through, behind the rod, by only applying a sufficiently large horizontal force?



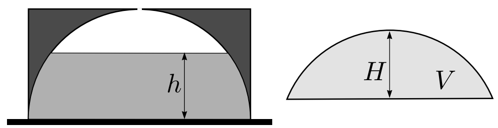

**PROB 58.** Sometimes a contraption is used to hang pictures etc. on the wall, whose model will be presented below. Against a fixed vertical surface is an immovable tilted plane, where the angle between the surface and the plane is $\alpha$. There is a gap between the surface and the plane, where a thin plate could be fit. The plate is positioned tightly against the vertical surface; the coefficient of friction between them can be considered equal to zero. In the space between the plate and the plane a cylinder of mass $m$ can move freely, its axis being horizontal and parallel to all considered surfaces. The cylinder rests on the plate and the plane and the coefficients of friction on those two surfaces are, respectively, $\mu_1$ and $\mu_2$. For which values of the friction coefficients the plate will assuredly not fall down regardless of its weight?




**PROB 59.** On top of a cylinder with a horisontal axis a plank is placed, whose length is $l$ and thickness is $h$. For which radius $R$ of the cylinder the horizontal position of the plank is stable?



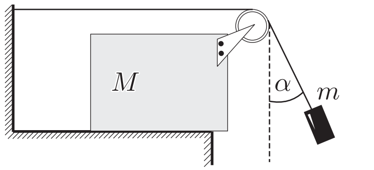

**PROB 60.** A vessel in the shape of a cylinder, whose height equals its radius $R$ and whose cavity is half-spherical, is filled to the brim with water, turned upside down and positioned on a horizontal surface. The radius of the half-spherical cavity is also $R$ and there is a little hole in the vessel's bottom. From below the edges of the freely lying vessel some water leaks out. How high will the remaining layer of water be, if the mass of the vessel is $m$ and the water density is $\rho$? If necessary, use the formula for the volume of a slice of a sphere (see Fig.): $V = \pi H^2(R - H/3)$.



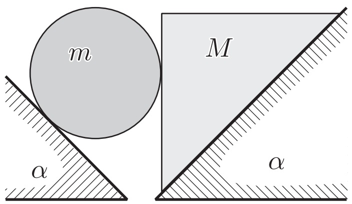

**PROB 61.** A vertical cylindrical vessel with radius $R$ is rotating around its axis

with the angular velocity $\omega$. By how much does the water surface height at the axis differ from the height next to the vessel's edges?

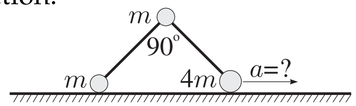
**PROB 62.** A block with mass $M$ is on a slippery horizontal surface. A thread extends over one of its corners. The thread is attached to the wall at its one end and to a little block of mass $m$, which is inclined by an angle $\alpha$ with respect to the vertical, at the other. Initially the thread is stretched and the blocks are held in place. Then the blocks are released. For which ratio of masses will the angle $\alpha$ remain unchanged throughout the subsequent motion?



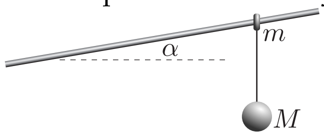

**PROB 63.** Two slippery ($\mu = 0$) wedge-shaped inclined surfaces with equal tilt angles $\alpha$ are positioned such that their sides are parallel, the inclines are facing each other and there is a little gap in between (see fig.). On top of the surfaces are positioned a cylinder and a wedge-shaped block, whereas they are resting one against the other and one of the block's sides is horizontal. The masses are, respectively, $m$ and $M$. What accelerations will the cylinder and the block move with? Find the reaction force between them.




**PROB 64.** Three little cylinders are connected with weightless rods, where there is a hinge near the middle cylinder, so that the angle between the rods can change freely. Initially this angle is a right angle. Two of the cylinders have mass $m$, another one at the side has the mass $4m$. Find the acceleration of the heavier cylinder immediately after the motion begins. Ignore friction.

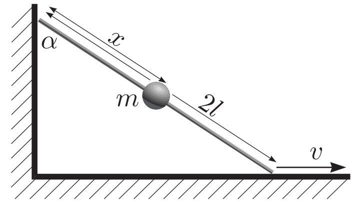



**PROB 65.** A slippery rod is positioned at an angle $\alpha$ with respect to the horizon. A little ring of mass $m$ can slide along the rod, to which a long thread is attached. A small sphere of size $M$ is attached to the thread. Initially the ring is held motionless, and the thread hangs vertically. Then the ring is released. What is the acceleration of the sphere immediately after that?

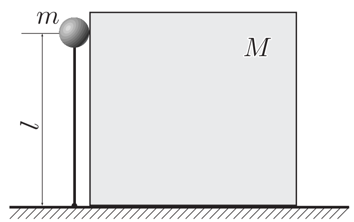



**PROB 66.** A block begins sliding at the uppermost point of a spherical surface. Find the height at which it will lose contact with the surface. The sphere is held in place and its radius is $R$; there is no friction.

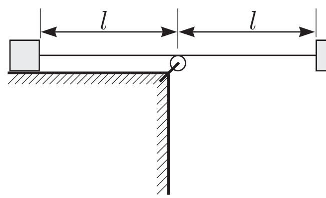
**PROB 67.** The length of a weightless rod is $2l$. A small sphere of mass $m$ is fixed at a distance $x = l$ from its upper end. The rod rests with its one end against the wall and the other against the floor. The end that rests on the floor is being moved with a constant velocity $v$ away from the wall. *a)* Find the force with which the sphere affects the rod at the moment, when the angle between the wall and the rod is $\alpha = 45°$ ; *(b)* what is the answer if $x \neq l$?




**PROB 68.** A light rod with length $l$ is connected to the horizontal surface with a hinge; a small sphere of mass $m$ is connected to the end of the rod. Initially the rod is vertical and the sphere rests against the block of mass $M$. The system is left to freely move and after a certain time the block loses contact with the surface of the block — at the moment when the rod forms an angle $\alpha = \pi/6$ with the horizontal. Find the ratio of masses $M/m$ and the velocity $u$ of the block at the moment of separation.

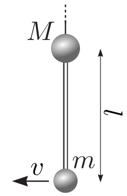



**PROB 69.** At a distance $l$ from the edge of the table lies a block that is connected with a thread to another exact same block. The length of the thread is $2l$ and it is extended around the pulley sitting at the edge of the table. The other block is held above the table such that the string is under tension. Then the second block is released. What happens first: does the first block reach the pulley or does the second one hit the table?

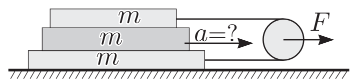



**PROB 70.** A cylindrical ice hockey puck with a uniform thickness and density is given an angular velocity $\omega$ and a translational velocity $u$. What trajectory will the puck follow if the ice is equally slippery everywhere? In which case will it slide farther: when $\omega = 0$ or when $\omega \neq 0$, assuming that in both cases $u$ is the same?

**PROB 71.** A little sphere of mass $M$ hangs at the end of a very long thread; to that sphere is, with a weightless rod, attached another little sphere of mass $m$. The length of the rod is $l$. Initially the system is in equilibrium. What horizontal velocity needs to be given to the bottom sphere for it to ascend the same height with the upper sphere? The sizes of the spheres are negligible compared to the length of the rod.



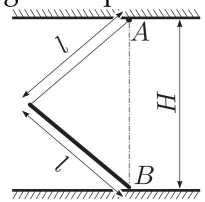

**PROB 72.** A block of mass $m$ lies on a slippery horizontal surface. On top of it lies another block of mass $m$, and on top of that — another block of mass $m$. A thread that connects the first and the third block has been extended around a weightless pulley. The threads are horizontal and the pulley is being pulled by a force $F$. What is the acceleration of the second block? The coefficient of friction between the blocks is $\mu$.




**PROB 73.** A boy with mass $m$ wants to push another boy standing on the ice, whose mass $M$ is bigger that his own. To that end, he speeds up, runs toward the

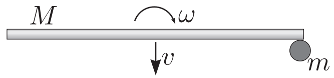
other boy and pushed him for as long as they can stand up. What is the maximal distance by which it is possible to push in this fashion? The maximal velocity of a run is $v$, the coefficient of friction between both boys and the ice is $\mu$.

**PROB 74.** A uniform rod with length $l$ is attached with a weightless thread (whose length is also $l$) to the ceiling at point $A$. The bottom end of the rod rests on the slippery floor at point $B$, which is exactly below point $A$. The length of $AB$ is $H$, $l < H < 2l$. The rod begins to slide from rest; find the maximal acceleration of its centre during subsequent motion.




**PROB 75.** A stick with uniform density rests with one end against the ground and with the other against the wall. Initially it was vertical and began sliding from rest such that all of the subsequent motion takes place in a plane that is perpendicular to the intersection line of the floor and the wall. What was the angle between the stick and the wall at the moment when the stick lost contact with the wall? Ignore friction.

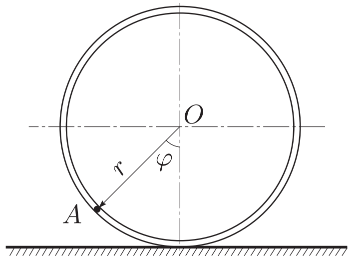
**PROB 76.** A log with mass $M$ is sliding along the ice while rotating. The velocity of the log's centre of mass is $v$, its angular velocity is $\omega$. At the moment when the log is perpendicular to the velocity of its centre of mass, the log hits a stationery puck with mass $m$. For which ratio of the masses $M/m$ is the situation, where the log stays in place while the puck slides away, possible? The collisions are perfectly elastic. The log is straight and its linear density is constant.



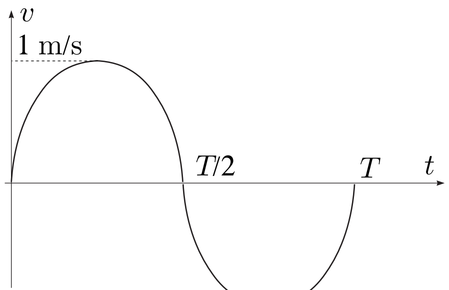

**PROB 77.** A ball falls down from height $h$, initially the ball's horizontal velocity was $v_0$ and it wasn't rotating. a) Find the velocity and the angular velocity of the ball after the following collision against the floor: the ball's deformation against the floor was absolutely elastic, yet there was friction at the contact surface such that the part of the ball that was in contact with the floor stopped. b) Answer the same question with the assumption that the velocities of the surfaces in contact never homogenized and that throughout the collision there was friction with coefficient $\mu$.

**PROB 78.** A ball is rolling down an inclined plane. Find the ball's acceleration. The plane is inclined at an angle $\alpha$, the coefficient of friction between the ball and the plane is $\mu$.

**PROB 79.** A hoop of mass $M$ and radius $r$ stands on a slippery horizontal surface. There is a thin slippery tunnel inside the hoop, along which a tiny block of mass $m$ can slide. Initially all the bodies are at rest and the block is at the hoop's uppermost point. Find the velocity and the acceleration of the hoop's central point at the moment when the angle between the imaginary line connecting the hoop's central point and the block's position and the vertical is $\varphi$.

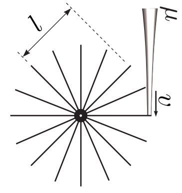



**PROB 80.** A block with mass $m = 10\,\mathrm{g}$ is put on a board that has been made such that, when sliding to the left, the coefficient of friction $\mu_1 = 0{,}3$, while when sliding to the right it is $\mu_2 = 0{,}5$. The board is repeatedly moved left-right according to the graph $v(t)$ (see fig.). The graph is periodic with period $T = 0{,}01\,\mathrm{s}$; the velocity $v$ of the board is considered

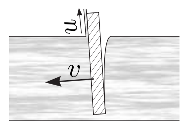



positive when directed to the right. Using the graph, find the average velocity that the block will move with.

**PROB 81.** A water turbine consists of a large number of paddles that could be considered as light flat boards with length $l$, that are at one end attached to a rotating axis. The paddles' free ends are positioned on the surface of an imaginary cylinder that is coaxial with the turbine's axis. A stream of water with velocity $v$ and flow rate $\mu$ (kg/s) is directed on the turbine such that it only hits the edges of the paddles. Find the maximum possible usable power that could be extracted with such a turbine.

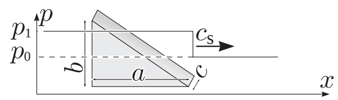



**PROB 82.** A flat board is inclined at an angle $\alpha$ to the vertical. One of its ends is in the water, the other one is outside the water. The board is moving with velocity $v$ with respect to its normal. What is the velocity of the water stream directed up the board?

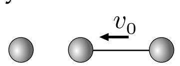



**PROB 83.** A motor-driven wagon is used to transport a load horizontally by a distance $L$. The load is attached to the side of the wagon by a cable of length $l$. Half of the time the wagon is uniformly accelerated, the other half — uniformly decelerated. Find the values of the acceleration $a$ such that, upon reaching the destination, the load will be hanging down motionlessly. You can assume that $a \ll g$.

**PROB 84.** A shockwave could be considered as a discontinuous jump of the air pressure from value $p_0$ to $p_1$, propagating with speed $c_s$. Find the speed which will be obtained, when influenced by the shockwave, *(a)* a wedge-shaped block: a prism whose height is $c$, whose base is a right triangle with legs $a$ and $b$ and which is made out of material with density $\rho$; *b)* an body of an arbitrary shape with volume $V$ and density $\rho$.



**PROB 85.** A dumbbell consisting of two elastic spheres connected with a thin steel rod is moving parallel to its axis with a velocity $v$ toward another exact same spheres. Find the velocity of the dumbbell after a central collision. Is the kinetic energy of the system conserved?




## HINTS

**1.** Write out the balance of torques for the contact point $O$ of the hoop and the shaft. What is the angle that the tangent to the shaft at point $O$ forms with the horizon (given that the wire slips on the shaft)?

**2.** Write down the equation for the torques for the cylinder & block system with respect to the contact point of the cylinder and the inclined plane. What angle with respect to the horizon is formed by the tangent to the cylinder constructed at the position of the little block?

**3.** According to the idea 4, consider the system "rod $CD$ + the mass $m$" as a whole; there are four forces acting on it: $m\vec{g}$, $\vec{F}$, and the tension forces of the rods, $\vec{T}_{AC}$ and $\vec{T}_{BD}$. The tension forces are the ones which we don't know and don't want to know. According to the idea 2, these will drop out from the balance of torques acting on the rod $CD$ with respect to the intersection point of $AC$ and $BD$. Indeed, due to the fact 2, the tension force in the rod $AC$ is parallel to $AC$; the same applies to the rod $BD$. Now, what must be the torque of force $F$? For what direction of the force will this torque be achieved with the minimum magnitude?

**4.** The vector sum of the forces $\vec{F}$ and $m\vec{g}$ has to compensate the sum of the friction and the normal force $\vec{f} = \vec{N} + \vec{F}_h$, i.e. has to be at an angle $\arctan\mu$ with respect to the normal to the plane. Let us draw the force triangle $m\vec{g} + \vec{f} + \vec{F} = 0$: the vector $m\vec{g}$ can be drawn immediately (its direction and magnitude are known), the direction of $\vec{f}$ can be noted by a straight line passing through the terminal point of $m\vec{g}$. $\vec{F}$ has to connect that straight line to the initial point of $m\vec{g}$. For which direction is its magnitude minimal?

**5.** Go to the reference frame of the inclined surface (invoke Ideas 6 and 7) and use the same method as for problem 4 ($\vec{a}+\vec{g}$ functions as the effective gravity $\vec{g}_e$).

**6.** Use a rotating reference frame associated with the cylinder (where the block is at rest, and the centrifugal force $\vec{f}_t$ is constant and pointing downwards). *(a)* The terminal point of the net force of gravity and centrifugal force is moving on a circle and has to be equal to the net force $\vec{f}$ of the normal and frictional forces. What is the maximum allowed angle between the vectors $\vec{f}_t$ and $\vec{f}$ so that there be no slipping? For which direction of $m\vec{g}$ is the angle between the vectors $\vec{f}_t$ and $\vec{f}$ maximal? *(b)* There are still only three forces; as long as there is an equilibrium, these three vectors must form a triangle and hence, must lay on the same plane. According to the idea 9, we'll depict the force balance in this plane, i.e. in the plane defined by the vectors $\vec{g}$ and $\vec{f}_t$. The approach used in part (a) can still be used, but the terminal point of $\vec{f}_t + m\vec{g}$ draws only an arc of a full circle. Determine the central angle of that arc. Depending on the arc length, it may happen that the maximal angle between the surface normal (= the direction of $\vec{f}_t$) and $\vec{f}$ is achieved at one of the endpoints of the arc.

**7.** Based on the Fact no. 4, on which line does the intersection point of the frictional forces have to lie? What can be said about the two angles formed by the frictional force vectors and the thread's direction. Given the Idea no. 1 (the axis is perpendicular with the tension in the thread)? Now combine the two conclusions above. Where is the intersection point of the friction force vectors? What is the direction of the cylinder's velocity vectors at the points where the cylinder rests on the rough band? Where is the cylinder's instantaneous rotation axis (see how to find it in the kinematics brochure)? What is the velocity vector of the cylinder's centre point? *(b)* Will the equilibrium condition found above be violated if the surface is uniformly rough?

**8.** Draw a circle whose diameter is the straight line connecting the points of support. Use Fact no. 6: which curve can the ball move along? Where is the bottom-most point of this curve?

**9.** Consider the torques acting on the rod with respect to the hinge. For which angle $\alpha$ will the net force of the normal and frictional forces push the rod harder against the board?

**10.** By how much will the block descend if the thread is extended by $\delta$?

**11.** Let's assume that the horizontal component of the tension in the rope is $T_x$. What is the vertical component of the tension next to the ceiling? Next to the weight? Write down the condition for the balance of the forces acting on a) the weight and b) the system of weight & rope (cf. Idea no. 4).

**12.** Seeing as $H \ll L$, clearly the curvature of the rope is small, and the angle between the tangent to the rope and horizon remains everywhere small. From the horizontal force balance for the rope, express the horizontal component of the tension force $T_x$ as a function of the length $l$ (note that while $T_x$ remains constant over the entire hanging segment of the rope, we'll need its value at the point $P$ separating the hanging and lying segments). Write down the balance of torques acting on the hanging piece of the rope with respect to the holding hand (according to what has been mentioned above, the arm of the gravity force can be approximated as $l/2$). As a result, you should obtain a quadratic equation for the length $l$.

**13.** Use Idea 8: change into the reference frame of the rotating hinge. *a)* Following the idea 15, write down the condition of torque balance with respect to the hinge (Idea no. 2) for a small deviation angle $\varphi$. Which generates a bigger torque, $m\vec{g}$ or the centrifugal force? (Note that alternatively, the idea 17 can be also used to approach this problem). *b)* Following the idea 17, express the net potential energy for the small deviation angles $\varphi_1$ and $\varphi_2$ using the energy of the centrifugal force (which resembles elastic force!) and the gravitational force; according to the idea 16, keep only the quadratic terms. You should obtain a quadratic polynomial of two variables, $\varphi_1$ and $\varphi_2$. The equilibrium $\varphi_1 = \varphi_2 = 0$ is stable if it corresponds to the potential energy minimum, i,e, if the polynomial yields positive values for any departure from the equilibrium point; this condition leads to two inequalities. First, upon considering $\varphi_2 = 0$ (with $\varphi_2 \neq 0$) we conclude that the multiplier of $\varphi_1^2$ has to be positive. Second, for any $\varphi_2 \neq 0$, the polynomial should be strictly positive, i.e. if we equate this expression to zero and consider it as a quadratic equation for $\varphi_1$, there should be no real-valued roots, which means that the discriminant should be negative.

**14.** Apply the ideas 15 ja 18 for such a angular position of the beam, for which the magnitude of the buoyant force doesn't change (i.e. by assuming a balance of vertical forces). From Idea no. 2, draw the axis through the centre of mass. While computing the torque of the buoyant force, use Ideas 19, 20; the cross-section of the underwater part of the beam could be represented as a superposition of a rectangle and two narrow triangles (one of them of negative mass).

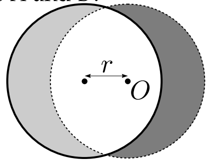
**15.** The container & water system is affected by the gravity and the normal reaction force of the horizontal surface on the liquid. Since we know the pressure of the liquid at the base of the container, we can express the mass of the container from the vertical condition for equilibrium.

**16.** To compute the first correction using the perturbation method we use the Fact 49 and the reference system of the block sliding down uniformly and rectilinearly: knowing the magnitude and the direction of the frictional force we can find its component in $\vec{w}$ and $\vec{u}$ direction. The sign of the latter flips after half a period, and so it cancels out upon averaging.

**17.** Let us choose the origin of the vertical $x$-axis to be a point on the surface of the ocean very far from the iron deposit. For the zero reference point of the Earth's gravitational potential we shall choose $x = 0$ (i.e. $\varphi_{\text{earth}} = gx$), for that of the iron deposit we shall take a point at infinity. Then, for the points on the ocean's surface very far from the iron deposit, the gravitational potential is zero. It remains to find an expression for the potential above the iron deposit as a function of $x$ (using the principle of superposition) and equate it to zero.

**18.** Let us employ the reference frame of the platform. Let us the consider the balance of torques with respect to the axis of the small disk (then the lever arm of the force exerted by that axis is zero). Let us divide the disk into little pieces of equal size. The frictional forces acting on the pieces are equal by magnitude and are directed along the linear velocities of the points of the disk (in the chosen reference frame). Since the motion of the disk can be represented as a rotation around an instantaneous axis, then concentric circles of frictional force vectors are formed (centred at the instantaneous rotation axis). Clearly, the net torque of these vectors with respect to the disk's axis is the smaller, the smaller is the circles' curvature (i.e. the farther the instantaneous rotation axis is): the torque is zero when the instantaneous rotation axis is at infinity and the concentric circles become parallel straight lines. An instantaneous rotation axis at infinity means that the motion is translational, $\omega_3 = 0$ (since the linear velocity $v = \omega_3 r$ of a given point is finite, but $r = \infty$).

**19.** The instantaneous axis of rotation is at a distance $r = v/\omega$ from the disk's axis. Let's use the same imaginary slicing as in the previous problem. Now compute the component of the net force in the direction of motion. Notice that the frictional forces on the points that are symmetrical with respect to the instantaneous rotation axis balance each other across a whole circular region of radius $R - r$. The non-balanced region is unfortunately shaped for calculation. Let us imagine extending the "balanced" region up to $R$ (the dashed circle in the figure). The part of this extended balanced region, where there is no actual rotating disk underneath (the dark gray crescent in the figure), could be represented as a superposition of the two disks, one rotating clockwise and the other – anticlockwise. In that case the clockwise component partakes in the balancing, whereas the anticlockwise component remains unbalanced. To sum up, two thin crescent-shaped regions remain unbalanced: one corresponds to the the real disk (light gray in the figure), the other — to a disk rotating anticlockwise (dark gray); normal to $\vec{v}$, the width of these regions is everywhere equal to $r$. The net force is the easiest to find by integrating across the crescent-shaped regions using the polar coordinate $\varphi$: $|d\vec{F}| = A \cdot dS$, where $dS$ is the area of the surface element; $dF_x = A\cos\varphi dS = B\cos^2\varphi d\varphi$, $F_x = \int dF_x = B\int_0^{2\pi}\cos^2\varphi d\varphi$. What are the values of the constants $A$ and $B$?




**20.** Consider the unit vector $\vec{\tau}$ directed along the infinitesimal displacement vector of the centre of the mass at the instant when the pencil begins moving. Let's express its coordinates in the Cartesian axes $(x, y, z)$, where $x$ is parallel to the pencil and the $(x, y)$-plane is parallel to the inclined slope. Using the spatial rotations formulae we represent it in the new coordinates $(x', y', z)$, which are rotated with respect to $(x, y, z)$ around the $z$-axis by an angle $\varphi$ (so that the axis $x'$ is horizontal). Using the spatial rotations formulae we express the vector's $\vec{\tau}$ vertical coordinate $z'$ in the $(x', y', z')$ coordinate axes, which is obtained from the axes $(x', y', z)$ by rotating about the $x'$ by the angle $\alpha$.

**21.** The string connects the two points with the shortest distance along the cylinder's side; when unfolded, the cylinder is a rectangle. Consider the vertical plane touching the surface of the cylinder that includes the hanging portion of the string. This plane and the cylinder touch along a straight line $s$. If you imagine unfolding the cylinder, the angle between the string and the straight line $s$ is equal to the cylinder's inclination angle $\alpha$. Given this, $l$ is easy to find. When the weight oscillates, the trace of the string still stays straight on the unfolded cylinder. Therefore the length of the hanging string (and thus the weight's potential energy) do not depend in any oscillatory state on whether the surface of the cylinder is truly cylindrical or is unfolded into a planar vertical surface (as long as the spatial orientation of the axis $s$ is preserved).

**22.** Write down the two equations describing the balance of force and torques, and then another one that describes the linear relation between the elongations of the string: $T_1 - T_2 = T_2 - T_3$.

**23.** Initially only the vertical forces affect the hanging block, therefore the initial displacement vector is also vertical. If the acceleration of the large block is $a_1$, that of the block on top of it — $a_2$ and that of the hanging block — $a_3$, then $a_1 + a_2 = a_3$ holds. Now we can write down Newton's 2nd law for each body. The fourth and the final unknown is the tension in the string.

**24.** Go to the reference frame of the wedge-block. In the borderline case, the force of inertia's and gravity's net force on the ball $m$ is normal to the left slope (so that the ball stay at rest there). Consider the net forces acting on the balls. Their components normal to the surface they rest on are $\vec{F}_{\perp 1}$ and $\vec{F}_{\perp 2}$. These are equal to the normal forces $\vec{N}_1$ and $\vec{N}_2$ acting on the balls and therefore have to have equal magnitudes ($F_{\perp 1} = F_{\perp 2}$) to ensure that the force balance is achieved horizontally for the wedge-block.

**25.** Let's take the displacement $\xi$ of the wedge as coordinate describing the system's position. If the wedge moves by $\xi$, then the block moves the same amount with respect to the wedge, because the rope is unstretchable, and the kinetic energy changes by $\Pi = mg\xi\sin\alpha$. The velocity of the wedge is $\dot{\xi}$ and that of the block is $2\dot{\xi}\sin\frac{\alpha}{2}$ (found by adding velocities, where the two vectors $\dot{\xi}$ are at an angle $\alpha$), therefore the net kinetic energy $K = \frac{1}{2}\dot{\xi}^2(M + 4m\sin^2\frac{\alpha}{2})$. Then we find $\Pi'(\xi) = mg\sin\alpha$ and $\mathcal{M} = M + 4m\sin^2\frac{\alpha}{2}$; their sum gives the answer.

**26.** Again, let's take the wedge's displacement as the coordinate $\xi$; if the displacement of the block along the surface of the wedge is $\eta$, then the centre of mass being at rest gives $\eta(m_1\cos\alpha_1 + m_2\cos\alpha_2) = (M + m_1 + m_2)\xi$. From here one can extract $\eta$ as a function of $\xi$, but to keep the formulae brief it's better not to substitute this expression everywhere. The kinetic energies of the block can be found as sums of horizontal $[\frac{1}{2}m_i(\dot{\xi} - \dot{\eta}\cos\alpha_i)^2]$ and vertical $[\frac{1}{2}m_i(\dot{\eta}\sin\alpha_i)^2]$ energies.

**27.** When writing down energy conservation, note that the block's velocity is twice the cylinder's velocity horizontal component and that the latter is equal to the vertical component, too (why?). Project Newton's 2nd law onto the axis that passes through the top corner of the step and the cylinder's centre: this axis is perpendicular both to the normal force between the block and the cylinder and to the cylinder's tangential acceleration. Second question: the ratio of two normal forces is constant (why? what is it equal to? Hint: compare the horizontal accelerations of the cylinder and the block and remember Newton's 2nd law), therefore they will be equal to zero at the same instant.

**28.** By projecting Newton's 2nd law on the axis in the direction of the normal force we see that the normal force is the smallest at the bottommost point of the trajectory's arch-shaped part. (There, the centripetal acceleration is the largest, gravitational force's component along the axis is the smallest).

**29.** The energy of the "pellet & block" system is always conserved; momentum will only start to be conserved once the pellet passes the bottommost point. When it arrives there for the second time, the block's velocity is maximal (why?).

**30.** Let's apply Idea no. 44 for $\vec{P}$: the system's net momentum is $P = \omega lm + 2\omega lM$, net force $F = (m + M)g - T$. The same using rotational considerations: with respect to the leftmost ball's initial position, the angular momentum is $l(2\omega l)M$ (velocity is $2\omega l$, the velocity's lever arm — $l$); net torque is $(T + Mg)l$. Now, for the formula given in Idea no. 44 we need the angular acceleration $\varepsilon = \dot{\omega}$. Let's find it using Method no. 6: $\Pi = l\varphi(m + 2M)$, $K = \frac{1}{2}\dot{\varphi}^2 l^2(m + 4M)$. Another solution route: the ratio of accelerations is 1:2; there are four unknowns (two normal forces, acceleration and string tension); equations: three force balances (for either ball and the rod) and one torque balance (wrt the left endpoint of the rod).

**31.** Method no. 6: for the generalized coordinate $\xi$ we can use the displacement of the thread's endpoint. Ideas no. 32,20: the change of the system's CM $y$-coordinate is $\xi\rho h/M$ ($h$ — the difference in the heights of the thread's endpoints, $M$ — the net mass of the system; assume that $\xi \ll h$). For the $x$-coordinate it's $2\xi\rho R/M$.

**32.** $\langle T(1+\cos\alpha)\rangle = 2mg$, $T = \langle T\rangle + \tilde{T}$, where $|\tilde{T}| \ll T$. Based on the Idea no. 16 we ignore the tiniest term $\langle \tilde{T}\alpha^2\rangle$ and note that $\langle \alpha^2\rangle > 0$.

**33.** We have to consider two options: either all the bodies move together, or the rightmost large block moves separately. Why cannot the situations occur where *(a)* all three components move separately, or *(b)* the left large block moves separately?

**34.** After the collision the ball's trajectories are orthogonal crossing straight lines; the angle with respect to the initial trajectory is determined by how much the collision was off-centre.

**35.** For slightly non-central motion: what will be the direction of momentum of the ball that was first to be hit? Now apply the Idea no. 50 again. Central motion: express the velocities after the collision via the horizontal component of the momentum $p_x$ that has been transferred to one of the balls. What is the transferred vertical component $p_y$? Energy conservation provides us an equation to find $p_y$ (it is convenient to express the energy as $p^2/2m$).

**36.** The graph looks like $n$ intersecting straight lines; the intersection point of a pair of straight lines corresponds to a collision of two balls (the graph of either ball's motion is a jagged line; at a collision point the angles of the two jagged lines touch one another so that it looks as if the two straight lines intersect).

**37.** Initial velocities in the centre of mass: $\frac{mv}{m+M}$, $\frac{Mv}{m+M}$, final velocities are zero; friction does work: $\mu mgL$.

**38.** Based on the figure we immediately obtain (to within a multiplicative constant) the magnitudes and directions of the momenta, but not which momentum is which ball's. It is necessary to find out where the ball marked with an arrow will proceed after the collision. Fact no. 13 will help choose from the three options.

**39.** Energy: in time $dt$ the distribution of the liquid will change: there is still some water at the centre, but a certain mass $dm$ has been displaced from above to the level of the tap (and then through the tap), so the change in the system's potential energy is $gH \cdot dm$. Momentum: the water in the barrel obtains the total momentum $\rho gHS \cdot dt$ from the walls. This momentum is passed on to the stream of water with the mass $\rho Sv \cdot dt$.

**40.** Energy is not conserved: the grains of sand slip and experience friction. In time $dt$ the sand landing on the conveyor belt receives momentum $dp = v \cdot dm = v\mu \cdot dt$ from the belt: the force between the freshly fallen sand and the belt is $F_1 = dp/dt$. The sand already lying on the belt experiences the gravitational force $mg$ which is compensated by the component of the friction parallel to the belt, $F_2 = mg\cos\alpha$, where $m = \sigma L$ is the mass of the sand on the belt and $\sigma v = \mu$. The minimization has to be done over $v$.

**41.** During the collision $\Delta p_\perp = \sqrt{2gh}$.

**42.** Consider a short section of the path along the hill with length $dl$. In addition to the change in the potential energy work is done to overcome friction, $dA_h = \mu mg\tan\alpha \cdot dl$. WE find $dA_h = C \cdot dx$, where $C$ is a constant. Summing over all such little path increments $dl$ we find $A_h = C\Delta x$.

**43.** The kinetic energy $K = \frac{m}{2}\dot{x}^2 + M\dot{x}^2$, where $x$ is the displacement along the slanted surface; $\Pi = (M+m)\sin\alpha$. Having found the acceleration $a$ we change into a reference frame (of the cylinder) moving with acceleration $a$ (Ideas no. 6 and 7), where the block is being displaced along the effective acceleration due to gravity — as low as possible.

**44.** According to the Ideas no. 59 and 60, the angular momentum of the rod before the collision is $L_0 = Mlv - \frac{1}{3}Ml^2\omega$; after the collision $L_1 = Mlv' - \frac{1}{3}Ml^2\omega'$; $L_1 = L_2$. The expression for energy is $K = \frac{1}{2}Mv^2 + \frac{1}{6}Ml^2\omega^2$. The condition for being at the end: $v' + l\omega' = 0$ (we consider $\omega$ to be positive if the rotation is in the direction marked in the figure).

**45.** The angular momentum with respect to the impact point before the collision: $mv(x - \frac{l}{2}) - I_0\omega$, where $v = \omega\frac{l}{2}$ and $I_0 = \frac{1}{12}ml^2$.

**46.** The instantaneous rotation axis passes the contact point of the cylinder and the floor; its distance from the centre of mass does not change, so we can use Idea no. 63; $I = \frac{3}{2}mR^2$.

**47.** Let us direct the $z$ axis upward (this will fix the signs of the angular momenta). The final moment of inertia with respect to the $x$-axis is $-\frac{7}{5}mv_yR - muR$ and with respect to the $y$-axis is $\frac{7}{5}mv_xR$.

**48.** Immediately after the first collision the centres of masses of both dumbbells are at rest, the velocities of the colliding balls reverse direction, the non-colliding balls' velocities don't change. Both dumbbells act like pendula and complete half an oscillation period, after which the second collision occurs – analogous to the first one.

**49.** The grains of sand perform harmonic oscillations in the plane perpendicular to the cylinder's axis — like a mathematical pendulum of length $l = R$ in the gravitational field $g\cos\alpha$; along the axis there is uniform acceleration ($a = g\sin\alpha$). Focussing occurs if the time to cross the trough along its axis is an integer multiple of the oscillation's half-period.

**50.** Observing the equilibrium position we conclude that the centre of mass lies on the symmetry axis of the hanger. The three suspension points must be located on the two concentric circles mentioned by Idea no. 67. Therefore one of the circles must accommodate at least two points out of the three, while the circles' centre (the hanger's centre of mass) must lie inside the region bounded by the hanger's wires on its symmetry axis. There is only one pair of circles that satisfies all these conditions. Computing the radii $l_1$ and $l_2$ of the circles using trigonometry we determine the reduced length of the pendulum $l_1 + l_2$ and, using that, the oscillation period.

**51.** The effective mass of the moving water can be found using the acceleration of the falling ball. For the rising bubble the effective mass is exactly the same, the mass of the gas, compared to that, is negligibly small.

**52.** The water stream could be mentally divided into two parts: the leftmost stream will turn to the left upon touching the trough, the rightmost — to the right. Thus, two imaginary 'water tubes' form. In either tube the static pressure is equal to the external pressure (since there is the liquid's outer surface in the vicinity): according to Bernoulli's law, the velocity of the liquid cannot change. Based on the conservation of momentum horizontally, the momentum flows of the left- and right-flowing streams have to add up to the original stream's momentum flow's horizontal component. Note that due to continuity, $\mu = \mu_v + \mu_p$.

**53.** Due to continuity $(u+v)(H+h) = Hu$ Const, where $h = h(x)$ is the height of the water at point $x$ and $v = v(x)$ is the velocity. We can write down Bernoulli's law for an imaginary 'tube' near the surface (the region between the free surface and the stream lines not far from the surface): $\frac{1}{2}\rho(u+v)^2 + \rho g(H+h) = \frac{1}{2}\rho u^2 + \rho gH = \text{Const}$. We can ignore that small second order terms (which include the factors $v^2$ or $vh$)

**54.** The phase trajectory is a horizontal rectangle with sides $L$ and $2mv$, where $L$ is the distance from the block to the wall; the adiabatic invariant is thus $4Lmv$.

**55.** Consider the balance of torques. For the net force vectors of the normal and frictional forces, when you extend them, their crossing point must be above the centre of mass.

**56.** Let's write down Newton's 2nd law for rotational motion with respect to the crossing point of the normal forces: the angular momentum of the bug is $L = mvl\sin\alpha\cos\alpha$, the speed of change of this angular momentum will be equal to the torque due to gravity acting on the bug (the other forces' lever arms are zero). When computing the period, note that the acceleration is negative and proportional to the distance from the bottom endpoint, i.e. we are dealing with harmonic oscillations.

**57.** The blocking occurs if the net force of normal and frictional forces pulls the rod downwards.

**58.** Once the blocking occurs we can ignore all the forces apart from normal and frictional ones. Suppose it has occurred. Then the net frictional and normal forces acting from the left and from the right have to balance each other both as forces and torques, i.e. lie on the same straight line and have equal magnitudes. Thus we obtain the angle between the surface normal and the net force of friction and normal force.

**59.** Consider the direction of the torque acting on the plank with respect to the point of contact, when the plank has turned by an angle $\varphi$: the contact point shifts by $R\varphi$, the horizontal coordinate of the centre of mass shifts by the distance $\frac{h}{2}\varphi$ from the original position of the contact point.

**60.** The only force from the surface on the system vessel & water is equal to the hydrostatic pressure $\rho g h \pi R^2$; it balances the gravitational force $(m + \rho V)g$. Note that $H = R - h$.

**61.** The gravitational potential of the centrifugal force is $\frac{1}{2}\omega^2 r^2$, where $r$ is the distance from the rotation axis.

**62.** Assume the reference frame of the large block (which moves with acceleration $a$). Where does the effective gravity (the net force of the gravity and the force of inertia) have to be directed? What is $a$? With which acceleration does the little block fall in this reference frame? What is the tension $T$ of the thread? Having answers to these questions we can write down the equilibrium condition for the large block $ma = T(1 - \sin\alpha)$.

**63.** Let us use the displacement of the sphere (down the inclined surface) as the generalized coordinate $\xi$. What is the displacement of the sphere (up the other inclined surface)? Evidently $\Pi = (m - M)g\xi \sin\alpha$. The normal force between the two bodies can be found by projecting Newton's second law onto the inclined surface's direction.

**64.** Let the displacement of the large cylinder be $\xi$, the horizontal displacement of the middle and the leftmost cylinder, respectively, $x$ and $y$. What is the relationship between them given that the centre of mass is at rest? What is the relationship between them given that the length of the rods does not change? From the two equations thus obtained we can express $x$ and $y$ via $\xi$. If we assume the displacement to be tiny, what is the relationship between the vertical displacement $z$ of the middle cylinder and the horizontal projection of the rod's length, $\xi - x$? Knowing these results, applying Method no. 6 is straightforward.

**65.** Where is the small displacement $\xi$ of the sphere directed (see Idea no. 30)? What is the displacement of the ring expressed via $\xi$? Use Method no. 6.

**66.** Use Idea no. 38 along with energy conservation by projecting the force and the acceleration in the Newton's 2nd law radially.

**67.** Let us use some ideas from kinematics to find the acceleration of the sphere (K1, K29 and K2: by changing into the reference frame moving with velocity $v$ we find the component of the sphere's acceleration along the rod and by noticing that the horizontal acceleration of the sphere is zero, we obtain, using trigonometry, the magnitude of the acceleration). Now use Newton's 2nd law.

**68.** Using the velocity $v$ of the sphere we can express the velocity of the block at the moment being investigated (bearing in mind that their horizontal velocities are equal). Using Idea no. 38 we find that the block's (and thus the sphere's) horizontal acceleration is zero; by using Newton's 2nd law for the sphere and the horizontal direction we conclude that the tension in the rod is also zero. From the energy conservation law we express $v^2$ and from Newton's 2nd law for the sphere and the axis directed along the rod we obtain an equation wherein hides the solution.

**69.** Using Newton's 2nd law investigate whither the system's centre of mass will move — to the left or to the right (if the centre of mass had not move, then the both events would have happened at the same time).

**70.** To answer the first part: show that the force perpendicular to velocity is zero (use Method no. 3 and Idea no. 26). To answer the second part use Method no. 3 and idea 54.

**71.** Due to the length of the thread there are no horizontal forces, i.e. the horizontal component of momentum is conserved, and so is the energy. From the two corresponding equation the limiting velocity $v = v_0$ can be found, for which the bottom sphere ascends exactly to the height of the top one. Note that at that point its vertical velocity is zero, cf. Idea no. 42.

**72.** Use Idea no. 49. Options: all block keep together; everything slides; the top one slides and and the bottom two stay together (why is it not possible that the top two keep together and the bottom one slides?).

**73.** Which conservation law acts when the two boys collide (during a limited time of collision) — do we consider the collision absolutely elastic or inelastic (can momentum be lost and where? If it is inelastic, where does the energy go?), see Idea no. 56? After the collision: the common acceleration of the two boys is constant, knowing the initial and final velocities finding the distance becomes an easy kinematics problem.

**74.** Prove that for a vertical thread the velocity is maximal (by applying Idea no. 42 for the rotation angle of the rod show that its angular velocity is zero in that position; use Idea no. 59). Then it only remains to apply energy conservation (remember that $\omega = 0$).

**75.** Find the instantaneous rotation axis (make sure that its distance from the centre of mass is $\frac{l}{2}$). Prove that the centre of mass moves along a circle centred at the corned of the wall and the floor, whereas the polar coordinate of the centre of mass on that circle is the same as the angle $\varphi$ between the wall and the stick. Express the kinetic energy as a function of the derivative $\dot{\varphi}$ of the generalized coordinate $\varphi$ using the parallel-axis (Steiner's) theorem and express the energy conservation law as $\omega^2 = f(\varphi)$; using Method no. 6 we obtain $\varepsilon = \dot{\omega} = \frac{1}{2}f'(\varphi)$. When the normal force against the wall reaches zero, the acceleration of the centre of mass is vertical: present this condition using the tangential and radial accelerations of the centre of mass on its circular orbit ($\frac{l}{2}\varepsilon$ and $\frac{l}{2}\omega^2$ respectively) and use it as an equation to find $\varphi$.

**76.** Based on Idea no. 62 we find that $\omega = 6v/l$. Using energy and momentum conservation we eliminate the puck's velocity after the collision and express the mass ratio.

**77.** The forces along the normal to the surface are elastic forces, so the energy in vertical direction is conserved during the collision: after the collision the corresponding velocity component is the same as before. To find the other two unknowns, the horizontal and angular velocities, we can obtain one equation using Idea no. 62. The second equation arises from *(a)* the condition that the velocity of the ball's surface is zero at the contact point (no sliding; *(b)* the equation arising from 58).

**78.** Using the idea 49 we investigate the sliding and rolling regimes. In the latter case the quickest way to find the answer is to use Idea no. 63.

**79.** The velocity can be found from the conservation laws for energy and momentum (note that the hoop is moving translationally). To find the acceleration it is convenient to use the non-inertial reference frame of the hoop, where the centripetal acceleration of the block is easily found. The condition for the radial balance of the block gives the normal force between the block and the hoop (don't forget the force of inertia!); the horizontal balance condition for the hoop provides an equation for finding the acceleration.

**80.** Let us assume the block's velocity to be approximately constant. For a certain time $t_l$ the base slides to the left with respect to the block and the momentum imparted by the frictional force at that time is also directed to the left. During the remaining time $t_r$ the base slides to the right with respective momentum directed to the right as well. The equilibrium condition is that the two momenta have equal magnitudes; hence we ding the equilibrium value of $t_l/t_r$. From the graph we find the velocity for which that ratio has the needed value.

**81.** As the water flows against the paddles it obtain the same vertical velocity $u$ as the paddles themselves. This allows to compute the momentum imparted to the paddle per unit time (i.e. the force), which ends up being proportional to the difference: $F \propto v - u$. From there, it is not very hard to find the maximum of the power $Fu$.

**82.** In the reference frame of the board the problem is equivalent to the problem no. 52.

**83.** Go into the (accelerated) reference frame of the wagon, where the effective gravity $\sqrt{a^2 + g^2}$ is at a small angle with respect to the vertical. The load will oscillate yet remain motionless at the end if the cable is vertical at the stopping moment and the load's velocity is zero. It is possible when the corresponding position is the maximal deviation during the oscillation. Therefore the oscillation amplitude has to be the same both during the acceleration and deceleration, so that even when the deceleration begins the cable has to be vertical. In that case, how are the acceleration time and the oscillation period related?

**84.** If the shockwave is at the point where the intersection area of its wavefront and the considered body is $S$, then what is the force acting on the body? Let us assume that the body stays (almost) at the same place as the shockwave passes it. Then the momentum imparted during the time $dt$ can be found using the cross-sectional area $S$ and the distance $dx = c_s \cdot dt$ covered by the wavefront. Note that $S \cdot dx$ is the volume element. Finally we sum over all imparted momenta.

**85.** The rod will act like a spring (since the rod is thin and made out of steel, while steel is elastic). After the left sphere has collided with the stationary sphere, the latter will acquire velocity $v_0$ and the former will stay at rest. Then the dumbbell, as a system of spheres and springs, will begin oscillating around its centre of mass. What is the velocity of the centre of mass? Convince yourself that after half a period the single sphere is already far enough that the left sphere is not going to collide with it again. The oscillations of the dumbbell will decay little by little — so some energy will be lost there.

## ANSWERS

**1.** $\arcsin \frac{R\mu}{(R+l)\sqrt{\mu^2+1}}$.

**2.** $\arcsin \frac{m}{M+m}\frac{\mu}{\sqrt{\mu^2+1}}$.

**3.** $mg/2$.

**4.** a) $\mu mg/\sqrt{1+\mu^2}$; b) $mg\sin(\arctan\mu - \alpha)$.

**5.** $\mu \geq \frac{|g\sin\alpha - a\cos\alpha|}{g\cos\alpha + a\sin\alpha}$, if $g + a\tan\alpha > 0$.

**6.** a) $\omega^2 R \geq g\sqrt{1+\mu^{-2}}$;
b) $\omega^2 R \geq g\sqrt{1+\mu^{-2}}$, if $\mu < \cot\alpha$ and $\omega^2 R \geq g(\cos\alpha + \mu^{-1}\sin\alpha)$ if $\mu > \cot\alpha$

**7.** $v/2$.

**8.** $\tan 2\alpha = h/a$

**9.** $\mu_1 \geq \sqrt{l^2 - h^2}/h$

**10.** $3mg$

**11.** $2\arctan[(1+\frac{m}{M})\cot\alpha]$

**12.** $\sqrt{2HL\mu + \mu^2H^2} - \mu H \approx \sqrt{2HL\mu} - \mu H \approx 7{,}2\,\mathrm{m}$.

**13.** a) $\omega^2 < g/l$ ; b) $(2-\sqrt{2})g/l$

**14.** $\frac{1}{2}(1 - 3^{-1/2})\rho_v \approx 211\,\mathrm{kg/m^3}$

**15.** $\frac{\pi}{3}\rho R^3$

**16.** $v/\sqrt{\mu^2\cot^2\alpha - 1}$

**17.** $\frac{4}{3}\pi G r^3 \Delta\rho/g(r+h) \approx 0{,}95\,\mathrm{cm}$

**18.** $-\omega$

**19.** $\mu mgv/\omega R$

**20.** $\cos\varphi\tan\alpha < \tan 30°$

**21.** $L - \pi R/2\cos\alpha$; $2\pi\sqrt{L/g}$

**22.** $\frac{1}{12}mg$, $\frac{1}{3}mg$, $\frac{7}{12}mg$

**23.** $mg/(2M+m)$

**24.** $m < M\cos 2\alpha$.

**25.** $mg\sin\alpha/[M + 2m(1-\cos\alpha)] = mg\sin\alpha/[M + 4m\sin^2\frac{\alpha}{2}]$.

**26.** $g\frac{(m_1\sin\alpha_1 - m_2\sin\alpha_2)(m_1\cos\alpha_1 + m_2\cos\alpha_2)}{(m_1+m_2+M)(m_1+m_2) - (m_1\cos\alpha_1 + m_2\cos\alpha_2)^2}$.

**27.** $mg(5\sqrt{2}-4)/6$); Simultaneously.

**28.** $\cos\alpha \geq \frac{1}{3}(2 + v^2/gR)$

**29.** $2\frac{m}{M+m}\sqrt{2gR}$

**30.** $mMg/(m+4M)$

**31.** $F_x = 2Ra\rho$, $F_y = \rho(m+\rho L)g - (L - \pi R - 2l)a$, where $a = \rho g(L - \pi R - 2l)/(m+\rho L)$.

**32.** The one that had not been pushed.

**33.** If $F \leq 2\mu mg\frac{m+M}{2m+M}$: $a_1 = a_2 = \frac{1}{2}\frac{F}{M+m}$; otherwise $a_1 = \frac{F}{M} - \mu g\frac{m}{M}$, $a_2 = \mu g\frac{m}{2m+M}$.

**34.** On a half-circle.

**35.** *(a)* $v/5$; *(b)* $v/4$.

**36.** $n(n-1)/2$

**37.** $\sqrt{2\mu gL(1+\frac{m}{M})}$

**38.** 3,5; was coming from below right.

**39.** A: $\sqrt{2gh}$ ; $\sqrt{gh}$.

**40.** $2R\mu\sqrt{gl\sin\alpha}$, $\sqrt{gl\sin\alpha}$.

**41.** $u - \mu\sqrt{2gh}$.

**42.** $mg(h + \mu a)$.

**43.** $\arctan\frac{2}{5} \approx 21°48'$.

**44.** *(a)* $(\omega l + 3v)/4$; *(b)* $(\omega l + v)/2$.

**45.** At a distance $2l/3$ from the holding hand, where $l$ is the length of the bat.

**46.** $\frac{2}{3}\frac{F}{M}\frac{a}{R}$

**47.** $(v_{x0}, v_{y0} - \frac{5}{7}u)$

**48.** $L/v_0 + \pi\sqrt{m/2k}$

**49.** $\frac{1}{2}\pi^2(n+\frac{1}{2})^2 R\tan\alpha$

**50.** 1,03 s

**51.** 2,0 g

**52.** $v_1 = v_2 = v$; $\cot^2\frac{\alpha}{2}$

**53.** $\sqrt{gH}$.

**54.** 5 m/s.

**55.** *(a)* $\tan \leq 2\mu$ ; *(a)* impossible.

**56.** $g(1-\frac{x}{l})\sin^{-1}\alpha$; $\frac{\pi}{2}\sqrt{l\sin\alpha/g}$

**57.** $\mu < \cot\alpha$.

**58.** $\mu_1 < \tan\frac{\alpha}{2}$ and $\mu_2 < \tan\frac{\alpha}{2}$.

**59.** $R > h/2$

**60.** $\sqrt[3]{3m/\pi\rho}$

**61.** $\omega^2 R^2/2g$

**62.** $M/m = \cot\alpha - 1$.

**63.** $\frac{2mM}{M+m}g\tan\alpha$

**64.** $g/9$.

**65.** $g\frac{m+M}{m+M\sin^2\alpha}\sin^2\alpha$.

**66.** $2/3R$

**67.** $m[g - v^2(2l-x)/\sqrt{2}l^2]$

**68.** $M/m = 4$, $u = \sqrt{gl/8}$.

**69.** The first one arrives first

**70.** A straight line; if $\omega \neq 0$

**71.** $\sqrt{2gl(1+m/M)}$

**72.** $\frac{F}{3m}$, if $\frac{F}{m\mu g} < 6$; $\frac{F}{4m} + \frac{1}{2}\mu g$, if $6 < \frac{F}{m\mu g} < 10$; $3\mu g$, if $\frac{F}{m\mu g} > 10$

**73.** $m^2v^2/2(M^2 - m^2)\mu g$

**74.** $\sqrt{(l - \frac{H}{2})g}$

**75.** $\arccos\frac{2}{3} \approx 48°12'$

**76.** $M/m = 4$.

**77.** *(a)* $\omega = 5v_0/7R$, $v_x = 5v_0/7$, $v_y = \sqrt{2gh}$;
*(b)* $v_y = \sqrt{2gh}$, $v_x = v_0 - 2\mu v_y$, $\omega = 5\sqrt{2gh}\mu/R$.

**78.** $\frac{5}{7}g\sin\alpha$, if $\mu > \frac{2}{7}\tan\alpha$, otherwise $g\sin\alpha - \mu g\cos\alpha$

**79.** $\sqrt{\frac{2gr}{m+M}\frac{1+\cos\varphi}{m\sin^2\varphi+M}}m\cos\varphi$;
$\frac{gm\sin 2\varphi}{m\sin^2\varphi+M}[\frac{1}{2} + \frac{m^2\cos\varphi(1+\cos\varphi)}{(m\sin^2\varphi+M)(m+M)}]$

**80.** 0,6 m/s

**81.** $\frac{1}{4}\mu v^2$

**82.** $v/\cos\alpha$

**83.** $n^{-2}Lg/4\pi^2 l$, $n = 1, 2, \ldots$

**84.** *(a),(b)* $(p_1 - p_0)V/mc_s$.

**85.** $\frac{1}{2}v_0$; no, a fraction goes into the longitudinal oscillations of the rod and then (as the oscillations die) into heat