PROBLEMS ON MECHANICS

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1 INTRODUCTION

This booklet is a sequel to a similar collection of problems on kinematics and has two main parts: *Section 3 — Statics* and *Section 4 — Dynamics*; *Section 5* contains revision problems. The main aim of this collection of problems is to present the most important solving ideas; using these, one can solve most (> 95%) of olympiad problems on mechanics. Usually a problem is stated first, and is followed by some relevant ideas and suggestions (letter 'K' in front of the number of an idea refers to the correspondingly numbered idea in the kinematics booklet; cross-linking works if the kinematics booklet is stored in the same folder as the mechanics one). The answers to the problems are listed at the end of the booklet (*Section 7*). They are preceded by quite detailed hints (*Section 6*), but it is recommended that you use the hints only as a last resort, after your very best efforts at tackling a problem fail (still, once you have solved a problem successfully by yourself, it is useful to check if your approach was the same as suggested by the hints).

The guiding principle of this booklet argues that almost all olympiad problems are "variations" on a specific set of topics — the solutions follow from corresponding solution ideas. Usually it is not very hard to recognize the right idea for a given problem, having studied enough solution ideas. Discovering all the necessary ideas during the actual solving would certainly show much more creativity and offer a greater joy, but the skill of conceiving ideas is unfortunately difficult (or even impracticable) to learn or teach. Moreover, it may take a long time to reach a new idea, and those relying on trying it during an olympiad would be in disadvantage in comparison to those who have mastered the ideas.

In science as a whole, solution ideas play a similar role as in olympiads: most scientific papers apply and combine known ideas for solving new (or worse, old) problems, at best developing and generalising the ideas. Genuinely new good ideas occur extremely rarely and many of them are later known as masterpieces of science. However, as the whole repertoire of scientific ideas encompasses immensely more than mere mechanics, it is not so easy to remember and utilise them in right places. The respective skill is highly valued; an especial achievement would be employing a well-known idea in an unconventional (unexpected, novel) situation.

In addition to ideas, the booklet also presents "facts" and "methods". The distinction is largely arbitrary, some ideas could have been called methods or facts and vice versa; attempt has been made to pursue the following categorization. Facts are fundamental or particular findings, the knowledge of which can be useful or necessary for problem solving, but which are not formulated as ready recipes. While in theory, all problems can be solved starting from the first principles (the fundamental "facts"), but typically such a "brute force" approach leads to long and sometimes unrealistically complex calculations; the "ideas" are recipes of how to solve problems more easily. The "methods" are powerful "ideas" of particularly wide applicability.

Several sources have been used for the problems: Estonian olympiads' regional and national rounds, Estonian-Finnish Olympiads, International Physics Olympiads, journal "Kvant", Russian and Soviet Union's olympiads; some problems have been modified (either easier or tougher), some are "folklore" (origins unknown).

Similarly to the kinematics booklet, problems are classified as being simple, normal, and difficult: the problem numbers are coloured according to this colour code (keep in mind that difficulty is a subjective category!).

Finally, don't despair if there are some things (or some sections) which you are not able to understand for the time being: just forward to the next topic or next problem; you can return to those parts which you didn't understand later.

2 FIRST LAWS — THEORETICAL BASIS

Those who are familiar with the basic laws of mechanics can skip this Section (though, you can still read it, you may get some new insight), and turn to Section 3. In fact, it is expected that majority of readers can do this because almost all the physics courses start with mechanics, and it is unlikely that someone is drawn to such a booklet aiming to develop advanced problem-solving skills without any prior experience in physics. However, attempts have been made to keep this series of study guides self-contained; this explains the inclusion of the current chapter. Still, the presentation in this Section is highly compressed and in some places involves such mathematical formalism which may *seem* intimidating for beginners (e.g. usage of the summation symbol $\sum$ and differentials), therefore it is not an easy reading. If you find this section to be too difficult to start with, take a high-school mechanics textbook and turn here to the Section "Statics".

2.1 Postulates of classical mechanics

Classical mechanics, the topic of this booklet, is a science based entirely on the three Newton's laws[2], formulated here as "facts".

fact 1: (Newton's 1$^{\text{st}}$ law.) While the motion of bodies depends on the reference frame (e.g. a body which moves with a constant velocity in one frame moves with an acceleration in another frame if the relative acceleration of the frames is non-zero), there exist so-called *inertial reference frames* where the facts 2–5 are valid for all the bodies.

fact 2: (Newton's 2$^{\text{nd}}$ law.) In an inertial frame of reference, a non-zero acceleration $\vec{a}$ of a body is always caused by an external *influence*; each body can be characterized by an *inertial mass* m (in what follows the adjective "inertial" will be dropped), and each influence can be characterized by a vectorial quantity $\vec{F}$, henceforth referred to as the *force*, so that equality $\vec{F} = m\vec{a}$ is valid for any influence-body pair.

[1] As compared with v. 1.0, introductory theory sections are added
[2] I. Newton 1687
[3] which will also serve as the mass unit
[4] or a cylinder of 39.17 mm height and diameter, made of platinum-iridium alloy Pt-10Ir (the official SI definition).

Please note that once we establish an etalon for the mass[3], e.g. define 1 kg as the mass of one cubic decimeter of water[4], the fact 2 serves us also as the **definitions** of the mass of a body, and of the magnitude of a force. Indeed, if we have a fixed *reference force* which is (a) guaranteed to have always the same magnitude, and (b) can be applied to an arbitrary body (e.g. a spring deformed by a given amount) then we can define the mass of any other body in kilograms numerically equal to the ratio of its acceleration to the acceleration of the etalon when the both bodies are subject to the reference force. Newton's 2$^{\mathrm{nd}}$ law is valid if this is a self-consistent definition, i.e. if the obtained mass is independent of what reference force was used. Similarly, the magnitude of any force in *Newtons* (denoted as $\mathrm{N} \equiv \mathrm{kg\,m/s^2}$) can be defined to be equal to the product of the mass and the acceleration of a body subject to that force; Newton's 2$^{\mathrm{nd}}$ law is valid and this definition is self-consistent if the result is independent of which test body was used.

To sum up: the Newton's 2$^{\mathrm{nd}}$ law $\vec F = m\vec a$ serves us both as the definition of the mass of a body (assuming that we have chosen a mass etalon), and the force of an interaction; the law ensures that these are self-consistent definitions: the mass of a body and the magnitude of a force are independent of the measurement procedure.

fact 3: Forces are additive as vector quantities: if there are many forces $\vec F_i$ $(i = 1 \dots n)$ acting on a body of mass m then the fact 2 remains valid with $\vec F = \sum_i \vec F_i$.

The vector sum $\sum_i \vec F_i$ can be calculated using either the triangle/parallelogram rule, or component-wise arithmetic addition: $F_x = \sum_i \vec F_{ix}$, where an index x denotes the x-component (projection onto the x-axis) of a vector; similar expressions can be written for the y- and z-axis.

fact 4: Masses are additive as scalar quantities: if a body is made up of smaller parts of masses m_j $(j = 1 \dots m)$ then the total mass of the compound body equals to the sum of the masses of its components, $m = \sum_j m_j$.

fact 5: (Newton's 3$^{\mathrm{rd}}$ law.) If a body A exerts a force $\vec F$ on a body B then the body B exerts simultaneously the body A with an equal in modulus and antiparallel force $-\vec F$.

2.2 Basic rules derived from the postulates

The facts 1–5 can be considered to be the postulates of classical (Newtonian) mechanics, confirmed by experiments. *All the subsequent "facts", theorems, etc. can be derived mathematically using these postulates.*

Thus far we have used a vague concept of the *acceleration of a body*. Everything is fine as long as a body moves *translationally*, i.e. so that all its points have the same acceleration vector. However, if a body has a considerable size and rotates then different points have different accelerations, so that we need to clarify, the acceleration of which point needs to be used. In order to overcome this difficulty and keep our set of postulates 1–5 as simple as possible, let us assume that the fact 2 is valid for so-called *point masses*, i.e. for very small bodies the dimensions of which are much smaller than the characteristic travel distances; then, the position of a point mass is described by a single point which has unambiguously defined velocity and acceleration. We can generalize the fact 2 to real finite-sized-bodies by dividing these fictitiously into tiny pieces, each of which can be treated as a point mass.

To begin with, one can derive (see appendix 1) the generic formulation of the Newton's 2$^{\mathrm{nd}}$ law.

fact 6: (Momentum conservation law/generalized Newton's 2$^{\mathrm{nd}}$ law.) For the net momentum $\vec{\mathcal P} = \sum_i m_i \vec v_i$ of a system of point masses[5],

$$\frac{\mathrm d}{\mathrm dt}\vec{\mathcal P} = \vec{\mathcal F}, \tag{1}$$

where $\vec{\mathcal F}$ is the net force (the sum of *external forces*) acting on the system. In particular, **the net momentum is conserved ($\vec{\mathcal P}$ =const) if $\vec{\mathcal F} = 0$.**

By substituting $\vec{\mathcal P} = \sum_i m_i \vec v_i = \sum_i m_i \frac{\mathrm d\vec r_i}{\mathrm dt} = \frac{\mathrm d}{\mathrm dt}\sum_i m_i \vec r_i$ (where $\vec r_i$ denotes the position vector of the i-th point mass), we can rewrite Eq (1) as

$$M\frac{\mathrm d^2\vec r_C}{\mathrm dt^2} = \vec{\mathcal F},$$

where

$$\vec r_C = \frac{\sum_i m_i \vec r_i}{\sum_i m_i} \tag{2}$$

is called *the centre of mass*. This result clarifies: in the case of macroscopic bodies, the fact 2 remains valid if we use the acceleration of the centre of mass.

According to the Newton's 2$^{\mathrm{nd}}$ law, once we know how the interaction forces between bodies depend on the inter-body distances and on the velocities, we can (in theory) calculate how the system will evolve in time (such systems are refereed to as *determinstic systems*). Indeed, we know the accelerations of all the bodies, and hence, we can determine the velocities and positions after a small time increment: if the time increment Δt is small enough, the changes in accelerations $\Delta \vec a$ can be neglected, which means that the new velocity for the i-th body will be $\vec v_i' = \vec v_i + \vec a_i \Delta t$, and the new position vector $\vec r_i' = \vec r_i + \vec v_i \Delta t$; the whole temporal dependences $\vec v_i(t)$ and $\vec r_i(t)$ (with $i = 1 \dots N$ where N is the number of bodies) can be obtained by advancing in time step-by-step. In mathematical terms, this is a numerical integration of a *system of ordinary differential equations*: the second time derivatives of coordinates $\ddot x_i$, $\ddot y_i$, and $\ddot z_i$ are expressed in terms of the coordinates x_i, y_i, z_i, and the first derivatives $\dot x_i$, $\dot y_i$, $\dot z_i$ [6]. While in principle, these calculations can be always made, at least numerically and assuming that we have enough computational power, in practice the mathematical task can be very difficult[7]. Apart from the facts 1–5, the Newtonian mechanics is a collection of recipes for easier solution of these differential equations.

Among such recipes, finding and applying conservation laws has a central role. This is because according to what has been said above, the evolution of mechanical systems is described by a system of differential equations, and each conservation

[5] such as rigid bodies in which case the point masses are bound by inter-molecular forces together into a macroscopic body

[6] Here we assume that the force depends only on the coordinates and velocities of the bodies; with the exception of the *Abraham-Lorentz* force (accounting for the cyclotron radiation), this is always satisfied.

[7] Also, there is the issue of possibly chaotic behaviour when in many-body-systems, small differences in initial conditions lead to exponentially growing differences — in the same way as it is impossible to put a sharp pencil vertically standig on its tip onto a flat surface.

law reduces the order of that system by one; this makes the mathematical task much simpler. The conservation laws can be derived mathematically from the Newton's laws; while it is definitely useful to know how it is done, majority of mechanics problems can be solved without being familiar with this procedure. Because of that, the conservation laws are derived in Appendices 1,2, and 3; here we just provide the formulations. We have already dealt with the momentum conservation law (see fact 6), so we can proceed to the next one.

fact 7: (Angular momentum conservation law.) For the net angular momentum $\vec{\mathcal{L}} = \sum_i m_i \vec{r}_i \times \vec{v}_i$ of a system of bodies,

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{\mathcal{L}} = \vec{\mathcal{T}}, \tag{3}$$

where

$$\vec{\mathcal{T}} = \sum_i \vec{r}_i \times \vec{F}_i$$

is the net torque acting on the system; here $\vec{F}_i$ stands for the net force acting on the i-th point mass. In particular, the net angular momentum of the system is conserved if $\vec{\mathcal{T}} = 0$.

Eq. (3) is derived in appendix 2, and can be considered to be the Newton's 2$^{\text{nd}}$ law for rotational motion of bodies.

In three-dimensional geometry, calculating the vector products to determine the net torque and angular momentum may be quite difficult. Luckily, most of the Olympiad problems involve two-dimensional geometry: velocities, momenta, and radius vectors lie in the $x-y$-plane, and vector products (torques and angular momenta) are parallel to the z-axis, i.e. we can consider $\vec{L} = \vec{r} \times \vec{p}$ and $\vec{T} = \vec{r} \times \vec{F}$ as scalars, characterized by their projection to the z-axis (in what follows denoted as L and T, respectively). According to the definition of the vector product, the sign of such torque is positive if the rotation from the vector $\vec{r}$ to the vector $\vec{F}$ corresponds to a clockwise motion, and negative otherwise. Thus we can write $T = |\vec{r}| \times |\vec{F}| \sin\alpha$, where α is the angle between the radius vector and the force and can be either positive (rotation from $\vec{r}$ to $\vec{F}$ is clockwise) or negative. We can introduce the *arm of the force* $h = |\vec{r}|\sin\alpha$ (see figure), in which case

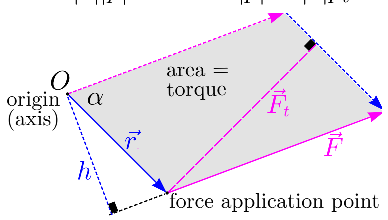
$$T = |\vec{F}|h;$$

similarly we can use the tangential component of the force $F_t = |\vec{F}|\sin\alpha$ and obtain

$$T = |\vec{r}|F_t.$$

Similar procedure can be applied to the angular momenta:

$$L = |\vec{r}||\vec{p}|\sin\alpha = h|\vec{p}| = |\vec{r}|p_t.$$



The discipline of *statics* studies *equilibria of bodies*, i.e. conditions when there is an inertial frame of reference where a body remains motionless. It is clear that both the momentum and angular momentum of a body at equilibrium needs to be constant, hence *the sum of all the forces, as well as the sum of all the torques acting on a body must be zero*; this applies also to any fictitious part of a body. While there are statics problems which study deformable bodies (which change shape when forces are applied to it), there is an important idealization of *rigid body*: a body which preserves its shape under any (not-too-large) forces.

While for the Newton's 2$^{\text{nd}}$ law [Eq. (1)], and for the static force balance condition, it doesn't matter where the force is applied to, in the case of angular momentum [Eq. (3)] and for the static torque balance condition, it becomes important. In classical mechanics, the forces are divided into *contact forces* which are applied at the contact point of two bodies (elasticity forces in its various forms, such as normal and friction forces, see below), and body forces which are applied to every point of the solid body (such as gravity and electrostatic forces). The application point of contact forces is obviously the contact point; in the case of body forces, the torque can be calculated by dividing the entire body (system of bodies) into so small parts (point masses) and by integrating the torques applied to each of these. It is easy to see that with the total body force (i.e. the sum of all the body forces applied to different parts of the body) $\vec{\mathcal{F}}$ and total torque $\vec{\mathcal{T}}$ applied to a body, one can always find such a radius vector $\vec{r}$ that $\vec{\mathcal{T}} = \vec{r} \times \vec{\mathcal{F}}$, i.e. although the body forces are applied to each point of the body, the net effect is as if the net force $\vec{\mathcal{F}}$ were to applied to a certain effective application centre; in some cases, there are simple rules for finding such effective application centres, e.g. in the case of an homogeneous gravity field, it appears to be the centre of mass.[8].

At the microscopic level of quantum mechanics, such a division of forces becomes meaningless, because on the one hand, fields which mediate body forces are also material things and in this sense, all the forces are contact forces. On the other hand, classical contact forces are also mediated at the microscopic level via fields so that in a certain sense, all the forces are body forces. Regardless, at the macroscopic level such a division still remains helpful.

fact 8: (Energy conservation law; for more details, see appendix 3.) If we define the kinetic energy for a system of point masses (or translationally moving rigid bodies) as

$$K = \frac{1}{2}\sum_i m_i \vec{v}_i^2,$$

and the total work done by all the forces during infinitely small displacements $\mathrm{d}\vec{r}_i$ of the point masses as

$$\mathrm{d}W = \sum_i \vec{F}_i \cdot \mathrm{d}\vec{r}_i$$

then the change of the kinetic energy equals to the total work done by all the forces,

$$\mathrm{d}K = \mathrm{d}W;$$

here, $\vec{F}_i$ denotes the total force acting on the i-th point mass. The work done by so-called *conservative forces* depends only on the initial and final states of the system (i.e. on the positions of the point masses), and not along which trajectories the point masses moved. This means that the work done by conservative

[8]Similarly, if two bodies make contact over a finite-sized area (rather than at just few contact points), we would need to find the total torque by integrating over the contact area, and one can always find the effective application point of these forces

[9]$\mathrm{d}\Pi \equiv \Pi(\vec{r}_1 + \mathrm{d}\vec{r}_1, \vec{r}_2 + \mathrm{d}\vec{r}_1, \ldots) - \Pi(\vec{r}_1, \vec{r}_2, \ldots)$

forces can be expressed as the decrease of a certain function of state $\Pi(\vec{r}_1, \vec{r}_2 \ldots)$ which is referred to as the *potential energy*; for infinitesimal displacements we can write $\mathrm{d}W_{\text{cons}} = -\mathrm{d}\Pi$[9]. Therefore, if we define the full mechanical energy as $E = K + \Pi$ then

$$\mathrm{d}E = \mathrm{d}W',$$

where $\mathrm{d}W'$ stands for the work done by the non-conservative forces; if there are no such forces then $\mathrm{d}E = 0$, hence

$$E = \Pi + K = \text{const.} \tag{4}$$

Here few comments are needed. First, while the momentum of a body is the momentum of its centre of mass, the kinetic energy of a composite body *is not* just the kinetic energy of its centre of mass: the kinetic energy in the frame of the centre of mass ("CM-frame") needs to be added, as well,

$$K = \frac{1}{2}Mv_C^2 + \frac{1}{2}\sum_i m_i\vec{u}_i^2,$$

where where $\vec{u}_i = \vec{v}_i - \vec{v}_C$ is the velocity of the i-th point mass in the CM-frame [10]. How to calculate the kinetic energy in the frame of the centre of mass for rotating rigid bodies will be discussed somewhat later.

Finally, let us notice that forces depending on the velocities (e.g. friction forces) and/or on time (e.g. normal force exerted by a moving wall), cannot be conservative because the work done by such forces along a path depends clearly on how fast the bodies move. An exception is provided by those velocity-dependent forces which are always perpendicular to the velocity (e.g. for the Lorentz force and normal force) and for which $\mathrm{d}W = \vec{F}\cdot\mathrm{d}\vec{r} = \vec{F}\cdot\vec{v}\mathrm{d}t \equiv 0$.

2.3 Basic forces

Gravity. Now, let us consider the case of a gravity field in more details; it can be described by the free fall acceleration vector $\vec{g}$. From the "Kinematics" we know that then all the bodies move with the acceleration $\vec{g}$; then, according to the Newton's 2$^{\text{nd}}$ law, this should be caused by a force

$$\vec{F} = m\vec{g}.$$

This is called *the gravity force*. The fact that in a given gravity field, the gravity force is proportional to the mass of a body is to be considered as an experimental finding. Let us recall that the mass is introduced via the Newton's 2$^{\text{nd}}$ law and describes the *inertia* of a body, i.e. the capability of a body to retain its velocity; because of this we can call it also *the inertial mass*. Here, however, the mass enters a completely different law: the gravity force is proportional to the mass. It is easy to imagine that the gravity force is defined by another characteristic of a body, let us call it *the gravitational mass*, unrelated to the *inertial mass*. Experiments show that the gravitational mass is always equal to the inertial mass and thus we can drop the adjectives "gravitational" and "inertial". As a matter of fact, the equivalence of inertial and gravitational mass has a great significance in physics and represents the main postulate and cornerstone of the general theory of relativity.

From the Newton's 3$^{\text{rd}}$ law we know that each force is caused by some other body: a gravity force acting on a body A needs to be caused by a body B. We also know that the gravity force is caused by and is *proportional to* the mass of a body, and apparently this applies both to the body A, and body B. Hence, the force needs to be proportional to the product of masses, $F = cm_Am_B$, where the coefficient of proportionality c can be a function of distance. It appears that c is inversely proportional to the squared distance, $c = G/r^2$; let us consider this as an experimental finding. Here $G \approx 6.67\times10^{-11}\,\mathrm{m^3kg^{-1}s^{-2}}$ is called *the gravitational constant*. Now it is easy to guess that this force must be parallel to the only preferred direction for a system of two point masses, the line connecting the two points. This is indeed the case; furthermore, the gravitational force appears to be an attractive force.

fact 9: The gravitational force acting on the i-th point mass due to the j-th point mass can be expressed as

$$\vec{F}_i = \hat{r}_{ij}\frac{Gm_im_j}{r_{ij}^2}, \tag{5}$$

where $\hat{r}_{ij} = (\vec{r}_j - \vec{r}_i)/|\vec{r}_j - \vec{r}_i|$ stands for the unit vector pointing from the i-th body to the j-th body and $r_{ij} = |\vec{r}_j - \vec{r}_i|$[11]. The presence of a third body does affect the validity of this law, i.e. the superposition principle holds: total gravitational force can be found by summing up the contributions from all the gravitating bodies according to Eq. (5).[12] Eq. (5) remains also valid when the gravitationally interacting bodies have spherically symmetric mass distribution — in that case, $\vec{r}_i$ and $\vec{r}_j$ point to the respective centres of symmetry (which coincide with the centres of mass) [13] NB! in the case of arbitrarily shaped bodies, using the centres of mass would be incorrect; the force needs to be calculated by dividing the bodies into point masses and taking integral.

For the gravitational pull of the Earth, we can typically approximate r_{ij} with the radius of the Earth R_E, so that

$$\vec{F} = m\vec{g},\ \ \vec{g} = \hat{z}\frac{GM_E}{R_E^2} \approx \hat{z}\cdot 9.81\,\mathrm{m/s^2}, \tag{6}$$

where $\hat{z}$ stands for a downwards pointing unit vector and M_E denotes the Earth's mass.

Note that the force due to a homogeneous gravity field $\vec{g}$ is applied effectively to a centre of mass of a body, regardless of its shape. Indeed, the torque of the gravity force is calculated then as

$$\mathcal{T} = \sum_i \vec{r}_i\times\vec{g}m_i = \left(\sum_i \vec{r}_im_i\right)\times\vec{g} = \vec{r}_C\times\vec{g}M,$$

where $M = \sum_i m_i$.

Gravity force is a conservative force because for any pair of point masses, the force is directed along the line connecting these point masses and depends only on the distance between these (cf. appendix 3). The work done by a gravity force $\vec{F} = m\vec{g}$ due to a homogeneous gravity field can be expressed as $\mathrm{d}A = m\vec{g}\cdot\mathrm{d}\vec{r}$, hence $\Pi = -m\vec{g}\cdot\mathrm{d}\vec{r}$; upon integration we

[10] Indeed, we divide the body into point masses and write $K = \frac{1}{2}\sum_i m_i(\vec{v}_i - \vec{v}_C + \vec{v}_C)^2 = \frac{1}{2}\sum_i m_i(\vec{u}_i + \vec{v}_C)^2$; now we can open the braces and factorize $\vec{v}_C$: $\frac{1}{2}\sum_i(\vec{u}_i + \vec{v}_C)^2 = \frac{1}{2}\sum_i m_i\vec{u}_i^2 + \vec{v}_C\sum_i m_i\vec{u}_i + \frac{1}{2}\vec{v}_C^2\sum_i m_i$. Here, $\sum_i m_i\vec{u}_i$ is the total momentum in the CM-frame, which is zero according to the definition of the centre of mass.

[11] I. Newton 1687

[12] The superposition principle corresponds to the linearity of the non-relativistic equations of the gravity field and can be treated as an experimentally verified postulate.

[13] In the booklet of electromagnetism, this property will be derived from Eq. (5) using the superposition principle.

obtain

$$\Pi = -m\vec{g}\cdot(\vec{r}-\vec{r}_0),$$

where $\vec{r}_0$ is the vector pointing to an arbitrarily chosen reference point. Now, keeping in mind that the energy is additive, we can write an expression for the gravitational potential energy of N bodies:

$$\Pi = -\vec{g}\cdot\sum_i m_i(\vec{r}_i-\vec{r}_0) = -\vec{g}\cdot(\vec{r}_C-\vec{r}_0)\sum m_i. \qquad (7)$$

Gravitational energy of two point masses can be calculated similarly by integration; for two point masses, it is usually convenient to take the reference configuration (for which the potential energy is zero) such that the distance between the two bodies is infinite. For small displacements, the work done by the gravity forces acting on the both bodies

$$\mathrm{d}W = G\frac{m_1m_2}{r_{12}^2}\hat{r}_{12}\cdot(\mathrm{d}\vec{r}_1-\mathrm{d}\vec{r}_2) = -G\frac{m_1m_2}{r_{12}^2}\mathrm{d}r_{12}$$

[here we took into account that $\mathrm{d}\vec{r}_1-\mathrm{d}\vec{r}_2=\mathrm{d}(\vec{r}_1-\vec{r}_2)$, and $\hat{r}_{12}\cdot\mathrm{d}(\vec{r}_1-\vec{r}_2)=-\mathrm{d}r_{12}$]. So,

$$\Pi = \int_\infty^{r_{12}} G\frac{m_1m_2}{r^2}\mathrm{d}r = -G\frac{m_1m_2}{r_{12}}.$$

If there are more than two interacting bodies, we can use the superposition principle to find

$$\Pi = -G\sum_{i>j}\frac{m_im_j}{r_{ij}}; \qquad (8)$$

note that the interaction energy of any pair of bodies needs to be counted only once, hence we sum over pairs with $i>j$.

fact 10: Potential energy of two gravitationally interacting spherically symmetric bodies is given by Eq. (8); in the case of homogeneous downwards-directed gravity field of strength g, the change of potential energy of a body of mass m is $\Delta\Pi = g\Delta h$, where h is the change of height.

Elasticity. Similarly to gravity forces, elasticity forces can be met literally at each our step. While microscopically, all elasticity forces can be explained (at least in principle) in terms of electrostatic interactions using quantum mechanics, macroscopically it can take different forms. First of all, there is the Hooke's law which describes elasticity forces for deformable bodies (e.g. a rubber band or a spring); there are also normal force and dry friction force which seemingly have nothing to do with elasticity, but in reality, both normal force and dry friction force have microscopically the same origin as the Hooke's law.

fact 11: (Hooke's law[14]) If the deformation of an elastically deformable body is not too large, the deformed body exerts a force which (a) is antiparallel to the deformation vector $\vec{a}$ and (b) by modulus is proportional to the deformation, i.e.

$$\vec{F} = -k\vec{a}. \qquad (9)$$

This law is valid for small deformations of all elastic materials including rubber bands, springs, etc, as long as the deformation is not too large, and the body deformation includes only stretching (or compression), and does not involve bending or shear[15]. If bending and shear deformations are involved, with a fixed deformation direction (described by the unit vector $\hat{a}\equiv\vec{a}/|\vec{a}|$), the force modulus remains to be proportional to $|\vec{a}|$; however, then the stiffness (the proportionality coefficient k) depends also on $\hat{a}$, and force is not necessarily antiparallel to the displacement[16].

Elasticity forces are conservative, potential energy can be found by simple integration $\Pi = -\int\vec{F}\cdot\mathrm{d}\vec{a} = \int ka\mathrm{d}a = \frac{1}{2}ka^2$.

fact 12: Under the assumptions of fact 9, the potential energy of an elastically deformable body is given by

$$\Pi = \frac{1}{2}ka^2. \qquad (10)$$

It is quite clear that if we take, for instance, a rubber band of length l and stiffness k, making it twice longer will reduce its stiffness by a factor of two. Indeed, one can divide the longer band fictitiously into two halves, each of length l, which means that if we apply now the same force F to the endpoints of the long band, both halves will be deformed by $x=F/k$ and hence, the entire band is deformed by $x'=x+x=2F/k$ so that the stiffness $k'=F/x'=k/2$. Similarly, making the band twice thicker will increase the stiffness by a factor of two because we can consider the thicker band as being made of two parallel thinner bands. NB! this does not apply to the springs and bending deformation: while making a spring twice longer will, of course, still decrease k by a factor of two, making it from a twice thicker wire will increase the stiffness more than two-fold: by the same bending angle the thicker wire will be deformed more than a thinner wire. This paragraph can be summarized as the following fact.

fact 13: For a band of elastic material of length L and cross-sectional area A, the stiffness

$$k = AY/L, \qquad (11)$$

where Y denotes the so-called *Young modulus* of the material[17].

This equality makes it possible to give an alternative formulation of the Hooke's law. To that end, let us introduce the concepts of *the strain*, which is defined as the relative deformation $\varepsilon = a/L$, and stress which is defined as the elasticity force per unit area, $\sigma = F/A$. Then, the Hooke's law can be rewritten as

$$\sigma = \varepsilon Y.$$

One can also introduce the concept of *energy density* of a deformed material, the ratio of the potential energy and the volume, $w = \Pi/(LA) = \frac{1}{2}Y\varepsilon^2$.

Now we can also address the question, which deformations can be considered as "small enough" so that the Hooke's law remains valid. One might think that we need to have $\varepsilon \ll 1$, but typically this is a too loose requirement: majority of elastic materials break far before deformations $\varepsilon\sim 1$ are reached. *Typically, the Hooke's law starts failing when deformations are so large that the material is close to breaking.* Also, for such large deformations, the material is no longer fully elastic: when the force is removed, the material does not fully restore its initial shape and some residual deformation will remain; such deformations are referred to as *plastic deformations*. There are also materials (which can be referred to as *plastic materials* or

[14] R. Hooke 1660

[15] Deformation of springs, in fact, does involve bending, but the Hooke's law remains in that case nevertheless valid.

[16] A proper description of the elasticity forces when bending and shearing are involved requires tensorial description and is beyond the scope of this booklet

[17] L. Euler 1727, G. Riccati 1782

plastically deformable materials) which deform plastically over a very wide range of strain values before breaking into pieces. In such cases, the Hooke's law remains valid only for extremely small strains by which the deformation still remains elastic.

Note that there are *hyperelastic materials* for which the deformation can be very large, $\varepsilon > 1$; then, indeed, the condition $\varepsilon \ll 1$ is required for the applicability of the Hooke's law.

Majority of the materials have rather large values of Y which means that unless we have really long and thin threads or wires, moderate forces will cause only a really minute and un-noticeable deformations. This is typically the case for wires, ropes, rods, and solid surfaces. In those cases, while the geometrical effect of the deformations can be neglected, such a deformation will be formed that the elasticity force compensates *any* applied external force. If we are dealing with a solid surface, such an elasticity force is called *the normal force*; in the case of rods, wires and ropes, we call it *the tension force*.

Unless otherwise emphasized, it is assumed that the tension force is parallel to the rod, wire or rope. In the case of a rope or a thin wire, this is essentially always the case: there is no possibility of having an elastic shear or bending because the rope is typically very "soft" with respect to such deformations: if we try to create a perpendicular elasticity force by applying a peperpendicular external force, the rope will be bended without giving rise to any noticeable force. In the case of a rod, this is not true: if we apply an external perpendicular force, the rod resists elastically to the attempts of bending and creates a perpendicular component of the tension force. Still, if all the external forces applied to a rod are parallel to it, according to the Newton's 3$^{\rm rd}$ law, the tension force will be also parallel to the rod.

In the case of a stretched string (rod, rope, etc), we can divide it fictitiously into two parts. Then, at the division point P, the two pieces attract each other with a certain elasticity force. The direction of this force depends on which part of the string is considered, but due to the Newton's 3$^{\rm rd}$ law, the modulus of the force remains the same. The force with which the two fictitious parts of the rope interact with each other at the point P describes the state of the rope at that point, and is referred to as *the tension*. So, we'll distinguish the force $\vec{F}$ which is applied to the endpoint of a rope, and the tension T, which is defined for any point of the rope and describes the state of it; note that when external forces are applied only to the endpoints of a rope at equilibrium, $|\vec{F}| = T$ (this follows from the force balance condition for any fictitious part of the rope).

fact 14: Tension is an elasticity force in a linear construction element such as string (rod, wire, etc)[18]. For a non-stretchable string[19], if it is being pulled (or pushed, which can happen in the case of a rod) the tension adjusts itself to the externally applied force to prevent stretching. If the mass of a thread or rope can be neglected, the tension is constant along it. In a freely bending rope, the tension force at a point P is parallel to the tangent drawn to the rope at point P.

fact 15: Normal force is the perpendicular to the surface component of an elasticity force at the surface of a rigid (non-deformable) body with which the rigid body acts on a contacting body; it adjusts itself to the externally applied force preventing thereby the rigid body from being deformed.

Note that if the externally applied force is not perpendicular to the rigid body surface then due to the Newton's 3$^{\rm rd}$ law, the elasticity force will have both perpendicular to the surface and tangential (parallel to the surface) components. The latter manifests itself at the contact points of two solid bodies as *the friction force*. More accurately, the friction force is the force at the contact point of two bodies due to the interaction of the molecules of one body with the molecules of the other body when the bodies try to slip one over the other; the surface molecules are kept at their place due to the elasticity forces inside each of the bodies; these elasticity forces are caused by the (typically unnoticeably small) shear deformation of the bodies.

In the case of solid bodies, if the external tangential forces are not too large as compared with the normal forces, the bodies will not slip, and the friction force adjusts itself so as to compensate the external tangential forces; this is called the *static friction force*.

fact 16: (Amontons' law.) The maximal static friction force at the contact area of two bodies $F_{\max} = \mu_s N$, where N is the normal force at that contact area and μ_s is a constant dependent on the two contacting materials, referred to as *the static coefficient of friction*; it may also depend on the temperature, humidity, etc. Thus, $F_{\max}$ is independent of the contact area.

fact 17: (Coulomb's friction law.) When two bodies move with respect to each other, the friction force at the contact area of these bodies $F = \mu_k N$, where N is the normal force at that contact area and μ_k is a constant dependent on the two contacting materials, referred to as *the kinetic coefficient of friction*; it does depend slightly on the slipping speed, but this dependence is weak and typically ignored.

In the case of Olympiad problems, most often it is assumed that $\mu_s = \mu_k$, but sometimes these are taken to be different, with $\mu_s > \mu_k$.

While the friction laws are very simple and have been formulated a long time ago[20], deriving it from the microscopic (molecular) dynamics turns out to be a very difficult task (there are still papers being published on that topic in research papers, c.f. M.H. Müser et al., Phys. Rev. Lett. 86, 1295 (2000), and O.M. Braun et al., Phys. Rev. Lett. 110, 085503.

3 STATICS

When solving problems on statics, one can always use standard brute-force-approach: equations (1) and (3) tells us that for each body at equilibrium, $\vec{\mathcal{F}} = 0$ and $\vec{\mathcal{T}} = 0$. So, for each solid body, we have a force balance condition, and a torque balance condition. According to the standard procedure, these equations are to be projected onto x-, y- and z-axis yielding us

[18] In the bulk of three-dimensional elastic bodies, instead of tension, the concept of stress is used; the respective description is mathematically more complicated, e.g. the stress is a *tensor* quantity.

[19] More precisely, a string made of a material with a very large value of Young modulus Y.

[20] The friction laws were developed by L. da Vinci 1493, G. Amontons 1699, and C.A. Coulomb 1785.

six equations (assuming 3-dimensional geometry); if there are N interacting bodies, the overall number of equations is $6N$. For a correctly posed problem, we need to have also $6N$ unknown parameters so that we could solve this set of algebraic equations. The description of the procedure sounds simple, but solving so many equations might be fairly difficult. In the case of 2-dimensional geometry, the number of equations is reduced twice (while the number of force balance equations comes down to two, all torques will be perpendicular to the plane, so there is only one torque equation), but even with only two bodies, we have still 6 equations.

Luckily, there are tricks which can help us reducing the number of equations! Usually the main ingenuity lies in

idea 1: choose optimal axes to zero as many projections of forces as possible. It is especially good to zero the projections of the forces *we do not know and are not interested in*,

for instance, the reaction force between two bodies or the tensile force in a string (or a rod). To zero as many forces as possible it is worthwhile to note that *a)* the axes may not be perpendicular; *b)* if the system consists of several bodies, then a different set of axes may be chosen for each body.

idea 2: for the torques equation it is wise to choose such a pivot point that zeroes as many moment arms as possible. Again it is especially beneficial to zero the torques of *"uninteresting" forces.*

For example, if we choose the pivot to be at the contact point of two bodies, then the moment arms of the friction force between the bodies and of their reaction force are both zero.

As mentioned above, for a *two-dimensional system*, we can write two equations per body for the forces (x- and y-components) and one equation (per body) for the torques. We could increase the number of equations either by using more than two projections for force balance equations, or more than one pivot point ("axis" of rotation) for the torque balance. However,

fact 18: *the maximum number of linearly independent equations (describing force and torque balance) equals the number of degrees of freedom of the body* (three in the two-dimensional case, as the body can rotate in a plane and shift along the x- and y-axis, and six in the three-dimensional case).

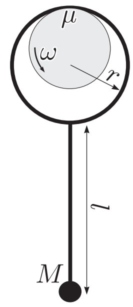
So, if we write down two force balance conditions with two torque balance conditions then one of the four equations would always be a redundant consequence of the three others.

Equations (1) and (3) seem to tell us that for 2-dimensional geometry, we should use one torques equation and two force equations; however, each force equation can be "traded for" one torques equation. So, apart for the "canonical" set of two force equations and one torque equation, we can use one force equation with two torque equations (with two different pivot points), and we can also use three torques equations with three different pivot points which must not lie on a single line. Indeed, let us have two torque balance conditions, $\sum_i \overrightarrow{OP_i} \times \vec{F}_i = 0$ and $\sum_i \overrightarrow{O'P_i} \times \vec{F}_i = 0$ where P_i is the application point of the force $\vec{F}_i$. Once we subtract one equation from the other, we obtain $\sum_i \overrightarrow{OO'} \times \vec{F}_i = \overrightarrow{OO'} \times \sum_i \vec{F}_i = 0$, which is the projection of the force balance condition to the perpendicular of OO'. It should be emphasized that at least one torques equation needs to be left into your set of equations: the "trading" of force equations for torques equations works because a rotation around O by a small angle $\mathrm{d}\varphi$ followed by a rotation around O' by $-\mathrm{d}\varphi$ results in a translational motion by $|OO'|\mathrm{d}\varphi$, but there is no such sequence of translational motions which could result in a rotational motion.

[21] unless there are parallel forces

idea 3: Using torque balance conditions is usually more efficient than using force balance conditions because for any force balance condition, we can eliminate only one force[21] by projecting the condition to the perpendicular of that force; meanwhile, if we choose the pivot point as the intersection point of the two lines along which two unparallel forces are applied, both forces disappear from the equation.

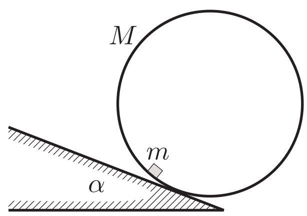
pr 1. An end of a light wire rod is bent into a hoop of radius r. The straight part of the rod has length l; a ball of mass M is attached to the other end of the rod. The pendulum thus formed is hung by the hoop onto a revolving shaft. The coefficient of friction between the shaft and the hoop is μ. Find the equilibrium angle between the rod and the vertical.



This problem is classified as a difficult one because most people who try to solve it have difficulties in drawing a qualitatively correct sketch. What really helps making a correct sketch is relying on the idea 2. Mathematical simplifications are further offered by

fact 19: on an inclined surface, slipping will start when the slope angle α fulfills $\tan\alpha = \mu$.

pr 2. On an incline with slope angle α there lies a cylinder with mass M, its axis being horizontal. A small block with mass m is placed inside it. The coefficient of friction between the block and the cylinder is μ; the incline is nonslippery. What is the maximum slope angle α for the cylinder to stay at rest? The block is much smaller than the radius of the cylinder.

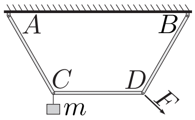


Here we can again use fact 19 and idea 2 if we add

idea 4: sometimes it is useful to consider a system of two (or more) bodies as one whole and write the equations for the forces and/or the torques for the whole system.

Then, the net force (or torque) acting on the compond body is the sum of *external* forces (torques) acting on the constituents. Our calculations are simplified because the internal forces (torques) between the different parts of the compound body can be ignored (due to Newton's 3rd they cancel each other out). For the last problem, it is useful to assemble a compound body from the cylinder and the block.

pr 3. Three identical rods are connected by hinges to each other, the outmost ones are hinged to a ceiling at points A and B. The distance between these points is twice the length of a rod. A weight of mass m is hanged onto hinge C. At least how strong a force onto hinge D is necessary to keep the system stationary with the rod CD horizontal?



Again we can use idea 2. The work is also aided by

fact 20: if forces are applied only to two endpoints of a rod and the fixture(s) of the rod at its endpoint(s) is (are) not rigid (the rod rests freely on its supports or is attached to a string or a hinge), then the tension force in the rod is directed along the rod.

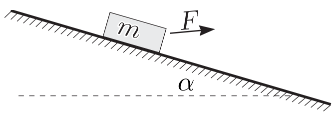
Indeed, at either endpoints, the applied net external force $\vec{F}$ must point along the rod, as its torque with respect to the other endpoint must be zero. Further, according to the Newton's 3rd law, the external force $\vec{F}$ must be met by an equal and opposite force exerted by the rod, which is the tension force $\vec{T}$, so $\vec{F} = -\vec{T}$.

Some ideas are very universal, especially the mathematical ones.

idea K-2 some extrema are easier to find without using derivatives,

for example, the shortest path from a point to a plane is perpendicular to it.

pr 4. What is the minimum force needed to dislodge a block of mass m resting on an inclined plane of slope angle α, if the coefficient of friction is μ? Investigate the cases when *a)* $\alpha = 0$; *b)* $0 < \alpha < \arctan\mu$.



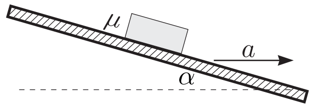
idea 5: force balance can sometimes be resolved vectorially without projecting anything onto axes.

Fact 19, or rather its following generalisation, turns out to be of use:

idea 6: if a body is on the verge of slipping (or already slipping), then the sum of the friction force and the reaction force is angled by $\arctan\mu$ from the surface normal.

This idea can be used fairly often, for instance in the next problem.

pr 5. A block rests on an inclined surface with slope angle α. The surface moves with a horizontal acceleration a which lies in the same vertical plane as a normal vector to the surface. Determine the values of the coefficient of friction μ that allow the block to remain still.



Here we are helped by the very universal

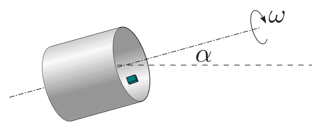
idea 7: many problems become very easy in a non-inertial translationally moving reference frame.

To clarify: in a translationally moving reference frame we can re-establish Newton's laws by imagining that each body is additionally acted on by an *inertial force* $-m\vec{a}$ where $\vec{a}$ is the acceleration of the frame of reference and m is the mass of a given body. Indeed, we have learned in kinematics that for translational motion of a reference frame, the accelerations can be added, cf. idea K-19; so, in a moving frame, all the bodies obtain additional acceleration $-\vec{a}$, as if there was an additional force $\vec{F} = -m\vec{a}$ acting on a body of mass m.

Note that that due to the equivalence of the inertial and heavy mass (cf. "Gravity", Section 2.3) the inertial foce is totally analogous to the gravitational force[22]. Because of that, we can use

idea 8: the net of the inertial and gravitational forces is usable as an effective gravitational force.

pr 6. A cylinder with radius R spins around its axis with an angular speed ω. On its inner surface there lies a small block; the coefficient of friction between the block and the inner surface of the cylinder is μ. Find the values of ω for which the block does not slip (stays still with respect to the cylinder). Consider the cases where *(a)* the axis of the cylinder is horizontal; *(b)* the axis is inclined by angle α with respect to the horizon.



idea 9: a rotating frame of reference may be used by adding a centrifugal force $m\omega^2\vec{R}$ (with ω being the angular speed of the frame and R being a vector drawn from the axis of rotation to the point in question) and Coriolis force. The latter is unimportant *(a)* for a body standing still or moving in parallel to the axis of rotation in a rotating frame of reference (in this case the Coriolis force is zero); *(b)* for energy conservation (in this case the Coriolis force is perpendicular to the velocity and, thus, does not change the energy).

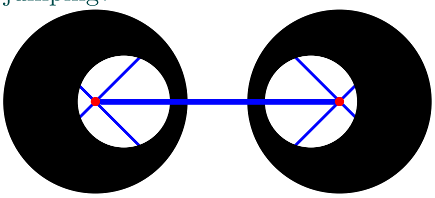
Warning: *in this idea, the axis of rotation must be actual, not instantaneous.* Expressions for the centrifugal force and Coriolis force are derived in appendix 4.

For the problem 6, recall also idea K-2b and idea 6; for part *(b)*, add

[22] Their equivalence is the cornerstone of the theory of general relativity (more specifically, it assumes the inertial and gravitational forces to be indistinguishable in any local measurement).

idea K-11: in case of three-dimensional geometry, consider two-dimensional sections. It is especially good if all interesting objects (for example, force vectors) lie on one section. The orientation and location of the sections may change in time.

pr 7. A cart has two cylindrical wheels connected by a weightless horizontal rod using weightless spokes and frictionless axis as shown in the figure. Each of the wheels is made of a homogeneous disc of radius R, and has a cylidrical hole of radius $R/2$ drilled coaxially at the distance $R/3$ from the centre of the wheel. The wheels are turned so that the holes point towards each other, and the cart is put into motion on a horizontal floor. What is the critical speed v by which the wheels start jumping?



This problem is somewhat similar to the previous one, and we would be able to solve it using those ideas which we have already studied. Indeed, if we consider the process in a frame co-moving with the cart, we can just apply Newton's 2nd law to the centripetal acceleration of the wheel's centre of mass. However, let us solve it using few more ideas.

idea 10: Gravity force (or a fictitious force which is proportional to the mass of a body) can be considered to be applied to the centre of mass of a body *only* in the following cases:
(a) the effective gravity field is homogeneous;
(b) the body has a spherically symmetric mass distribution;
(c) the effective gravity field is proportional to the radius vector, e.g. centrifugal force field if the motion is constrained to the plane perpendicular to the frame's axis of rotation.
In all the other cases, it may happen by coincidence that the gravity force is still applied to the centre of mass, but typically it is not. For instance, the Coriolis force can be considered to be applied to the centre of mass only if the body is not rotating (as seen from the rotating frame).

The part (a) of this claim has been motivated in the paragraph following the idea 9; parts (b) and (c) will be motivated in the booklet of electromagnetism (electrostatic and non-relativistic gravitational fields obey similar laws). In order to prove that the part (d) is valid, we need to show that total force and total torque exerted by the gravitational have the same magnitudes what would be obtained if the body were a point mass at the position of the centre of mass. So, using the attraction(repulsion) centre as the origin and assuming $\vec g = k\vec r$, let us express the total torque as $\vec{\mathcal{T}} = \vec r_i \sum k m_i \vec r_i \times \vec r_i \equiv 0$; the same result would be obtained for a centre of mass as the gravity force would have a zero arm. Further, let us express the total force as $\vec{\mathcal{F}} = \sum k m_i \vec r_i = kM(\sum m_i \vec r_i/M) = kM\vec r_C$; here $M = \sum m_i$ is the total mass of the body.

There are two more ideas which can be used here,

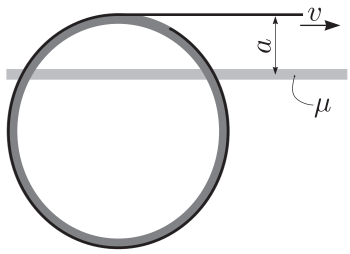
idea 11: In order to achieve a more symmetric configuration or to make the situation simpler in some other way, it is sometimes useful to represent a region with zero value of some quantity as a superposition of two regions with opposite signs of the same quantity.

This quantity can be mass density (like in this case), charge or current density, some force field etc. Often this trick can be combined with

idea 12: Make the problem as symmetric as possible.

This goal can be reached by applying idea 11, but also by using appropriate reference frames, dividing the process of solving into several phases (where some phases use symmetric geometry), etc.

pr 8. A hollow cylinder with mass m and radius R stands on a horizontal surface with its smooth flat end in contact the surface everywhere. A thread has been wound around it and its free end is pulled with velocity v in parallel to the thread. Find the speed of the cylinder. Consider two cases: *(a)* the coefficient of friction between the surface and the cylinder is zero everywhere except for a thin straight band (much thinner than the radius of the cylinder) with a coefficient of friction of μ, the band is parallel to the thread and its distance to the thread $a < 2R$ (the figure shows a top-down view); *(b)* the coefficient of friction is μ everywhere. *Hint:* any planar motion of a rigid body can be viewed as rotation around an instant centre of rotation, i.e. the velocity vector of any point of the body is the same as if the instant centre were the real axis of rotation.



This is quite a hard problem. It is useful to note

idea 13: if a body has to move with a constant velocity, then the problem is about statics.

Also remember ideas 1 and 2. The latter can be replaced with its consequence,

idea 14: if a body in equilibrium is acted on by *three forces at three separate points*, then their lines of action intersect at one point (note that the intersection point can be infinitely far — lines intersecting at infinity means that the lines are parallel to each other). *If there are only two points of action*, then the corresponding lines coincide.

This very useful idea follows directly from the torque balance condition if the intersection point of two lines of action is taken for the pivot point (with two arms and the total torque being equal to zero, the third arm must be also equal to zero).

Another useful fact is

fact 21: the friction force acting on a given point is always antiparallel to the velocity of the point in the frame of reference of the body causing the friction.

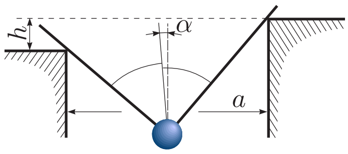
From time to time some mathematical tricks are also of use; here it is the property of inscribed angles, and more specifically the particular case of the Thales' theorem (among geometrical theorems, this is probably the most useful one for solving physics problems),

fact 22: a right angle is subtended by a semicircle (in general: an inscribed angle in radians equals half of the ratio between its arc-length and radius).

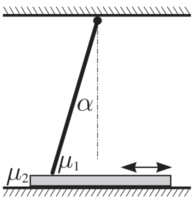
The property of inscribed angles is also useful in the next problem, if we add (somewhat trivial)

idea 15: in stable equilibrium the potential energy of a body is minimum.

pr 9. A light wire is bent into a right angle and a heavy ball is attached to the bend. The wire is placed onto supports with height difference h and horizontal distance a. Find the position of the wire in its equilibrium. Express the position as the angle between the bisector of the right angle and the vertical. Neglect any friction between the wire and the supports; the supports have little grooves keeping all motion in the plane of the wire and the figure.



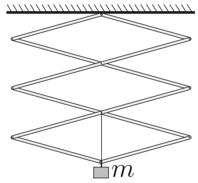
pr 10. A rod with length l is hinged to a ceiling with height $h < l$. Underneath, a board is being dragged on the floor. The rod is meant to block the movement the board in one direction while allowing it move in the opposite direction. What condition should be fulfilled for it to do its job? The coefficient of friction is μ_1 between the board and the rod, and μ_2 between the board and the floor.



Let's remember fact 6: if the relative sliding between two bodies has a known direction then the direction of the sum of the friction and reaction force vectors is always uniquely determined by the coefficient of friction. If a force makes one of the bodies move in such a way that the reaction force grows then they jam: the larger the forces we try to drag the bodies with, the larger friction and reaction forces restrain them.

idea 16: Friction can block movement. In such a case, all forces become negligible except for the friction force, reaction force and the externally applied force that tries to make the system move, because gravitational (and such) forces are fixed, but the said forces become the larger the harder we push or pull.

pr 11. Four long and four half as long rods are hinged to each other forming three identical rhombi. One end of the contraption is hinged to a ceiling, the other one is attached to a weight of mass m. The hinge next to the weight is connected to the hinge above by a string. Find the tension force in the string.



This problem is the easiest to solve using *the method of virtual displacement.*

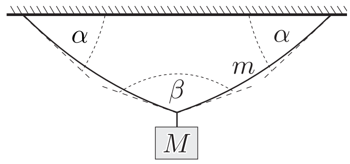
method 1: Imagine that we are able to change the length of the string or rod the tension in which is searched for by an infinitesimal amount Δx. Equating the work $T\Delta x$ by the change $\Delta\Pi$ of the potential energy, we get $T = \Delta\Pi/\Delta x$.

Generalisation: if some additional external forces $\vec{F}_i$ ($i = 1, 2, \ldots$) act on the system with the displacements of their points of action being $\delta\vec{x}_i$, while the interesting string or rod undergoes a virtual lengthening of Δx, then

$$T = (\Delta\Pi - \sum_i \delta\vec{x}_i \cdot \vec{F}_i)/\Delta x.$$

The method can also be used for finding some other forces than tension (for example, in problems about pulleys): by imaginarily shifting the point of action of the unknown force one can find the projection of this force onto the direction of the virtual displacement.

pr 12. A rope with mass m is hung from the ceiling by its both ends and a weight with mass M is attached to its centre. The tangent to the rope at its either end forms angle α with the ceiling. What is the angle β between the tangents to the rope at the weight?



Let us recall the fact 14: The tension in a freely hanging string is directed along the tangent to the string. In addition, we can employ

idea 17: for hanging ropes, membranes etc. it is useful to consider a piece of rope separately and think about the componentwise balance of forces acting onto it.

In fact, here we do not need the idea as a whole, but, rather, its consequence,

fact 23: the horizontal component of the tension in a massive rope is constant.

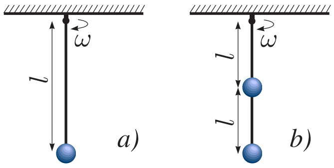
Using the idea 17 and fact 23, it is relatively easy to show that the following approximation is valid.

idea 18: If the weight of a hanging part of a rope is much less than its tension then the curvature of the rope is small and its horizontal mass distribution can quite accurately be regarded as constant.

This allows us to write down the condition of torque balance for the hanging portion of the rope (as we know the horizontal coordinate of its centre of mass). The next problem illustrates that approach.

pr 13. A boy is dragging a rope with length $L = 50\,\text{m}$ along a horizontal ground with a coefficient of friction of $\mu = 0.6$, holding an end of the rope at height $H = 1\,\mathrm{m}$ from the ground. What is the length l of the part of the rope not touching the ground?

pr 14. A light rod with length l is hinged in such a way that the hinge folds in one plane only. The hinge is spun with angular speed ω around a vertical axis. A small ball is fixed to the other end of the rod. *(a)* Find the angular speeds for which the vertical orientation is stable. *(b)* The ball is now attached to another hinge and, in turn, to another identical rod (see the figure below); the upper hinge is spun in the same way. What is now the condition of stability for the vertical orientation?



Here the following idea is to be used.

idea 19: For analysing stability of an equilibrium, there are two options.
First, presume that the system deviates a little from the equilibrium, either by a small displacement Δx or by a small angle $\Delta\varphi$, and find the direction of the appearing force or torque — whether it is towards the equilibrium or away from it.
Second, express the change of total potential energy in terms of the small displacement to see if it has a minimum or maximum (for a system at equilibrium, its potential energy must have an extremum); minimum corresponds to stability, and maximum — to instability (for a motivation and generalization of this method, see appendix 5).

NB! compute approximately: when working with forces (torques), it is almost always enough to keep only those terms which are linear in the deviation; when working with potential energy, quadratic approximation is to be used.

It is extremely important in physics to be able to apply linear, quadratic, and sometimes also higher order approximations, which is based on

idea 20: Taylor series:

$$f(x+\Delta x) \approx f(x) + f'(x)\Delta x + f''(x)\frac{\Delta x^2}{2} + \ldots,$$

for instace: $\sin\varphi \approx \tan\varphi \approx \varphi$; $\cos\varphi \approx 1 - \frac{x^2}{2}$; $e^x \approx 1 + x + \frac{x^2}{2}$, $(1+x)^a \approx 1 + ax + \frac{a(a-1)}{2}x^2$, $\ln 1 + x \approx 1 + x - \frac{x^2}{2}$. Analogous approach can be use for multivariable expressions, e.g. $(x+\Delta x)(y+\Delta y) \approx xy + x\Delta y + y\Delta x$. Consider using such approximations wherever initial data suggest some parameter to be small.

The case *(b)* is substantially more difficult as the system has two degrees of freedom (for example, the deviation angles $\Delta\varphi_1$ and $\Delta\varphi_2$ of the rods). Although idea 19 is generalisable for more than one degrees of freedom, apparently it is easier to start from idea 15.

idea 21: The equilibrium $x = y = 0$ of a system having two degrees of freedom is stable if (and only if[23]) the potential energy $\Pi(x, y)$ as a function of two variables has a local minimum at $x = y = 0$, i.e. for *any pair of values* x, y within a small neigbourhood of the equilibrium point $(0, 0)$, inequality $\Pi(x, y) > \Pi(0, 0)$ must hold.

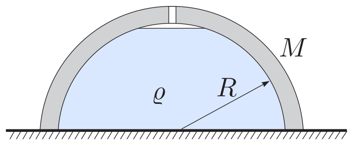
pr 15. If a beam with square cross-section and very low density is placed in water, it will turn one pair of its long opposite faces horizontal. This orientation, however, becomes unstable as we increase its density. Find the critical density when this transition occurs. The density of water is $\rho_w = 1000\,\mathrm{kg/m^3}$.

idea 22: The torque acting on a body placed into a liquid is equal to torque from buoyancy, if we take the latter force to be acting on the centre of the mass of the displaced liquid.

The validity of the idea 22 can be seen if we imagine that the displaced volume is, again, filled with the liquid, and the body itself is removed. Then, of course, the re-filled volume is at equilibrium (as it is a part of the resting liquid). This means that the torque of the buoyancy force must be balancing out the torque due to the weight of the re-filled volume; the weight of the re-filled volume is applied to its centre of mass, and according to the idea 14, the buoyancy force must be therefore also acting along the line drawn through the centre of mass.

Apart from the idea 22, solution of the problem 15 can be simplified by using the ideas 11 and 12.

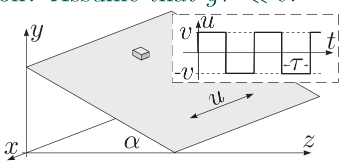
pr 16. A hemispherical container is placed upside down on a smooth horizontal surface. Through a small hole at the bottom of the container, water is then poured in. Exactly when the container gets full, water starts leaking from between the table and the edge of the container. Find the mass of the container if water has density ρ and radius of the hemisphere is R.



idea 23: If water starts flowing out from under an upside down container, normal force must have vanished between the table and the edge of the container. Therefore force acting on the system container+liquid from the table is equal solely to force from hydrostatic pressure.

The latter is given by pS, where p is pressure of the liquid near the tabletop and S is area of the container's open side.

pr 17. A block is situated on a slope with angle α, the coefficient of friction between them is $\mu > \tan\alpha$. The slope is rapidly driven back and forth in a way that its velocity vector $\vec{u}$ is parallel to both the slope and the horizontal and has constant modulus v; the direction of $\vec{u}$ reverses abruptly after each time interval τ. What will be the average velocity w of the block's motion? Assume that $g\tau \ll v$.



[23] We assume that apart from the energy, there are no other conserved quantities for this system.

idea 24: If the system changes at high frequency, then it is often pratical to use time-averaged values $\langle X\rangle$ instead of detailed calculations. In more complicated situations a high-frequency component $\tilde{X}$ might have to be included (so that $X = \langle X\rangle + \tilde{X}$).

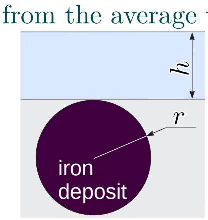
method 2: (*perturbation method*) If the impact of some force on a body's motion can be assumed to be small, then solve the problem in two (or more) phases: first find motion of the body in the absence of that force (so-called zeroth approximation); then pretend that the body is moving just as found in the first phase, but there is this small force acting on it. Look what correction (so-called first correction) has to be made to the zeroth approximation due to that force.

In this particular case, the choice of zeroth approximation needs some explanation. The condition $g\tau \ll v$ implies that within one period, the block's velocity cannot change much. Therefore if the block is initially slipping downwards at some velocity w and we investigate a short enough time interval, then we can take the block's velocity to be constant in zeroth approximation, so that it is moving in a straight line. We can then move on to phase two and find the average value of frictional force, based on the motion obtained in phase one.

For problem 17, recall also a lesson from kinematics,

idea K-7 If friction affects the motion then usually the most appropriate frame of reference is that of the environment causing the friction.

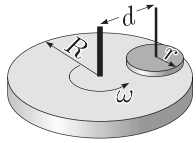
pr 18. Let us investigate the extent to which an iron deposit can influence water level. Consider an iron deposit at the bottom of the ocean at depth $h = 2\,\mathrm{km}$. To simplify our analysis, let us assume that it is a spherical volume with radius 1 km with density greater from the surrounding rock by $\Delta\rho = 1000\,\mathrm{kg/m^3}$. Presume that this sphere touches the bottom of the ocean with its top, i.e. that its centre is situated at depth $r + h$. By how much is the water level directly above the iron deposit different from the average water level?



idea 25: The surface of a liquid in equilibrium takes an equipotential shape, i.e. energies of its constituent particles are the same at every point of the surface.

If this was not the case, the potential energy of the liquid could be decreased by allowing some particles on the surface to flow along the surface to where their potential energy is smaller (cf. idea 15). Recall also the fact 10.

pr 19. A horizontal platform rotates around a vertical axis at angular velocity ω. A disk with radius r can freely rotate and move up and down frictionlessly along a vertical axle which is fixed to a distance $d > r$ from the platform's axis. The disk is pressed against the rotating platform due to gravity, the coefficient of friction between them is μ. Find the angular velocity acquired by the disk. Assume that pressure is distributed evenly over the entire base of the disk.



idea 26: If we transform into a rotating frame of reference, then we can add angular velocities about instantaneous axes of rotation in the same way as we usually add velocities.

Thus $\vec{\omega}_3 = \vec{\omega}_1 + \vec{\omega}_2$, where $\vec{\omega}_1$ is angular velocity of the reference frame, $\vec{\omega}_2$ angular velocity of the body in the rotating frame of reference and $\vec{\omega}_3$ that in the stationary frame. In this question, we can use fact 21, ideas 2, 9, 13, K-7, and also

idea K-33 Arbitrary motion of a rigid body can be considered as rotation about an instantaneous centre of rotation (in terms of velocity vectors of the body).

method 3: (differential calculus) Divide the object into infinitesimally small bits or the process into infinitesimally short periods (if necessary, combine this with idea 20).

Within an infinitesimal bit (period), quantities changing in space (time) can be taken constant (in our case, that quantity is the direction of frictional force vector). If necessary (see the next question), these quantities may be summed over all bits — this is called integration.

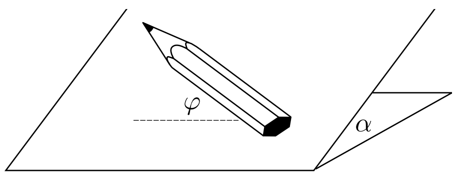
pr 20. A waxing machine consists of a heavy disk with mass M densely covered with short bristles on one side, so that if it lies on the floor, then its weight is evenly distributed over a circular area with radius R. An electrical motor makes the disk rotate at angular velocity ω, the user compensates for the torque from frictional forces by a long handle. The same handle can be used to push the machine back and forth along the floor. With what force does the machine have to be pushed to make it move at velocity v? Assume that angular velocity of the disk is large, $\omega R \gg v$, and that the force needed to compensate for the torque can be neglected. The coefficient of friction between the bristles and the floor is μ.

Here we need fact 21, ideas K-33, 11, and additionally

idea 27: Try to determine the region of space where forces (or torques etc) cancel at pairs of points.

These pairs of points are often symmetrically located. Idea 12 is relevant as well.

pr 21. A hexagonal pencil lies on a slope with inclination angle α; the angle between the pencil's axis and the line of intersection of the slope and the horizontal is φ. Under what condition will the pencil not roll down?

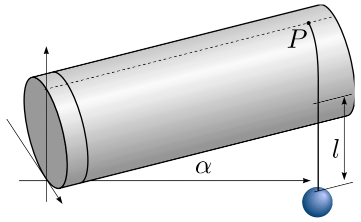


idea 28: When solving three-dimensional problems, sometimes calculating coordinates in appropriately chosen axes and

applying formulae of spatial rotations can be of use. For the rotation around z-axis by angle φ, $x' = x\cos\varphi - y\sin\varphi$ and $y' = y\cos\varphi + x\sin\varphi$.

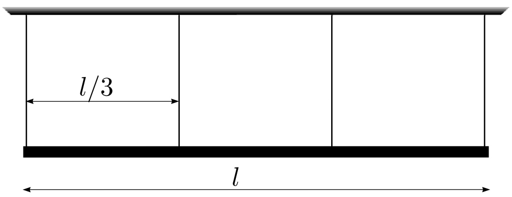
What (which vector) could be expressed in terms of its components in our case? The only promising option is the small shift vector of centre of mass when its starts to move; ultimately we are only interested in its vertical component.

pr 22. A slippery cylinder with radius R has been tilted to make an angle α between its axis and the horizontal. A string with length L has been attached to the highest point P of some cross-section of the cylinder, the other end of it is tied to a weight with mass m. The string takes its equilibrium position, how long (l) is the part not touching the cylinder? The weight is shifted from its equilibrium position in such a way that the shift vector is parallel to the vertical plane including the cylinder's axis; what is the period of small oscillations?



idea 29: Unfolding a part of the surface of a three-dimensional object and looking at the thereby flattened surface can assist in solving problems, among other things it helps to find shortest distances.

pr 23. A uniform bar with mass m and length l hangs on four identical light wires. The wires have been attached to the bar at distances $\frac{l}{3}$ from one another and are vertical, whereas the bar is horizontal. Initially, tensions are the same in all wires, $T_0 = mg/4$. Find tensions after one of the outermost wires has been cut.



idea 30: If more fixing elements (rods, strings, etc) than the necessary minimum have been used to keep a body in static equilibrium (i.e. more than the number of degrees of freedom) and fixing elements are absolutely rigid, then tensions in the elements cannot be determined. In order to make it possible, the elements have to be considered elastic (able to deform); recall the fact 13.

Let us note that this statement is in accordance with fact 18 that gives the number of available equations (there can be no more unknowns than equations). In this particular case, we are dealing with effectively one-dimensional geometry with no horizontal forces, but the body could rotate (in absence of the wires). Thus we have two degrees of freedom, corresponding to vertical and rotational motion. Since the wires are identical, they must have the same stiffness as well; the word "wire" hints at large stiffness, i.e. deformations (and the inclination angle of the bar) are small.

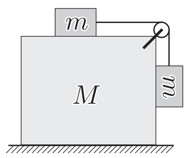
4 DYNAMICS

A large proportion of dynamics problems consist of finding the acceleration of some system of bodies, or finding the forces acting upon the bodies. There are several possible approaches for solving such problems, here we consider three of them.

method 4: For each body, we find all the forces acting on it, including normal forces and frictional forces[24], and write out Newton's 2nd law in terms of components (i.e. by projecting the equation on x, y, and possibly z-axes). NB! Select the directions of the axes carefully, cf. idea 1. In some cases, it may be possible (and more convenient) to abstain from using projections and work with vectorial equalities.

Keep in mind that for a correctly posed problem, it should be possible to write as many linearly independent equations as there are unknowns (following idea 1 may help to reduce that number). The guideline for figuring out how many equations can be found remains the same as in the case of statics problem, see idea 18 (for time being we consider problems where bodies do not rotate, so we need to count only the translational degrees of freedom). If the number of equations and the number of unknowns don't match, it is either an ill-posed problem, or you need to make additional physical assumptions (like in the case of problem 23).

pr 24. A block with mass M lies on a slippery horizontal surface. On top of it there is another block with mass m which in turn is attached to an identical block by a string. The string has been pulled across a pulley situated at the corner of the big block and the second small block is hanging vertically. Initially, the system is held at rest. Find the acceleration of the big block immediately after the system is released. You may neglect friction, as well as masses of the string and the pulley.



This question can be successfully solved using method 4, but we need three more ideas.

idea 31: If a body is initially at rest, then its shift vector is parallel to the force acting on it (and its acceleration) right after the start of its motion.

idea 32: If bodies are connected by a rope or a rod or perhaps a pulley or one is supported by the other, then there is a linear[25] arithmetic relation between the bodies' shifts (and velocities, accelerations) that describes the fact that length of the string (rod, etc.) is constant.

The relation for shifts is usually the easiest to find; if the motion is along a straight line, this relation can be differentiated

[24] It is convenient to make a sketch and draw all the force vectors from their application points.

[25] It is linear in shifts but may contain coefficients which are expressed in terms of nonlinear functions, e.g. trigonometric functions of angles.

once or twice with respect to time, to obtain the relation for velocities or accelerations. For curved motion care needs to be taken. In the case of velocities and for rigid bodies, we result in the idea K-35 (for two points of a rigid body, the projections of velocities of both points on the line connecting them are equal). In the case of accelerations, situation is more complicated as there will be also centripetal accelerations. However, if the motion starts from the rest, for a very short time period t we can assume that the acceleration is constant and the centripetal acceleration v^2/r is negligible (the speed v is still very small), hence there is a simple linear relation $\vec{s} = \frac{t^2}{2}\vec{a}$ between the shift $\vec{s}$ and the acceleration $\vec{a}$ of a certain point of a certain body; then the factor $\frac{t^2}{2}$ cancels out from the linear relation for the shifts, so that it carries directly over to the accelerations.

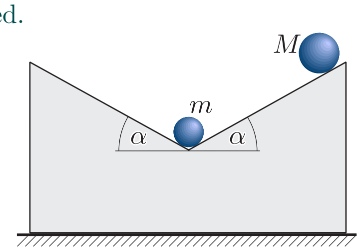
idea 33: If a light rope is thrown over an ideal pulley (both of negligible mass) then tension in the rope has the same magnitude to either side of the pulley; if the rope bends at the pulley, there is a normal force between the pulley and the rope which can be found as the vectorial sum of the tension forces.

Indeed, consider that piece of rope which is in contact with the pulley; its mass can be neglected, hence the inertial term in the Newton's 2$^{\text{nd}}$ law can be also neglected, hence the normal force acting on the rope must be equal and opposite to the vectorial sum of the tension forces.

method 5: Otherwise the same as method 4, but motion is investigated in a non-inertial frame of reference (see idea 7) where one of the bodies is at rest.

Method 5 is useful in many questions concerning wedges: it is difficult to write out the condition for an object to stay on the wedge in the laboratory frame. Applying idea 32 is also often easier in the wedge's frame of reference than in the laboratory frame. Don't forget that the body defining the reference frame is at rest: *we have one or more equations expressing its static equilibrium.*

pr 25. A wedge has been made out of a very light and slippery material. Its upper surface consists of two slopes making an angle α with the horizontal and inclined towards one another. The block is situated on a horizontal plane; a ball with mass m lies at the bottom of the hole on its upper surface. Another ball with mass M is placed higher than the first ball and the system is released. On what condition will the small ball with mass m start slipping upwards along the slope? Friction can be neglected.



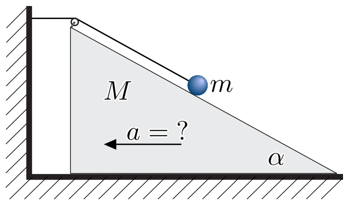
The final method is based on using *generalised coordinates* and originates from analytical mechanics. There, it is known as *Lagrangian formalism* and is introduced using relatively advanced mathematical apparatus (partial derivatives, variational analysis), but for most problems, its simplified version outlined below will suffice. More detailed discussion of the Lagrangian formalism is provided in appendix 6.

method 6: Let us call ξ a generalised coordinate if the entire state of a system can be described by this single number. Say we need to find the acceleration $\ddot{\xi}$ of coordinate ξ. If we can express the potential energy Π of the system as a function $\Pi(\xi)$ of ξ and the kinetic energy in the form $K = \mathcal{M}\dot{\xi}^2/2$ where coefficient $\mathcal{M}$ is a combination of masses of the bodies (and perhaps of moments of inertia), then

$$\ddot{\xi} = -\Pi'(\xi)/\mathcal{M}.$$

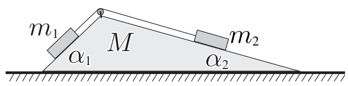
Here, a dot denotes differentiation w.r.t. time and dash w.r.t. coordinate ξ. Indeed, due to conservation of energy $\Pi(\xi) + \mathcal{M}\dot{\xi}^2/2 =$Const. Differentiating that w.r.t. time and using the chain rule, we obtain $\Pi'(\xi)\dot{\xi} + \mathcal{M}\dot{\xi}\ddot{\xi} = 0$. We reach the aforementioned formula after dividing through by $\dot{\xi}$.

pr 26. A small block with mass m lies on a wedge with angle α and mass M. The block is attached to a rope pulled over a pulley attached to the tip of the wedge and fixed to a horizontal wall (see the figure). Find the acceleration of the wedge. All surfaces are slippery (there is no friction).



Full solution of this problem is given in the hints' section to illustrate method 6

pr 27. A wedge with mass M and acute angles α_1 and α_2 lies on a horizontal surface. A string has been drawn across a pulley situated at the top of the wedge, its ends are tied to blocks with masses m_1 and m_2. What will be the acceleration of the wedge? There is no friction anywhere.



It may seem that there is more than one degree of freedom in this question: the wedge can move and the string can shift w.r.t. the wedge. However, we are saved by

idea 34: If x-components of the sum of external forces and of centre of mass velocity are both zero, then the x-coordinate of the centre of mass remains constant.

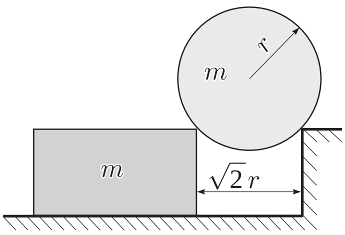
We can use this circumstance to reduce the effective number of degrees of freedom. In our particular case, the system consists of two components and thus the shift of component can be expressed by that of the other.

idea 35: The x-coordinate of the centre of mass of a system of bodies is

$$X_C = \sum x_i m_i / \sum m_i,$$

where m_i denotes mass of the i-th component and x_i the coordinate of its centre of mass. The formula can be rewritten in integral form, $X_C = \int x dm / \int dm$, where $dm = \rho(x,y,z)dV$ is differential of mass.

pr 28. Two slippery horizontal surfaces form a step. A block with the same height as the step is pushed near the step, and a cylinder with radius r is placed on the gap. Both the cylinder and the block have mass m. Find the normal force N between the cylinder and the step at the moment when distance between the block and the step is $\sqrt{2}r$. Initially, the block and the step were very close together and all bodies were at rest. Friction is zero everywhere. Will the cylinder first separate from the block or the step?



It is easy to end up with very complicated expressions when solving this problem, this may lead to mistakes. Therefore it is wise to plan the solution carefully before writing down any equations.

idea 36: Newton's laws are mostly used to find acceleration from force, but sometimes it is clever to find force from acceleration.

But how to find acceleration(s) in that case? It is entirely possible if we use method 6, but this path leads to long expressions. A tactical suggestion: if you see that the solution is getting very complicated technically, take a break and think if there is an easier way. There is a "coincidence" in this particular problem: straight lines drawn from the sphere's centre to points of touching are perpendicular; can this perhaps help? It turns out that it does.

idea 37: Pay attention to special cases and use simplifications that they give rise to!

Let us remind what we learned in kinematics:

idea K-34 In case of motion along a curve, the radial component (perpendicular to the trajectory) of a point's acceleration v^2/R is determined by velocity v and radius of curvature R; the component along the trajectory is linear acceleration (equal to εR in case of rotational motion, ε is angular acceleration).

The centre of mass of the cylinder undergoes rotational motion, method 6 is necessary to find angular acceleration — but we hoped to refrain from using it. An improvement on idea 1 helps us out:

idea 38: Project Newton's 2^{nd} law on the axis perpendicular to an unwanted vector, e.g. an unknown force or the tangential component of acceleration.

We can easily find the cylinder's velocity (and thus the radial component of acceleration) if we use

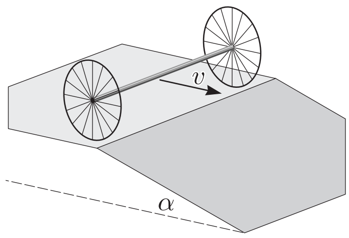
idea 39: If energy is conserved (or its change can be calculated from work done etc), write it out immediately. Energy is conserved if there is no dissipation (friction, inelastic collisions etc) and external forces acting on the system are static (e.g. a stationary inclined plane);

forces changing in time (force acting on a moving point, moving inclined plane) change energy as well. Idea 32 helps to write out conservation of energy (relation between bodies' velocities!). To answer the second question, we need

idea 40: Normal force vanishes at the moment when a body detaches from a surface.

Also, review idea 32 for horizontal components of accelerations.

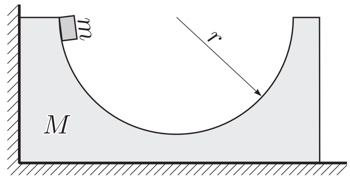
pr 29. Light wheels with radius R are attached to a heavy axle. The system rolls along a horizontal surface which suddenly turns into a slope with angle α. For which angles α will the wheels move without lifting off, i.e. touch the surface at all times? Mass of the wheels can be neglected. The axle is parallel to the boundary between horizontal and sloped surfaces and has velocity v.



idea 41: To answer the question whether a body lifts off, we have to find the point on the non-lifting-off trajectory with smallest normal force.

If normal force has to be negative at that point, then the body lifts off; the critical value is zero — compare with idea 40). Also, review ideas 1, 39 and K29.

pr 30. A block with mass M lies on a horizontal slippery surface and also touches a vertical wall. In the upper surface of the block, there is a cavity with the shape of a half-cylinder with radius r. A small pellet with mass m is released at the upper edge of the cavity, on the side closer to the wall. What is the maximum velocity of the block during its subsequent motion? Friction can be neglected.



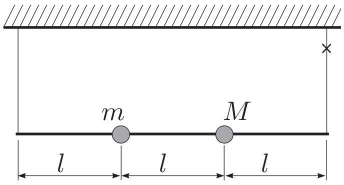
idea 42: Conservation law can hold only during some period of time.

idea 43: Momentum is conserved if the sum of external forces is zero; sometimes momentum is conserved only along one axis.

You will also need idea 39.

idea 44: Velocity is maximal (or minimal) when acceleration (and net force) is zero (since $0 = \frac{dv}{dt} = a$); shift is extremal when velocity is zero. Possible other pairs: electrical charge (capacitor's voltage)-current, current-inductive emf, etc.

pr 31. A light rod with length $3l$ is attached to the ceiling by two strings with equal lengths. Two balls with masses m and M are fixed to the rod, the distance between them and their distances from the ends of the rod are all equal to l. Find the tension in the second string right after the first has been cut.



There are several good solutions for this problem, all of which share applying idea 36 and the need to find the angular acceleration of the rod. Firstly, angular acceleration of the rod can be found from method 6 by choosing angle of rotation φ to be the generalised coordinate. Secondly, we may use Newton's 2^{nd} law for rotational motion: we find the torque on the rod about the point of attachment of the second string and equate it to $I\varepsilon$ with angular acceleration ε and moment of inertia $I = ml^2 + 4Ml^2$. More generally,

idea 45: When a body is rotating around the axis s, the net torque it experiences is $M = I\varepsilon$ (not to be confused with the body's mass), where I is its moment of inertia with respect to the axis s, $I = \sum m_i r_i^2 = \int r^2 \cdot dm = \int r^2\rho \cdot dV$ and r_i is the distance of i-th particle from the axis s (the sum is evaluated over all particles of the body). Kinetic energy is $K = \frac{1}{2}I\omega^2$.

Once the angular acceleration is found, in order to apply the idea 36 it may be helpful to use

idea 46: The more general and sometimes indispensable form of Newton's 2^{nd} law is $\vec{F} = \frac{d\vec{P}}{dt}$, where $\vec{P}$ is the net momentum of the system and $\vec{F}$ is the sum of external forces acting on the system. An analogous formula is $\vec{M} = \frac{d\vec{L}}{dt}$, where $\vec{L}$ is the net angular momentum of the system (with respect to a given point) and $\vec{M}$ is the sum of external torques.

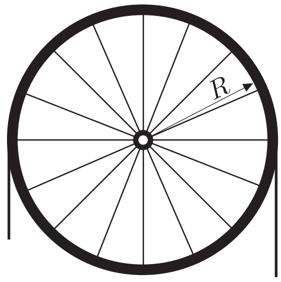
In our case this last method is fruitful when applied both to forces and torques.

Another solution method is to consider the rod and the balls as three different (interacting) bodies. Then the balls' accelerations can be found as per idea 32; one can also employ

idea 47: Net force and torque acting on very light bodies (compared to other bodies) are zero.

Clearly if this were not true, a non-zero force would generate an infinite acceleration for a massless body.

pr 32. An inextensible rough thread with mass per unit length ρ and length L is thrown over a pulley such that the length of one hanging end is l. The pulley is comprised of a hoop of mass m and radius R attached to a horizontal axle by light spokes. The initially motionless system is let go. Find the force on the axle immediately after the motion begins. The friction between the pulley and the axle is negligible.



Why not proceed as follows: to find the force, we will use idea 36; the acceleration of the system will be found using Method 6. To apply idea 36 most handily, let us employ

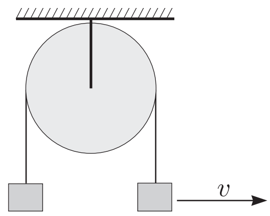
idea 48: Newton's 2^{nd} law can be written as $\vec{F} = M\vec{a}_C$, where $\vec{a}_C$ is the acceleration of the centre of mass.

This idea is best utilised when a part of the system's mass is motionless and only a relatively small mass is moved about (just like in this case: the only difference after a small period of time is that a short length of thread is "lost" at one end and "gained" at the other end). Obviously idea 34 will be useful here, and idea 11 will save you some effort. Bear in mind that in this case we are not interested in the centre of mass coordinate per se, but only in its change as a function of time; therefore in the expression for this coordinate we can omit the terms that are independent of time: their time derivatives will vanish. The time-dependent part of the centre of mass coordinate should be expressed using the same coordinate that we will use with Method 6 (since Method 6 will produce its second derivative with respect to time). A technical bit of advice may help: a vector is specified by *(a)* its magnitude and direction; *(b)* its projections onto coordinate axes in a given coordinate system;

idea 49: sometimes it is easier to compute the components of a vector, even if we are interested in its magnitude only.

Above all, this applies when the direction of the vector is neither known nor apparent. In this instance, we should find F_x and F_y in a suitable coordinate system.

pr 33. A thread is thrown over a pulley. At its both ends there are two blocks with equal masses. Initially the two blocks are at the same height. One of them is instantaneously given a small horizontal velocity v. Which of the two blocks will reach higher during the subsequent motion? The pulley's mass is negligible.



This problem is really tough, because the key to the solution is a very specific and rarely used

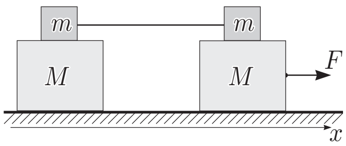
idea 50: If the centre of mass of a system cannot move, then the net force acting on it is zero.

Here the centre of mass can move about a little bit, but in the longer term (averaged over one period of the pendulum-like motion of the kicked block — cf. idea 24) it is motionless: the blocks have the same mass and if one of them rises, then in the expression for the centre of mass this will be compensated by the descent of the other block. This is also true for the horizontal coordinate of the centre of mass, but it is enough to consider the vertical coordinate only to solve the problem. Let us also bring up the rather obvious

fact 24: the tension in a weightless thread thrown over a weightless pulley or pulled along a frictionless surface is the same everywhere.

The solution algorithm is then as follows: we write down Newton's 2^{nd} law for *(a)* the system made out of two blocks and *(b)* one block; we average both equations and use the equality apparent from *(a)* to find the average tension in the thread, which we then substitute into equation *(b)*. Based on idea 24, we partition the tension in the thread into the average and the high-frequency component and use idea 20.

pr 34. A system of blocks sits on a smooth surface, as shown in the figure. The coefficient of friction between the blocks is μ, while that between the blocks and the surface is $\mu = 0$.



The bottom right block is being pulled by a force F. Find the accelerations of all blocks.

idea 51: When bodies are connected by frictional forces, then to answer some questions fully one needs to consider all possible combinations of there being relative slipping between all possible touching surfaces.

For example, if we are to assume that there is no slipping between two touching bodies, then they could be treated as a whole. Then one should find the frictional force F_h between the bodies and determine when the assumption holds, or when is F_h less that the maximum static friction force μN.

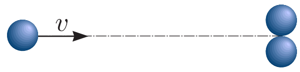
pr 35. A billiard ball hits another stationary billiard ball. At which collection of points could the stationary ball be positioned such that it would be possible to achieve the situation where both balls will fall into two (different) pockets on the table? The collisions are perfectly elastic, the balls are perfectly slippery (hence the rotation of the balls is negligible).

idea 52: If an absolutely elastic ball hits another motionless identical ball and the rotation (rolling) of the balls can be ignored, then upon impact there will be a right angle between the velocity vectors of the two balls.

To prove this, note that the three velocity vectors (velocity before and the two velocities after the impact) form a triangle because of the momentum conservation law. The conservation of energy means that the sides of the triangle satisfy Pythagore's theorem. A special case of this result is (see the problem after next)

fact 25: When an elastic ball undergoes a central collision with another identical stationary ball, then the first ball stops and the second gains the velocity of the first ball.

pr 36. An absolutely elastic and slippery billiard ball is moving with velocity v toward two motionless identical balls. The motionless balls are touching and their centres lie on a straight line that is perpendicular to the incoming ball's velocity vector. The moving ball is directed exactly toward the touching point of the two balls. Which velocity will the incoming ball have after the collisions? Consider two scenarios: *(a)* the incoming ball hits exactly in the middle between the balls; *(b)* its trajectory is a little bit off and it hits one of the stationary balls marginally earlier.



To answer the first question, it is necessary to use

idea 53: collisions (and other many-body interactions, like the motion of balls connected by threads or springs) are easier to treated in the centre of mass system, because in that system the momentum conservation is the easiest to write down (the net momentum is zero).

Also, do not forget idea 39! For the second question, let us use

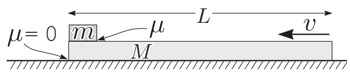
idea 54: if a force acting on a body during a known time does not change direction, then the transferred momentum has the same direction as the force.

pr 37. n absolutely elastic beads are sliding along the frictionless wire. What is the maximum possible number of collisions? The sizes of the beads are negligible, and so is the probability that more than two beads will collide at the same time.

idea 55: Representing the process visually, e.g. with a graph, tends to be great help.

Here is an auxiliary question: what would the elastic collision of two balls on an $x-t$ diagram look like?

pr 38. A plank of length L and mass M is lying on a smooth horisontal surface; on its one end lies a small block of mass m. The coefficient of friction between the block and the plank is μ. What is the minimal velocity v that needs to be imparted to the plank with a quick shove such that during the subsequent motion the block would slide the whole length of the board and then would fall off the plank? The size of the block is negligible.

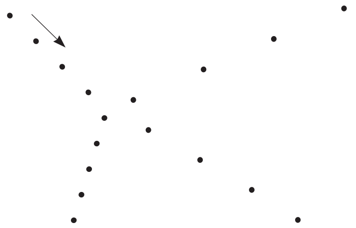


This problem has two more or less equivalent solutions. First, we could solve it using idea 7. Second, we could use ideas 39 and 53, further employing

idea 56: if a body slides along a level surface, then the energy that gets converted to heat is equal to the product of the friction force and the length of the sliding track.

Indeed, the friction force has a constant magnitude and, as seen in the reference frame of the support, it is always parallel to displacement.

pr 39. The given figure has been produced off a stroboscopic photograph and it depicts the collision of two balls of equal diameters but different masses. The arrow notes the direction of motion of one of the balls before the impact. Find the ratio of the masses of the two balls and show what the direction of motion for the second ball was before the impact.



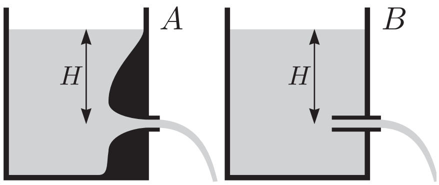
idea 57: sometimes it is beneficial to treat momenta as vectors, treating their vectorial sums and differences using triangle or parallelogram rules (this is also true of other vectorial quantities: displacements, velocities, accelerations, forces etc.)

To be more specific: when two bodies interact, the vector of the impulse is equal to the vectorial difference of their two momenta. Cf. idea 5.

fact 26: In a stroboscopic photograph, the vector from one position of the body to the next is proportional to its velocity (vector).

fact 27: (Newton's 3$^{\text{rd}}$ law) if two bodies have interacted, the changes of momenta of the two bodies are equal and opposite.

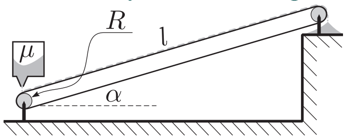
pr 40. There are two barrels (A and B) whose taps have different design, see figure. The tap is opened, the height of the water surface from the tap is H. What velocity does the water stream leave the barrels with?



idea 58: If it seems that it is possible to solve a problem using both energy and momentum conservation, then at least one of these is not actually conserved!

It could not be otherwise: the answers are, after all, different. It pays to be attentive here. While designing the tap A, there was a clear attempt to preserve the laminarity of the flow: energy is conserved. However, if, motivated by method 3, we were to write down the momentum given to the stream by the air pressure during an infinitesimal time dt — $pSdt$ (where S is the tap's area of cross-section), we would see that, owing to the flow of water, $p \neq \rho g$ (cf. dynamical pressure, Bernoulli's law!). On the other hand, for tap B the laminar flow is not preserved; there will be eddies and loss of energy. We could nonetheless work with momentum: we write the expression for the pressure exerted on the liquid by the walls of the barrel (generally the pressures exerted by the left and the right hand side walls of the barrel cancel each other out, but there remains an uncompensated pressure $p = \rho g H$ exerted to the left of the cross-section of the tap S).

pr 41. Sand is transported to the construction site using a conveyor belt. The length of the belt is l, the angle with respect to the horizontal is α; the belt is driven by the lower pulley with radius R, powered externally. The sand is put onto the belt at a constant rate μ (kg/s). What is the minimal required torque needed to transport the sand? What is the velocity of the belt at that torque? The coefficient of friction is large enough for the sand grains to stop moving immediately after hitting the belt; take the initial velocity of the sand grains to be zero.



fact 28: To make anything move — bodies or a flow (e.g. of sand) — force needs to be exerted.

For this problem, idea 58 and methode 3 will come in handy in addition to

idea 59: (the condition for continuity) for a stationary flow the flux of matter (the quantity of stuff crossing the cross-section of the flow per unite time) is constant and is independent of the cross-section: $\sigma v = \text{Const}$ [$\sigma(x)$ is the matter density per unit distance and $v(x)$ — the velocity of the flow].

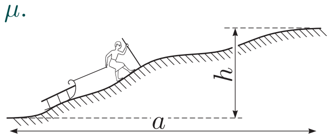
For a flow of incompressible (constant density) liquid in a pipe, such a density is $\sigma = \rho S$ and therefore $vS = \text{Const}$. For a region of space where the flow is discharged — a sink — the mass increases: $\frac{dm}{dt} = \sigma v$ — this equation, too, could be called the condition for continuity.

pr 42. A ductile blob of clay falls against the floor from the height h and starts sliding. What is the velocity of the blob at the very beginning of sliding if the coefficient of friction between the floor and the blob is μ? The initial horizontal velocity of the blob was u.

idea 60: If during an impact against a hard wall there is always sliding, then the ratio of the impulses imparted along and perpendicular to the wall is μ.

Indeed, $\Delta p_\perp = \int N(t)dt$ (integrated over the duration of the impact) and $\Delta p_\parallel = \int \mu N(t)dt = \mu \int N(t)dt$.

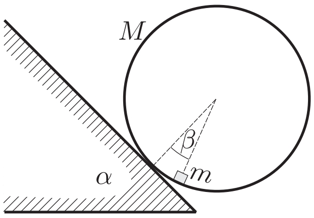
pr 43. A boy is dragging a sled by the rope behind him as he slowly ascends a hill. What is the work that the boy does to transport the sled to the tip of the hill if its height is h and the horizontal distance from the foot of the hill to its tip is a? Assume that the rope is always parallel to the tangent of the hill's slope, and that the coefficient of friction between the sled and the snow is μ.



fact 29: if the exact shape of a certain surface or a time dependence is not given, then you have to deal with the general case: prove that the proposition is true for an arbitrary shape.

Clearly, to apply the fact 29, one will need idea 3.

pr 44. An empty cylinder with mass M is rolling without slipping along a slanted surface, whose angle of inclination is $\alpha = 45°$. On its inner surface can slide freely a small block of mass $m = M/2$. What is the angle β between the normal to the slanted surface and the straight line segment connecting the centre of the cylinder and the block?



Clearly the simplest solution is based on idea 6, but one needs to calculate the kinetic energy of a rolling cylinder.

idea 61: $K = K_c + M_\Sigma v_c^2/2$, where K_c is the kinetic energy as seen in the centre of mass frame and M_Σ — is the net mass of the system. Analogously: $\vec{P} = M_\Sigma \vec{v}_c$ (since $\vec{P}_c \equiv 0$) and the angular momentum $\vec{L} = L_c + \vec{r}_c \times \vec{P}$. Parallel-axis (Steiner) theorem holds: $I = I_0 + M_\Sigma a^2$, where I is the moment of inertia with respect to an axis s and I_0 — that with respect to an axis through the centre of mass (parallel to s) while a is the distance between these two axes.

We will have to compute angular momentum already in the next problem, so let us clarify things a little.

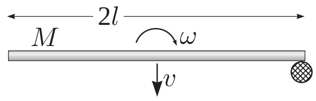
idea 62: Angular momentum is additive. Dividing the system into point-like masses, $\vec{L} = \sum \vec{L}_i$, where for i-th point-like mass $\vec{L}_i = \vec{r}_i \times \vec{p}_i$ (generally) or $L_i = h_i p_i = r_i p_{ti}$ (motion in a plane), $h_i = r_i \sin\alpha_i$ is the lever arm and $p_{ti} = p_i \sin\alpha$ — is the tangential component of the momentum). *Kinetic energy, momentum etc. are also additive.*

If in a three-dimensional space the angular momentum is a vector, for a motion in a plane this vector is perpendicular to the plane and is therefore effectively a scalar (and thus one can abandon cross products). It is often handy to combine ideas 61 and 62: we do not divide the system into particles but, instead, into rigid bodies ($L = \sum L_i$), we compute the moment of inertia L_i of each body according to idea 61: the moment of inertia of the centre of mass plus the moment of inertia as measured in the centre of mass frame.

idea 63: Here are moments of inertia for a few bodies, with respect to the centre of mass. A rod of length of l: $\frac{1}{12}Ml^2$, solid sphere: $\frac{2}{5}MR^2$, spherical shell: $\frac{2}{3}MR^2$, cylinder: $\frac{1}{2}MR^2$, square with side length a, axis perpendicular to its plane: $\frac{1}{6}Ma^2$.

If the the rotation axis does not go through the centre of mass, then one can *(a)* find the moment of inertia with respect to the axis of interest using the parallel-axis (Steiner) theorem; *(b)* apply idea 61 to calculate kinetic energy or angular momentum (in which case it is only enough to know the moment of inertia with respect to the centre of mass).

pr 45. A rod of mass M and length $2l$ is sliding on ice. The velocity of the centre of mass of the rod is v, the rod's angular velocity is ω. At the instant when the centre of mass velocity is perpendicular to the rod itself, it hits a motionless post with an end. What is the velocity of the centre of mass of the rod after the impact if *(a)* the impact is perfectly inelastic (the end that hits the post stops moving); *(b)* the impact is perfectly elastic.



In case of an absolutely elastic collision one equation follows from energy conservation; if the collision is inelastic, then another condition arises: that of a motionless end of the rod. Still, we have two variables. The second equation arises from

idea 64: if a body collides with something, then its angular momentum is conserved with respect to the point of impact.

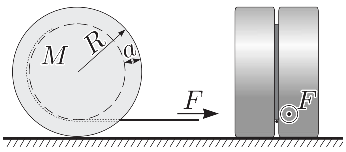
Indeed, during the impact the body's motion is affected by the normal and frictional forces, but both are applied through the point of impact: their lever arm is zero. If a body is moving in a gravitational or similar field, then in the longer term the angular momentum with respect to the point of impact may begin to change, but immediately before and after the collision it is nonetheless the same (gravity is not too strong as opposed to the normal forces that are strong yet short-lived; even though gravity's lever arm is non-zero, it cannot change the angular momentum in an instant).

pr 46. If one hits something rigid — e.g. a lamppost — with a bat, the hand holding the bat may get stung (hurt) as long as the impact misses the so-called centre of percussion of the bat (and hits either below or above such a centre). Determine the position of the centre of percussion for a bat of uniform density. You may assume that during an impact the bat is rotating around its holding hand.

method 7: Convert a real-life problem into the formal language of physics and math — in other words, create a model.

Phrased like that, it may seem that the method is rather pointless. However, converting and interpreting real-life scenarios — *modelling* the problem — is one of the most challenging and interesting aspects of physics. It is interesting because it supplies more creative freedom than solving an existing model using well-established ideas. Still, this freedom has limits: the model has to describe the reality as best as possible, the approximations have to make sense and it is desirable that the model were solvable either mentally or with aid of a computer. For a given problem, there is not much freedom left and the business is simplified: there clear hints as to sensible assumptions. Let us begin translating: "A rigid rod of length l and uniform density is rotating around one end with the angular velocity ω, the rotation axis is perpendicular to the rod. At a distance x from the axis there is a motionless post that is parallel to the axis of rotation. The rod hits the post." Now we encounter the first obstacle: is the impact elastic or inelastic? This is not brought up in the text of the problem. Let us leave it for now: maybe we can get somewhere even without the corresponding assumption (it turns out that this is the case). Now we encounter the central question: what does it mean for the hand "not to get stung"? We know it hurts when something hits our hand — if this something gets an impulse from the hand during a short period of time (the impact), as this implies a large force. The hand is stationary, so the hand-held end of the bat should come to halt without receiving any impulse from the hand. Thus our interpretation of the problem is complete: "Following the impact, the rotation is reversed, $0 \geq \omega' \geq -\omega$; during the impact the axis of rotation imparts no impulse on the rod. Find x." The penultimate sentence hints at the usage of idea 64.

pr 47. A massive cylinder of radius R and mass M is lying on the floor. A narrow groove of depth a has been chiselled along the circumference of the cylinder. A thread has been wrapped around the groove and is now being pulled by its free end, held horizontally, with a force F. The cylinder is positioned such that the thread is being freed from below the cylinder. With what acceleration will the cylinder start moving? The friction between the floor and the cylinder is large enough for there to be no slipping.



There are multiple ways to tackle this problem, but let us use the following idea.

idea 65: The relation $I\varepsilon = M$ is clearly valid only if the centre of rotation is motionless; however, it turns out that it also holds when the instantaneous axis of rotation is moving translationally such that the distance of the body's centre of mass from the axis does not change (eg when rolling a cylindrical or spherical object).

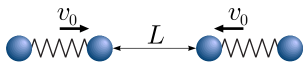
To prove this idea, recall idea 6: kinetic energy appears when work is done, $K = \frac{1}{2}I\omega^2 = M\varphi$ (φ is the angle of rotation of the body, $\omega = d\varphi/dt$). If the moment of inertia with respect to the instantaneous axis of rotation I does not depend on time, then $dK/dt = \frac{1}{2}Id\omega^2/dt = I\omega\varepsilon = dM\varphi/dt = M\omega$, which gives $I\varepsilon = M$.

pr 48. A ball is rolling along a horizontal floor in the region $x < 0$ with velocity $\vec{v}_0 = (v_{x0}, v_{y0})$. In the region $x > 0$ there is a conveyor belt that moves with velocity $\vec{u} = (0, u)$ (parallel to its edge $x = 0$). Find the velocity of the ball $\vec{v} = (v_x, v_y)$ with respect to the belt after it has rolled onto the belt. The surface of the conveyor belt is rough (the ball does not slip) and is level with the floor.

idea 66: For cylindrical or spherical bodies rolling or slipping on a horizontal surface, the angular momentum is conserved with respect to an arbitrary axis lying in the plane of the surface.

Indeed, the points where the normal force and the gravity are applied are on the same straight line with the forces themselves and their sum is zero, meaning that their net torque is also zero; the force of friction is lying in the plane of the surface, and so its lever arm with respect to an axis in the same plane is zero.

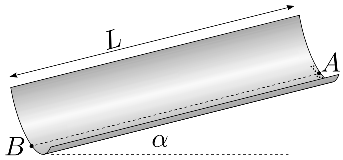
pr 49. A "spring-dumbbell" comprises two balls of mass m that are connected with a spring of stiffness k. Two such dumbbells are sliding toward one another, the velocity of either is v_0. At some point the distance between them is L (see fig.). After which time is the distance between them equal to L again? The collisions are perfectly elastic.



idea 67: If a system consisting of elastic bodies, connected by springs, threads etc., interacts with other bodies, then the duration of impact of the elastic bodies is significantly smaller than the characteristic times of other processes. The whole process can then be divided into simpler stages: an almost instantaneous collision of elastic bodies (that could be considered free, as e.g. the spring exerts an insignificant force compared to that exerted in an elastic collision) and the subsequent (or preceding, or in between the collisions) slow process: the oscillations of the spring etc.

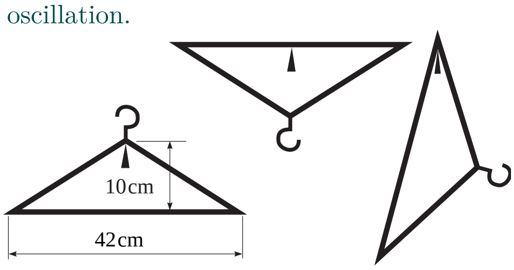
Note: this is a rather general idea, division into simpler steps can be useful if rapid (almost instantaneous) processes can occur in a dynamical system; see next problem for an example (also recall idea 53)

pr 50. Small grains of sand are sliding without friction along a cylindrical trough of radius R (see fig.). The inclination angle of the trough is α. All grains have initial velocity zero and start near point A (but not necessarily at the point A itself). What should be the length of the trough such that all grains would exit it at the point B (i.e. exactly at the bottom of the trough)?



idea 68: If the motion of a spread collection of particles could be divided into oscillation in a known direction and an oscillation-free motion (so motion perpendicular to the oscillation), then the particles are focussed at certain points: where the oscillation phase of all particles is either zero or is an integer multiple of 2π.

pr 51. A coat hanger made of wire with a non-uniform density distribution is oscillating with a small amplitude in the plane of the figure. In the first two cases the longer side of the triangle is horizontal. In all three cases the periods of oscillation are equal. Find the position of the centre of mass and the period of oscillation.



Background info: A finite-size rigid body that oscillates around a fixed axis is known as the physical pendulum. Its frequency of small oscillations is easy to derive from the relation $I\ddot{\varphi} = -mgl\varphi$, where I is the moment of inertia with respect to the axis of oscillation and l is the distance of the centre of mass from that axis: $\omega^{-2} = I/mgl = I_0/mgl + l/g$ (here we employ the parallel-axis/Steiner theorem, see idea 61). The *reduced length* of the physical pendulum is the distance $\tilde{l} = l + I_0/ml$ such that the frequency of oscillation of a mathematical pendulum of that length is the same as for the given physical pendulum.

idea 69: If we draw a straight line of length $\tilde{l}$ such that it passes through the centre of mass and one of its ends is by the axis of rotation, then if we move the rotation axis to the other end of the segment (and let the body reach a stable equilibrium), then the new frequency of oscillation is the same as before. Conclusion: the set of points where the axis of rotation could be placed without changing the frequency of oscillation, consists of two concentric circles around the centre of mass.

Proof: the formula above could be rewritten as a quadratic equation to find the length l corresponding to the given frequency ω (i.e. to the given reduced length $\tilde{l} = g/\omega^2$): $l^2 - l\tilde{l} + I_0/m = 0$. According to Vieta's formulae, the solutions l_1 and l_2 satisfy $l_1 + l_2 = l$, so that l_1 and $l_2 = \tilde{l} - l_1$ result in the same frequency of oscillations.

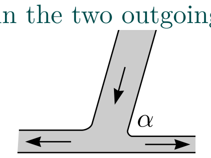
pr 52. A metallic sphere of radius 2 mm and density $\rho = 3000\,\mathrm{kg/m^3}$ is moving in water, falling freely with the acceleration $a_0 = 0{,}57$g. The water density is $\rho_0 = 1000\,\mathrm{kg/m^3}$. With what acceleration would a spherical bubble of radius 1 mm rise in the water? Consider the flow to be laminar in both cases; neglect friction.

idea 70: If a body moves in a liquid, the fluid will also move. (A) If the flow is laminar (no eddies), only the liquid adjacent to the body will move; (B) if the flow is turbulent, there will be a turbulent 'tail' behind the body. In either case the characteristic velocity of the moving liquid is the same as the velocity of the body.

Using method 6 we find that in the case (A) the kinetic energy of the system $K = \frac{1}{2}v^2(m + \alpha\rho_0 V)$, where the constant α is a number that characterizes the geometry of the body that correspond to the extent of the region of the liquid that will move (compared to the volume of the body itself). This expression is obtained by noticing that the characteristic speed of the liquid around the body is v, and the characteristic size of the region where the liquid moves (the speed is not much smaller than v) is estimated as the size of the body itself. If a body is acted on by a force F, then the power produced by this force is $P = Fv = \frac{dK}{dt} = va(m + \alpha\rho_0 V)$. Thus $F = a(m + \alpha\rho_0 V)$: the effective mass of the body increases by $\alpha\rho_0 V$. In the problem above, the constant α for the spherical body can be found using the conditions given in the first half of the problem.

In case (B), if we assume that the velocity of the body is constant, we find $K = \frac{1}{2}v^2\rho_0(\alpha S v t)$, where S is the cross-sectional area of the body and αS is the cross-sectional area of the turbulent 'tail'. This α, again, characterizes the body. From here, it is easy to find $Fv = \frac{dK}{dt} = \frac{\alpha}{2}v^3\rho_0 S$, which gives $F = \frac{\alpha}{2}v^2\rho_0 S$.

pr 53. A stream of water falls against a trough's bottom with velocity v and splits into smaller streams going to the left and to the right. Find the velocities of both streams if the incoming stream was inclined at an angle α to the trough (and the resultant streams). What is the ratio of amounts of water carried per unit time in the two outgoing streams?



This is a rather hard problem. Let us first state a few ideas and facts.

idea 71: For liquid flow, Bernoulli's (i.e. energy conservation) law is often helpful: $p + \rho g h + \frac{1}{2}\rho v^2 = \mathrm{Const}$, where p is the static pressure, h is the height of the considered point and v is the velocity of the flow at that point.

fact 30: Inside the liquid close to its free surface the static pressure is equal to the external pressure.

To solve the second half of the problem, the following is needed:

idea 72: Idea 46 can be generalized in a way that would hold for open systems (certain amounts of matter enter and leave the system): $\vec{F} = \frac{d\vec{P}}{dt} + \vec{\Phi}_{P\mathrm{in}} - \vec{\Phi}_{P\mathrm{out}}$, where $\vec{\Phi}_{P\mathrm{in}}$ and $\vec{\Phi}_{P\mathrm{out}}$ are the entering and the outgoing fluxes of momentum (in other words, the net momentum of the matter entering and leaving the system, respectively).

The momentum flux of the flowing liquid could be calculated as the product of momentum volume density $\rho\vec{v}$ with the flow rate (volume of liquid entering/leaving the system per unit time).

What is the open system we should be considering in this case? Clearly, a system that would allow relating the incoming flow rate μ (kg/s) to the outgoing fluxes (μ_l ja μ_r) using the formula above: a small imaginary region of space that would include the region where the stream splits into two.

fact 31: If we can ignore viscosity, the component of the force exerted by the stream bed (including the 'walls' limiting the flow) on the flow that is parallel to these walls is zero.

pr 54. Find the velocity of propagation of small waves in shallow water. The water is considered shallow if the wavelength is considerably larger than the depth of the water H. Thanks to this we can assume that along a vertical cross-section the horizontal velocity of all particles v_h is the same and that the horizontal velocity of water particles is significantly smaller than the vertical velocity. The smallness of the waves means that their height is significantly smaller than the depth of the water. This allows us to assume that the horizontal velocity of the water particles is significantly smaller than the wave velocity, u.

idea 73: A standard method for finding the velocity of propagation (or another characteristic) of a wave (or another structure with persistent shape) is to choose a reference system where the wave is at rest. In this frame, (a) continuity (idea 59) and (b) energy conservation (e.g. in the form of Bernoulli's law) hold. In certain cases energy conservation law can be replaced by the balance of forces.

(An alternative approach is to linearise and solve a system of coupled partial differential equations.)

pr 55. A small sphere with mass $m = 1\,\mathrm{g}$ is moving along a smooth surface, sliding back and forth and colliding elastically with a wall and a block. The mass of the rectangular block is $M = 1\,\mathrm{kg}$, the initial velocity of the sphere is $v_0 = 10\,\mathrm{m/s}$. What is the velocity of the sphere at the instant when the distance between the sphere and the wall has doubled as compared with the initial distance? By how many times will the average force (averaged over time) exerted by the sphere on the wall have changed?

idea 74: If a similar oscillatory motion takes place, for which the parameters of the system change slowly (compared to the period of oscillation), then the so-called adiabatic invariant I is conserved: it is the area enclosed by the closed contour traced by the trajectory of the system on the so-called phase diagram (where the coordinates are the spatial coordinate x and momentum p_x).

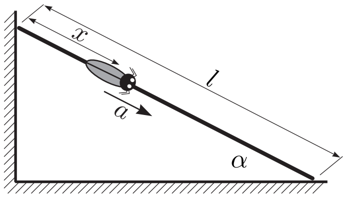
Let us be more precise here. The closed contour is produced as a parametric curve (the so-called phase trajectory) $x(t), p_x(t)$ if we trace the motion of the system during one full period T. The phase trajectory is normally drawn with an arrow that indicated the direction of motion. The adiabatic invariant is not exactly and perfectly conserved, but the precision with which it is conserved grows if the ratio τ/T grows, where τ is the characteristic time of change of the system's parameters.

Adiabatic invariant plays an instrumental role in physics: from the adiabatic law in gases (compare the result of the previous problem with the adiabatic expansion law for an ideal gas with one degree of freedom!) and is applicable even in quantum mechanics (the number of quanta in the system — e.g. photons — is conserved if the parameters of the system are varied slowly).

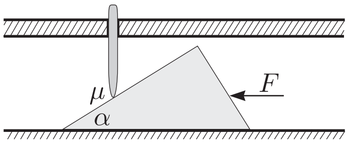
5 REVISION PROBLEMS

pr 56. A straight homogeneous rod is being externally supported against a vertical wall such that the angle between the wall and the rod is $\alpha < 90°$. For which values of α can the rod remain stationary when thus supported? Consider two scenarios: a) the wall is slippery and the floor is rough with the friction coefficient μ ; b) the floor is slippery and the wall is rough with the friction coefficient μ.

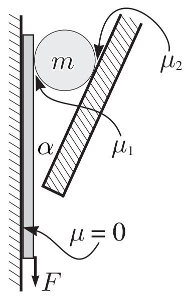
pr 57. A light stick rests with one end against a vertical wall and another on a horizontal floor. A bug wants to crawl down the stick, from top to bottom. How should the bug's acceleration depend on its distance from the top endpoint of the stick? The bug's mass is m, the length of the stick is l, the angle between the floor and the stick is α and the stick's mass is negligible; both the floor and the wall are slippery ($\mu = 0$). How long will it take the bug to reach the bottom of the stick having started at the top (from rest)?



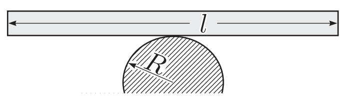
pr 58. A wedge with the angle α at the tip is lying on the horizontal floor. There is a hole with smooth walls in the ceiling. A rod has been inserted snugly into that hole, and it can move up and down without friction, while its axis is fixed to be vertical. The rod is supported against the wedge; the only point with friction is the contact point of the wedge and the rod: the friction coefficient there is μ. For which values of μ is it possible to push the wedge through, behind the rod, by only applying a sufficiently large horizontal force?



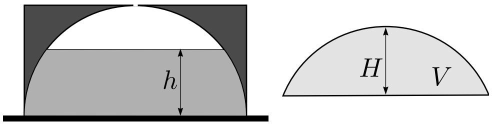
pr 59. Sometimes a contraption is used to hang pictures etc. on the wall, whose model will be presented below. Against a fixed vertical surface is an immovable tilted plane, where the angle between the surface and the plane is α. There is a gap between the surface and the plane, where a thin plate could be fit. The plate is positioned tightly against the vertical surface; the coefficient of friction between them can be considered equal to zero. In the space between the plate and the plane a cylinder of mass m can move freely, its axis being horizontal and parallel to all considered surfaces. The cylinder rests on the plate and the plane and the coefficients of friction on those two surfaces are, respectively, μ_1 and μ_2. For which values of the friction coefficients the plate will assuredly not fall down regardless of its weight?



pr 60. On top of a cylinder with a horisontal axis a plank is placed, whose length is l and thickness is h. For which radius R of the cylinder the horizontal position of the plank is stable?

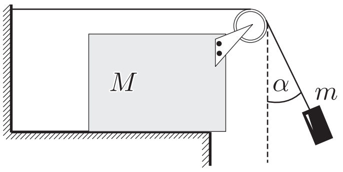


pr 61. A vessel in the shape of a cylinder, whose height equals its radius R and whose cavity is half-spherical, is filled to the brim with water, turned upside down and positioned on a horizontal surface. The radius of the half-spherical cavity is also R and there is a little hole in the vessel's bottom. From below the edges of the freely lying vessel some water leaks out. How high will the remaining layer of water be, if the mass of the vessel is m and the water density is ρ? If necessary, use the formula for the volume of a slice of a sphere (see Fig.): $V = \pi H^2(R - H/3)$.

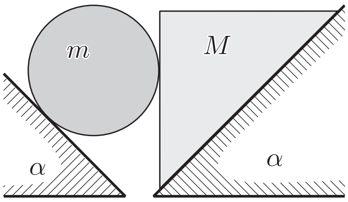


pr 62. A vertical cylindrical vessel with radius R is rotating around its axis with the angular velocity ω. By how much does the water surface height at the axis differ from the height next to the vessel's edges?

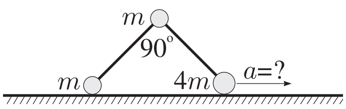
pr 63. A block with mass M is on a slippery horizontal surface. A thread extends over one of its corners. The thread is attached to the wall at its one end and to a little block of mass m, which is inclined by an angle α with respect to the vertical, at the other. Initially the thread is stretched and the blocks are held in place. Then the blocks are released. For which ratio of masses will the angle α remain unchanged throughout the subsequent motion?



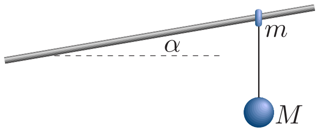
pr 64. Two slippery ($\mu = 0$) wedge-shaped inclined surfaces with equal tilt angles α are positioned such that their sides are parallel, the inclines are facing each other and there is a little gap in between (see fig.). On top of the surfaces are positioned a cylinder and a wedge-shaped block, whereas they are resting one against the other and one of the block's sides is horizontal. The masses are, respectively, m and M. What accelerations will the cylinder and the block move with? Find the reaction force between them.



pr 65. Three little cylinders are connected with weightless rods, where there is a hinge near the middle cylinder, so that the angle between the rods can change freely. Initially this angle is a right angle. Two of the cylinders have mass m, another one at the side has the mass $4m$. Find the acceleration of the heavier cylinder immediately after the motion begins. Ignore friction.

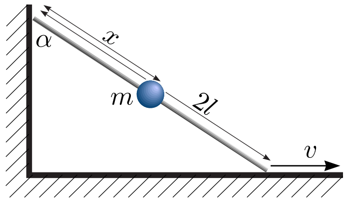


pr 66. A slippery rod is positioned at an angle α with respect to the horizon. A little ring of mass m can slide along the rod, to which a long thread is attached. A small sphere of size M is attached to the thread. Initially the ring is held motionless, and the thread hangs vertically. Then the ring is released. What is the acceleration of the sphere immediately after that?

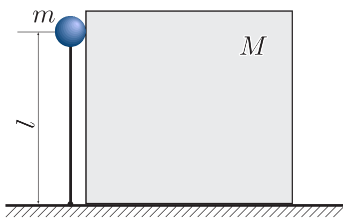


pr 67. A block begins sliding at the uppermost point of a spherical surface. Find the height at which it will lose contact with the surface. The sphere is held in place and its radius is R; there is no friction.

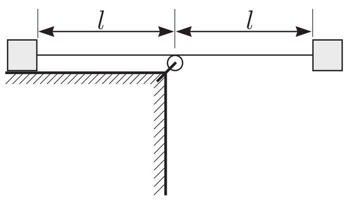
pr 68. The length of a weightless rod is $2l$. A small sphere of mass m is fixed at a distance $x = l$ from its upper end. The rod rests with its one end against the wall and the other against the floor. The end that rests on the floor is being moved with a constant velocity v away from the wall. *a)* Find the force with which the sphere affects the rod at the moment, when the angle between the wall and the rod is $\alpha = 45°$; *(b)* what is the answer if $x \neq l$?



pr 69. A light rod with length l is connected to the horizontal surface with a hinge; a small sphere of mass m is connected to the end of the rod. Initially the rod is vertical and the sphere rests against the block of mass M. The system is left to freely move and after a certain time the block loses contact with the surface of the block — at the moment when the rod forms an angle $\alpha = \pi/6$ with the horizontal. Find the ratio of masses M/m and the velocity u of the block at the moment of separation.



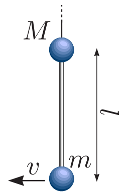
pr 70. At a distance l from the edge of the table lies a block that is connected with a thread to another exact same block. The length of the thread is $2l$ and it is extended around the pulley sitting at the edge of the table. The other block is held above the table such that the string is under tension. Then the second block is released. What happens first: does the first block reach the pulley or does the second one hit the table?



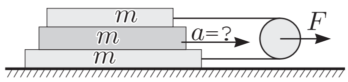
pr 71. A cylindrical ice hockey puck with a uniform thickness and density is given an angular velocity ω and a translational velocity u. What trajectory will the puck follow if the

ice is equally slippery everywhere? In which case will it slide farther: when $\omega = 0$ or when $\omega \neq 0$, assuming that in both cases u is the same?

pr 72. A little sphere of mass M hangs at the end of a very long thread; to that sphere is, with a weightless rod, attached another little sphere of mass m. The length of the rod is l. Initially the system is in equilibrium. What horizontal velocity needs to be given to the bottom sphere for it to ascend the same height with the upper sphere? The sizes of the spheres are negligible compared to the length of the rod.

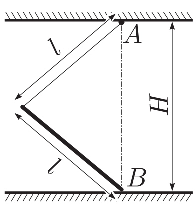


pr 73. A block of mass m lies on a slippery horizontal surface. On top of it lies another block of mass m, and on top of that — another block of mass m. A thread that connects the first and the third block has been extended around a weightless pulley. The threads are horizontal and the pulley is being pulled by a force F. What is the acceleration of the second block? The coefficient of friction between the blocks is μ.



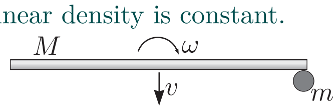
pr 74. A boy with mass m wants to push another boy standing on the ice, whose mass M is bigger that his own. To that end, he speeds up, runs toward the other boy and pushed him for as long as they can stand up. What is the maximal distance by which it is possible to push in this fashion? The maximal velocity of a run is v, the coefficient of friction between both boys and the ice is μ.

pr 75. A uniform rod with length l is attached with a weightless thread (whose length is also l) to the ceiling at point A. The bottom end of the rod rests on the slippery floor at point B, which is exactly below point A. The length of AB is H, $l < H < 2l$. The rod begins to slide from rest; find the maximal speed of its centre during subsequent motion. Also, find the acceleration of the rod's centre and tension in the thread at that moment when the rod's speed is maximal if the rod's mass is m.



pr 76. A stick with uniform density rests with one end against the ground and with the other against the wall. Initially it was vertical and began sliding from rest such that all of the subsequent motion takes place in a plane that is perpendicular to the intersection line of the floor and the wall. What was the angle between the stick and the wall at the moment when the stick lost contact with the wall? Ignore friction.

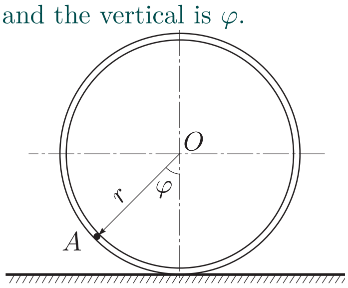
pr 77. A log with mass M is sliding along the ice while rotating. The velocity of the log's centre of mass is v, its angular velocity is ω. At the moment when the log is perpendicular to the velocity of its centre of mass, the log hits a stationery puck with mass m. For which ratio of the masses M/m is the situation, where the log stays in place while the puck slides away, possible? The collisions are perfectly elastic. The log is straight and its linear density is constant.



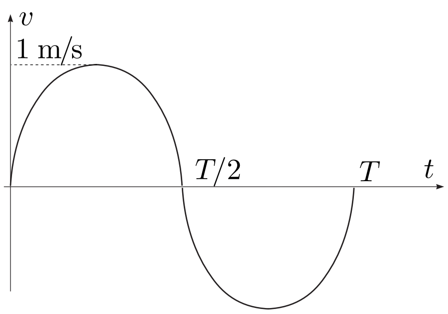
pr 78. A ball falls down from height h, initially the ball's horizontal velocity was v_0 and it wasn't rotating. a) Find the velocity and the angular velocity of the ball after the following collision against the floor: the ball's deformation against the floor was absolutely elastic, yet there was friction at the contact surface such that the part of the ball that was in contact with the floor stopped. b) Answer the same question with the assumption that the velocities of the surfaces in contact never homogenized and that throughout the collision there was friction with coefficient μ.

pr 79. A ball is rolling down an inclined plane. Find the ball's acceleration. The plane is inclined at an angle α, the coefficient of friction between the ball and the plane is μ.

pr 80. A hoop of mass M and radius r stands on a slippery horizontal surface. There is a thin slippery tunnel inside the hoop, along which a tiny block of mass m can slide. Initially all the bodies are at rest and the block is at the hoop's uppermost point. Find the velocity and the acceleration of the hoop's central point at the moment when the angle between the imaginary line connecting the hoop's central point and the block's position and the vertical is φ.

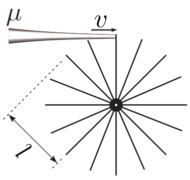


pr 81. A block with mass $m = 10\,\mathrm{g}$ is put on a board that has been made such that, when sliding to the left, the coefficient of friction $\mu_1 = 0{,}3$, while when sliding to the right it is $\mu_2 = 0{,}5$. The board is repeatedly moved left-right according to the graph $v(t)$ (see fig.). The graph is periodic with period $T = 0{,}01\,\mathrm{s}$; the velocity v of the board is considered

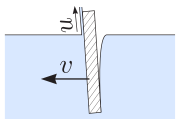


positive when directed to the right. Using the graph, find the average velocity that the block will move with.

pr 82. A water turbine consists of a large number of paddles that could be considered as light flat boards with length l, that are at one end attached to a rotating axis. The paddles' free ends are positioned on the surface of an imaginary cylinder that is coaxial with the turbine's axis. A stream of water with velocity v and flow rate μ (kg/s) is directed on the turbine such that it only hits the edges of the paddles. Find the maximum possible usable power that could be extracted with such a turbine.

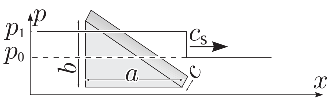


pr 83. A flat board is inclined at an angle α to the vertical. One of its ends is in the water, the other one is outside the water. The board is moving with velocity v with respect to its normal. What is the velocity of the water stream directed up the board?

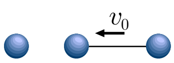


pr 84. A motor-driven wagon is used to transport a load horizontally by a distance L. The load is attached to the side of the wagon by a cable of length l. Half of the time the wagon is uniformly accelerated, the other half — uniformly decelerated. Find the values of the acceleration a such that, upon reaching the destination, the load will be hanging down motionlessly. You can assume that $a \ll g$.

pr 85. A shockwave could be considered as a discontinuous jump of the air pressure from value p_0 to p_1, propagating with speed c_s. Find the speed which will be obtained, when influenced by the shockwave, *(a)* a wedge-shaped block: a prism whose height is c, whose base is a right triangle with legs a and b and which is made out of material with density ρ; *b)* an body of an arbitrary shape with volume V and density ρ.



pr 86. A dumbbell consisting of two elastic spheres connected with a thin steel rod is moving parallel to its axis with a velocity v toward another exact same spheres. Find the velocity of the dumbbell after a central collision. Is the kinetic energy of the system conserved?



appendix 1: Momentum conservation law.

Let us consider a system of N point masses ("bodies"), and let us represent the force acting on the i-th point as a sum, $\vec F_i = \sum_j \vec F_{ij} + \vec{\mathcal F}_i$, where $\vec F_{ij}$ is the force exerted on the i-th body due to the j-th body [26] and $\vec{\mathcal F}_i$ is an external force, i.e. the net force exerted by such bodies which are not part of the given system. Then, the Newton's 2$^{\rm nd}$ law for the i-th body is written as

$$m_i\frac{d}{dt}\vec v_i = \sum_j \vec F_{ij} + \vec{\mathcal F}_i.$$

If we sum this equality over the index i, we obtain at the left-hand-side

$$\sum_i m_i\frac{d}{dt}\vec v_i = \frac{d}{dt}\sum_i m_i\vec v_i = \frac{d}{dt}\vec P,$$

where $\vec P = \sum_i m_i\vec v_i$ is called the momentum of the system of bodies. Here we have kept in mind that $\vec F_{ii} = 0$, and made use of the additivity of differentiation: derivative of a sum is the sum of derivatives. The internal forces at the right-hand-side cancel out:

$$\sum_i\left(\sum_j \vec F_{ij}\right) = \sum_{i,j}\vec F_{ij} = \sum_{i>j}(\vec F_{ij} + \vec F_{ji}) = 0.$$

Here we first represented the sum as being taken over all the index ij pairs, and then grouped the terms with symmetric indices (ij and ji) together ($\sum_{i>j}$ means that the sum is taken over all such ij-pairs where $i > j$); finally, we use the Newton's 3$^{\rm rd}$ law to conclude that $\vec F_{ij} + \vec F_{ji} = 0$. Upon introducing the net external force as $\vec{\mathcal F} = \sum_i \vec{\mathcal F}_i$, we obtain

$$\frac{\mathrm d}{\mathrm dt}\vec P = \vec{\mathcal F}.$$

The last equality is essentially a generalization of the Newton's 2$^{\rm nd}$ law to a system of bodies. In particular, if there are no external forces, $\vec{\mathcal F} = 0$, and the momentum $\vec P$ is conserved.

Notice that if there are no external forces, the equations of motion (equations which define how the system will evolve), i.e. equations expressing the Newton's 2$^{\rm nd}$ law, obey *translational symmetry*: we can translationally displace the reference frame by a vector $\vec a$ without any change to the equations of motion. Indeed, the new vectors pointing to the positions of the bodies (*the radius vectors*) are expressed in terms of the old ones as $\vec r_i' = \vec r_i - \vec a$. The internal forces $\vec F_{ij}$ depend only on the relative placement of the bodies, i.e. on the vectors $\vec r_i' - \vec r_j' = (\vec r_i - \vec a) - (\vec r_j - \vec a) = \vec r_i - \vec r_j$ which are expressed in terms of the new coordinates exactly in the same way as in terms of the old coordinates.

The discipline of *analytical mechanics* shows that each symmetry of the equations of motion containing a parameter (which can take arbitrarily small values) gives rise to a conservation law[27]. Here we have actually three independent parameters, the components of the displacement vector a_x, a_y, and a_z; because of that we have three conserved quantities — the respective components of the momentum vector $\vec P$.

appendix 2: Angular momentum conservation law.

Similarly to the momentum conservation law, we consider a system of N bodies, with the same designations. Then, we

[26] Due to the fact 5, $\vec F_{ii} = -\vec F_{ii}$, hence $\vec F_{ii} = 0$: a body cannot exert a force on itself

[27] This is the content of the Noether's theorem (E. Noether 1918).

can take the time derivative of the expression of the angular momentum of the i-th body:

$$\frac{\mathrm{d}}{\mathrm{d}t}m_i\vec{r}_i\times\vec{v}_i = m_i\left(\frac{\mathrm{d}\vec{r}_i}{\mathrm{d}t}\times\vec{v}_i+\vec{r}_i\times\frac{\mathrm{d}\vec{v}_i}{\mathrm{d}t}\right).$$

Here we have applied the product differentiation rule $(ab)' = a'b+ab'$ which is still valid in vector algebra: $(\vec{a}\cdot\vec{b})' = \vec{a}'\cdot\vec{b}+\vec{a}\cdot\vec{b}'$ and $(\vec{a}\times\vec{b})' = \vec{a}'\times\vec{b}+\vec{a}\times\vec{b}'$ (NB! we need to keep the order of the vectors since the cross product is *anticommutative*, $\vec{a}\times\vec{b} = -\vec{b}\times\vec{a}$) Notice that $\frac{\mathrm{d}}{\mathrm{d}t}\vec{r}_i = \vec{v}_i$ and $\vec{v}_i\times\vec{v}_i = 0$, hence the first term in right-hand-side drops out. Further, let us sum our first equality over the index i, and substitute the remaining terms at the right-hand-side using the Newton's 2[nd] law,

$$m_i\frac{\mathrm{d}\vec{v}_i}{\mathrm{d}t} = \sum_j\vec{F}_{ij}+\vec{\mathcal{F}}_i \Rightarrow m_i\vec{r}_i\times\frac{\mathrm{d}\vec{v}_i}{\mathrm{d}t} = \vec{r}_i\times\left(\sum_j\vec{F}_{ij}+\vec{\mathcal{F}}_i\right)$$

to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L} = \sum_{i,j}\vec{r}_i\times\vec{F}_{ij}+\sum_i\vec{r}_i\times\vec{\mathcal{F}}_i.$$

Now, let us notice that due to the Newton's 3[rd] law $\vec{F}_{ij} = -\vec{F}_{ji}$; all the macroscopic non-relativistic forces between two point masses take place either at the contact point when these two point masses touch each other (elasticity force, friction force), or is parallel to the line connecting these points (electrostatic force, gravitational force) [28] In either case, we can write $\vec{r}_j = \vec{r}_i + k\vec{F}_{ji}$; if we multiply this equality by $\vec{F}_{ji}$, we obtain $\vec{r}_i\times\vec{F}_{ij} = \vec{r}_j\times\vec{F}_{ij} = -\vec{r}_j\times\vec{F}_{ji}$, hence the internal torques cancel pair-wise out from the sum; what remains is the net external torque $\vec{\mathcal{T}} = \sum_i\vec{r}_i\times\vec{\mathcal{F}}_i$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{\mathcal{L}} = \vec{\mathcal{T}}.$$

This can be considered as the generalization of the Newton's II law to the rotational motion of a system of bodies; if the torque of external forces is zero ($\vec{\mathcal{T}} = 0$) then we end up with the conservation of angular momentum, $\vec{\mathcal{L}} =$const.

Note that within the discipline of analytical mechanics, the angular momentum conservation can be derived from rotational symmetry of the full energy of a mechanical system (while we rotate the reference frame by an angle α around an axis which goes through the origin, the expression for the full energy needs to remain unchanged).

appendix 3: Energy conservation law.

In order to derive the energy conservation law, let us consider the time derivative of the kinetic energy of a system of bodies, defined as $K = \frac{1}{2}\sum_j m_i\vec{v}_i^2 = \frac{1}{2}\sum_j m_i\vec{v}_i\cdot\vec{v}_i$ (here we wrote v_i^2 as a dot product of $\vec{v}_i$ with itself). For differentiation of dot products, the ordinary rule for the derivative of a product applies: $(\vec{a}\cdot\vec{b})' = \vec{a}'\cdot\vec{b}+\vec{a}\cdot\vec{b}'$; since the dot product is commutative (i.e. $\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$), we obtain $(\vec{a}\cdot\vec{a})' = 2\vec{a}\cdot\vec{a}'$. So,

$$\frac{\mathrm{d}}{\mathrm{d}t}K = \sum_j m_i\frac{\mathrm{d}\vec{v}_i}{\mathrm{d}t}\cdot\vec{v}_i = \sum_j\vec{F}_i\cdot\vec{v}_i,$$

which can be rewritten for the differential of the kinetic energy as $\mathrm{d}K = \sum_j\vec{F}_i\cdot\vec{v}_i\mathrm{d}t$, where the force acting on the i-th body can be expressed as the sum of internal and external forces, $\vec{F}_i = \sum_j\vec{F}_{ij}+\vec{\mathcal{F}}_i$ and $\vec{v}_i\mathrm{d}t = \mathrm{d}\vec{r}_i$. Then

$$\mathrm{d}K = \sum_{i,j}\vec{F}_{ij}\cdot\mathrm{d}\vec{r}_i+\sum_i\vec{\mathcal{F}}_i\cdot\mathrm{d}\vec{r}_i.$$

Here, $\vec{F}_{ij}\cdot\mathrm{d}\vec{r}_i$ is called *the work* done by the force $\vec{F}_{ij}$. If the forces $\vec{F}_{ij}$

(a) depend only on the coordinates $\vec{r}_i$ and *do not depend* on the velocities $\vec{v}_i$ and on time t;

(b) there exists such function $\Pi\equiv\Pi(\vec{r}_1,\vec{r}_2,\ldots\vec{r}_N)$ (henceforth referred to as the *potential energy*) that by any infinitesimal (i.e. infinitely small) displacements $\mathrm{d}\vec{r}_i$ of the bodies, the total work done by all the internal forces $\vec{F}_{ij}$ and external forces $\vec{\mathcal{F}}_i$ equals to the opposite of the *total differential*[29] of the potential energy, i.e.

$$-\mathrm{d}\Pi(\vec{r}_1,\vec{r}_2,\ldots\vec{r}_N) = \sum_i\left(\vec{\mathcal{F}}_i+\sum_j\vec{F}_{ij}\right)\cdot\mathrm{d}\vec{r}_i;$$

then for any displacement of the system, $\mathrm{d}K = -\mathrm{d}\Pi$ and hence, $\mathrm{d}(K+\Pi) = 0$, i.e. the total energy $E = K+\Pi = $ const; such internal forces $\vec{F}_{ij}$ and external forces $\vec{\mathcal{F}}_i$ which satisfy the above listed conditions are referred to as *conservative forces*.

Note that the condition (b) above is equivalent to saying that the work done by the forces depends only on the initial and final states of the system (i.e. on the positions of the point masses), and not along which trajectories the point masses moved. Mathematically, this condition can be also rewritten using partial derivatives[30],

$$-\frac{\partial\Pi(\vec{r}_1,\vec{r}_2,\ldots\vec{r}_N)}{\partial x_i} = \sum_j F_{ijx}+\mathcal{F}_{ix};\qquad(12)$$

right-hand-side of this condition is the x-component of the total force acting on the i-th point mass (x_i stands for the x-coordinate of the i-th particle); similar conditions need to be valid also for the y- and z-components.

An important case when the forces are conservative is the case of central force fields: the internal force between two point masses is parallel to the line connecting these point masses and depends by modulus only on the distance,

$$\vec{F}_{ij} = (\vec{r}_i-\vec{r}_j)f_{ij}(|\vec{r}_i-\vec{r}_j|),$$

and the external force acting upon i-th point mass has a similar property with respect to a reference point at the $\vec{r}_{i0}$,

$$\vec{\mathcal{F}}_i = (\vec{r}_i-\vec{r}_{i0})f_i(|\vec{r}_i-\vec{r}_{i0}|);$$

note that due to Newton's 3[rd] law, $f_{ij}(r) = f_{ji}(r)$. Then, with positive values of f_{ij} and f_i corresponding to repulsion and negative values to attraction, the potential energy is given by

$$\Pi = \sum_{i<j\leq N}g_{ij}(|\vec{r}_i-\vec{r}_j|)+\sum_{i\leq N}g_i(\vec{r}_i-\vec{r}_{i0}),\text{ where}$$

$$g_{ij}(r) = -\int^r f_{ij}(r')r'\mathrm{d}r',\quad g_i(r) = -\int^r f_i(r')r'\mathrm{d}r'\qquad(13)$$

(the lower bound of these integrals can be arbitrary). It is not too difficult to check that with such potential energy, equality (12) is satisfied, indeed, for all values of i.

[28] At non-relativistic speeds, the Lorentz force acting between two moving charges is much smaller than the electrostatic force and is therefore a relativistic effect; still, the Lorentz force can lead to situations with seeming violation of the conservation of angular momentum, e.g. in the case of a moving charge at the centre of a ring current.

[29] The total differential is defined as $\mathrm{d}\Pi(\vec{r}_1,\vec{r}_2,\ldots\vec{r}_N)\equiv\Pi(\vec{r}_1+\mathrm{d}\vec{r}_1,\vec{r}_2+\mathrm{d}\vec{r}_2,\ldots\vec{r}_N+\mathrm{d}\vec{r}_N)-\Pi(\vec{r}_1,\vec{r}_2,\ldots\vec{r}_N)$

[30] Partial derivative of a function: while taking derivative with respect to a given variable, all the other variables are assumed to be constant; for instance, $\frac{\partial f(x,y)}{\partial x}$ denotes the derivative of $f(x,y)$ with respect to x while y is considered to be a constant.

If there are also some non-conservative external forces present then we can separate conservative and non-conservative forces, $\vec{\mathcal{F}}_i = \vec{\mathcal{F}}_i' + \vec{\mathcal{F}}_i''$, leading to

$$\mathrm{d}(K + \Pi) = \sum_i \vec{\mathcal{F}}_i'' \mathrm{d}\vec{r}_i,$$

where $\vec{\mathcal{F}}_i''$ denotes the sum of all the non-conservative forces acting on the i-th point mass.

appendix 4: Centrifugal force and Coriolis force.

Consider a system of reference, which rotates around the origin O with an angular velocity $\vec{\Omega}$ (the vector defines the rotation axis according to the corkscrew rule). Consider a point P, which is motionless in the rotating system, and let us denote $\vec{r} = \overrightarrow{OP}$. In the lab system of reference, the point P moves with velocity $v = r\Omega$, and when studying the direction of the velocity $\vec{v} = \frac{\mathrm{d}\vec{r}}{\mathrm{d}t}$, one can see that $\vec{v} = \vec{\Omega} \times \vec{r}$. Now, if the point P moves in the rotating frame of reference with velocity $\vec{u} = \frac{\mathrm{d}\vec{r}}{\mathrm{d}\tau}$ (let us use τ to measure the time in the rotating system), then this additional velocity needs to be added to what would have been for a motionless point:

$$\frac{\mathrm{d}\vec{r}}{\mathrm{d}t} = \frac{\mathrm{d}\vec{r}}{\mathrm{d}\tau} + \vec{\Omega} \times \vec{r}.$$

So, we can conclude that the time-derivatives of vectors in rotating and lab frames of reference are related via equality

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}\tau} + \vec{\Omega} \times .$$

This is written in the form of an operator, which means that we can write any vector (e.g. $\vec{r}$ or $\vec{v}$) rightwards of all the three terms. In particular, we can apply this formula to the right- and left-hand-sides of the equality $\vec{v} = \vec{u} + \vec{\Omega} \times \vec{r}$:

$$\begin{aligned}\frac{\mathrm{d}\vec{v}}{\mathrm{d}t} =& \left(\frac{\mathrm{d}}{\mathrm{d}\tau} + \vec{\Omega}\times\right)\left(\vec{u} + \vec{\Omega} \times \vec{r}\right) \\ =& \frac{\mathrm{d}\vec{u}}{\mathrm{d}\tau} + \vec{\Omega} \times \vec{u} + \frac{\mathrm{d}(\Omega \times \vec{r})}{\mathrm{d}\tau} + \vec{\Omega} \times \left(\vec{\Omega} \times \vec{r}\right).\end{aligned}$$

Here we need to bear in mind that when taking derivatives of vectors and products of vectors, all the well-known rules can be applied; in particular, $\frac{\mathrm{d}}{\mathrm{d}t}(\vec{a} \times \vec{b}) = \frac{\mathrm{d}\vec{a}}{\mathrm{d}t} \times \vec{b} + \vec{a} \times \frac{\mathrm{d}\vec{b}}{\mathrm{d}t}$ and $\frac{\mathrm{d}}{\mathrm{d}t}(\vec{a} \cdot \vec{b}) = \frac{\mathrm{d}\vec{a}}{\mathrm{d}t} \cdot \vec{b} + \vec{a} \cdot \frac{\mathrm{d}\vec{b}}{\mathrm{d}t}$. We also need the rule for the double cross product, $\vec{a} \times (\vec{b} \times \vec{c}) = \vec{b}(\vec{a} \cdot \vec{c}) - \vec{c}(\vec{a} \cdot \vec{b})$; you can memorize this equality by keeping in mind that the double product is a linear combination of the vectors from the inner braces, and that the sign '+' comes with the vector from the middle position. And so, bearing in mind that $\frac{\mathrm{d}\vec{\Omega}}{\mathrm{d}\tau} = 0$ and $\frac{\mathrm{d}\vec{r}}{\mathrm{d}\tau} = \vec{u}$, and assuming that $\vec{r} \perp \vec{\Omega} \Rightarrow \vec{r} \cdot \vec{\Omega} = 0$, we obtain

$$\frac{\mathrm{d}\vec{v}}{\mathrm{d}t} = \frac{\mathrm{d}\vec{u}}{\mathrm{d}\tau} + 2\vec{\Omega} \times \vec{u} - \Omega^2\vec{r}.$$

Let us recall that $\frac{\mathrm{d}\vec{v}}{\mathrm{d}t}$ is the acceleration of the point P as seen in the lab frame of reference, and $\frac{\mathrm{d}\vec{u}}{\mathrm{d}\tau}$ is the same as seen in the rotating frame of reference. Now, if P is a point mass m, and there is an external force $\vec{F}$ acting on P, then $\vec{F} = m\frac{\mathrm{d}\vec{v}}{\mathrm{d}t}$ and hence,

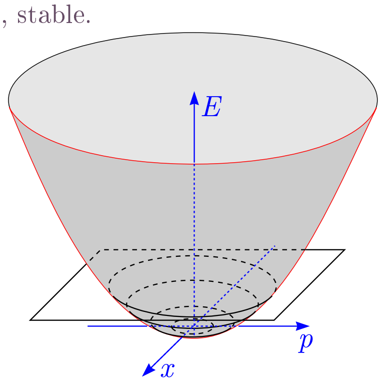
$$m\frac{\mathrm{d}\vec{u}}{\mathrm{d}\tau} = \vec{F} - 2\vec{\Omega} \times \vec{u}m + \Omega^2\vec{r}m,$$

i.e. in the rotating system of reference, the body behaves as if there were additional forces: the Coriolis force $-2\vec{\Omega} \times \vec{u}m$, and the centrifugal force $\Omega^2\vec{r}m$.

appendix 5: Stablity and conservation laws.

It is well-known that a system is stable at the minimum of its potential energy. But why? Why is a minimum different from a maximum? In the case of the Fermat' principle, there is a clear difference: there is no longest optical path between two points — the ray could just go "zig-zag" —, but there is definitely one which is the shortest!

The reason is simple — at an equilibrium state, the kinetic energy has always minimum (as long as masses are positive). What we actually do need for a stability is a **conditional extremum of one conserved quantity (such as the net energy), under the assumption that the other conserved quantities are kept constant** (unconditional extremum is OK, too). Consider the motion of a body along x-axis and let us describe it on the phase plane, with coordinates x and p (the momentum). The overall energy is $E = U(x) + p^2/2m$. Now, if we depict this energy as a surface in 3-dimensional space with coordinates x, p and E, the point describing the state of the system will move along the intersection line of that surface with a horizontal plane $E = \text{Const}$. At the minimum of $U(x)$, with $p = 0$, this intersection line would be just a single point, because this is the lowest point of that surface. The near-by trajectories will be obtained if we ascend the horizontal plane a little, $E = E_{\min} + \varepsilon$, so that it no longer just touches the surface, but cuts a tiny ellips from it. All the points of that trajectory (the ellips) are close to the equilibrium point, so the state is, indeed, stable.



It appears that a system can be stable also because of a conditional maximum of the net energy: while an unconditional extremum of the kinetic energy can only be a minimum, things are different for conditional extrema. Perhaps the simplest example is the rotation of a rigid body. Let us consider a rectangular brick with length a, width b, and thickness c ($a > b > c$). Let I_c be its moment of inertia for the axis passing its centre of mass and perpendicular to the (a, b)-plane; I_b and I_a are defined in a similar way. For a generic case, the moment of inertia I will depend on the orientation of the rotation axis, but it is quite clear that $I_c \geq I \geq I_a$ (it can be shown easily once you learn how to use tensor calculations). Now, let us throw the brick rotating into air and study the motion in a frame which moves together with the centre of mass of the brick (in that frame, we can ignore gravity). There are two conserved quantities: angular momentum L, and rotation energy $K = L^2/2I$. We see that for a fixed L, the system has minimal energy for $I = I_c$ (axis is parallel to the shortest edge of the brick), and maximal energy for $I = I_a$ (axis is parallel to the longest edge of the brick). You can easily check experimentally that both ways of rotation are, indeed, stable! Meanwhile, if the axis is parallel to the third edge, the rotation is unstable. This phenomenon is demonstrated in a video made by NASA on the International Space Station, https://mix.msfc.nasa.gov/abstracts.php?p=3873.

Well, actually the rotation with the minimal energy is still a little bit more stable than that of with the maximal energy; the reason is in dissipation. If we try to represent the motion of the system in the phase space (as described above), a bowl-shaped energy surface (as shown in the figure above) would be substituted by a hill-shaped one; at the equilibrium, the phase trajectory is contracted into a point — the point where the top the "hill" is touching a horizontal plane $E = E_{\max}$. Due to dissipation, the energy will decrease, $E = E_{\max} - \varepsilon$, and the phase trajectory would be a slowly winding-out spiral. So, while you are probably used to know that dissipation draws a system towards a stable state, here it is vice versa, it draws the system away from the stable state! This is what is known as *dissipative instability*.

appendix 6: Lagrangian formalism.

In our approach to mechanics, we postulated the Newton's laws; based on that, we derived energy conservation law which is valid for conservative forces, and using energy conservation law, we arrived at the method of generalized coordinates.

In *analytical mechanics*, the order is opposite. First, we postulate that any mechanical system has a certain potential energy, and a certain kinetic energy, both of which are additive; we also establish a formula for kinetic energy of point masses, and for potential energies of point mass interactions depending on the type of interaction (this is done similarly to how we established rules for calculating forces for different interaction types in chapter 2).

Second, let us consider a mechanical system obeying n degrees of freedom, i.e. in order to specify uniquely the state of the system, we need n parameters. We postulate that if this system evolves from one state described by a set of coordinates q_i, $i \in [1, n]$ at time $t = \tau$ [this state corresponds to a point in the $n+1$-dimensional *configuration space* with coordinates $(q_1, q_2, \ldots, q_n, t)$] to another state q_i' at $t = \tau'$ then the evolution of the system in time takes place along such a path $q_i(t)$ (a curved line connecting the initial and final states in the configuration space) that makes the value of a certain integral $\mathcal{S}$ as small as possible. This integral, referred to as the *action*, is defined via the full potential and kinetic energies of the system, denoted as V and T, respectively; V depends on the coordinates, $V = V(q_i), i \in [1, n]$, and T also on the changing rate of coordinates $\dot{q}_i$:

$$\mathcal{S} = \int_{\tau}^{\tau'} \mathcal{L}[q_i(t), \dot{q}_i(t), t]\mathrm{d}t, \tag{14}$$

where

$$\mathcal{L}(q_i, \dot{q}_i, t) = T(q_i, \dot{q}_i, t) - V(q_i, t) \tag{15}$$

is called the *Lagrangian* of the system, and the postulate itself as the *principle of least action*[31]

Using the methods of *variational analysis*, one can show that the integral[32] S has an extremum if

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} = \frac{\partial \mathcal{L}}{\partial q_i}; \tag{16}$$

here $\frac{\partial \mathcal{L}}{\partial \dot{q}_i}$ means that we take derivative of the Lagrangian $L(q_i(t), \dot{q}_i, t)$ with respect to only one of its $2n+1$ variables, $\dot{q}_i$, while considering all the other variables to be constant. $\frac{\partial \mathcal{L}}{\partial q_i}$ is defined analogously. Meanwhile, $\frac{\mathrm{d}}{\mathrm{d}t}$ denotes taking a full time derivative, i.e. we take into account that $\mathcal{L}$ depends on time both explicitly through its last argument t, as well as implicitly since the quantities q_i and $\dot{q}_i$ are also functions of time. Please note that Eq. (16) is valid for every i, so that we have a system of n equations. From the principle of least action to Equation (16), there is only one mathematical step, so we can say that analytical mechanics basically postulates the Equation (16).

Which way is better: the historical way of postulating the Newton's laws, or postulating Eq. 16? Both approaches have strong and weak points. While the classical approach is built up step-by-step, from immediate experimental findings, the approach of analytical mechanics takes the least action principle "out of thin air". Meanwhile, Eq. (16) gives us a very universal and powerful tool for theoretical analysis (the usage of which is not limited to mechanics): as soon as we have an expression for the Lagrangian, we can write down the evolution equation describing how the system will evolve. However, it should be kept in mind that only in the case of classical mechanics, $\mathcal{L} = T - V$, and one should keep vigilance even in the case of classical mechanics (see below).

As a matter of fact, the least action principle can be introduced more naturally (not "our of thin air") using quantum mechanics. Indeed, if we consider a point mass as a quantum-mechanical probability wave then using quasi-classical approximation, we can express the phase of the wave as

$$\varphi = \int (\vec{k} \cdot \mathrm{d}\vec{r} - \omega \mathrm{d}t) = \hbar^{-1} \int (\vec{p} \cdot \mathrm{d}\vec{r} - E\mathrm{d}t); \tag{17}$$

here $\vec{p}$ is the momentum and E — the energy of the particle. If we keep in mind that $\mathrm{d}\vec{r} = \vec{v}\mathrm{d}t$ and $\vec{v} \cdot \vec{p} = 2T$ then we can further write $\varphi = \hbar \int [2T - (T+V)]\mathrm{d}t = \hbar \mathcal{S}$. So, the action $\mathcal{S}$ gives us directly the phase of the wave. The waves add up constructively if they arrive at the same phase, and many waves coming along different paths arrive almost at the same phase if these paths are close to the path of least action. It should be noted that exactly the same phenomenon happens in the case of light propagation, and can be summarized as the Fermat' principle. [33] We can say that according to the Huygens principle for wave propagation, the amplitude of the probability wave can be found as the sum over the contributions from all the possible ray tracing paths; however, majority of these contributions cancel out due to opposite phases, and only the contribution of the "optimal path" (and its immediate neighbourhood) is left intact; "optimal" means corresponding to an extremum (which appears to be a minimum) of the action. So,

[31] Also as the *principle of a stationary action*.

[32] More precisely, *functional* — a scalar quantity which depends on which function(s) $q_i(t)$ we have.

[33] We can see rainbow exactly due to the same reason: the rainbow arc can be seen because the angle by which a light beam is deflected after a reflection inside a spherical water droplet has an extremum (as a function of the aim parameter); hence, near the deflection angle extremum, a wide range of aim parameters corresponds to a narrow range of deflection angles.

we can say that a point mass moves along the trajectory of least action.

Now, let us check if the least action principle is in agreement with the Newton's laws. To this end, let us consider a system of point masses m_i, $i \in [1, n]$, and use the ordinary Euclidean coordinates: let $\vec{r}_i$ point to the position of i-th point mass. Then we define (postulate) the Lagrangian as

$$\mathcal{L} = \frac{1}{2}\sum_{i,j} m_i \vec{v}_i^2 - \Pi(\vec{r}_1, \vec{r}_2, \ldots);$$

here we have denoted $\vec{v}_i = \frac{\mathrm{d}\vec{r}}{\mathrm{d}t}$ and assumed that all the interaction forces are conservative: $\Pi(\vec{r}_1, \vec{r}_2, \ldots)$ denotes the total potential energy as a function of coordinates of all the particles. Then, if we apply Eq. 16 to this Lagrangian, and keep in mind that $\frac{\partial \mathcal{L}}{\partial v_{ix}} = m_i v_{ix}$ (where index x denotes a projection of a vector to the x-axis), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} m_i v_{ix} = -\frac{\partial \Pi(\vec{r}_1, \vec{r}_2, \ldots \vec{r}_N)}{\partial x_i}.$$

According to Eq. (12), what we have at the right-hand-side is the x-component of the force acting on the i-th particle (obviously a similar expression is obtain for y- and z-components). So, we conclude that if written for Euclidian coordinates, equation (16) is equivalent to the Newton's laws. Meanwhile, equation (16) being satisfied is equivalent to the least action principle being valid. Now, let us notice that the least action principle is formulated independently of the coordinate system: if a certain trajectory $\vec{r}_i = \vec{r}_i(t)$ has minimal action in Euclidian coordinates $\vec{r}_i$ then it remains being minimal even if expressed in terms of the generalized coordinates $q_i = q_i(t)$. Since the trajectory has an extremal action in terms of generalized coordinates q_i then (according to the results of the variational analysis), the Lagrangian equation (16) must be also valid when the generalized coordinates q_i are used. This completes our proof[34] that Newton's laws and Eq. (16) are equivalent.

Although we kind of completed the proof, we need to make a comment regarding the cases when the number of degrees of freedom is recuced due to various constraints. As an example, let us consider a rigid body made of N molecules; this set of molecules has $3N$ degrees of freedom. However, the relative distances between molecules are fixed by molecular forces, so that there are only six degrees of freedom left: three numbers fix the position of the centre of mass, and the orientation of the body is fixed by three angular coordinates. Previously we have proved the least action principle for a set of point masses (molecules), so we know that our system evolves in the $3N+1$-dimensional configuration space along such a curve σ connecting the starting point A with the destination point B which minimizes the action. In this configuration space, the Lagrangian needs to account for the inter-molecular interaction energies, as well. While the expression for the inter-molecular interaction energies may be fairly complicated, as long as we are interested only in the macroscopic dynamics, we just need to fix the distances. The distances can be fixed with a simplified Lagrangian: we say that the inter-molecular interaction energy is zero, if the distance between two molecules equals to what it should be, and becomes very large otherwise. Due to the inter-molecular distances being fixed, the state of this system of molecules can be fully described by six generalized coordinates; this means that all the trajectories in the $3N+1$-dimensional configuration space are constrained into a six-dimensional hypersurface $\mathcal{M}$ (points A and B also need to lie on that hypersurface). We know that the trajectory σ minimizes the action between A and B in the $3N+1$-dimensional configuration space; the hypersurface $\mathcal{M}$ is a part of that space, so it surely minimizes the action between A and B in the hypersurface $\mathcal{M}$, as well. Therefore, Eq. (16) must remain valid when we use six generalized coordinates to describe the state of a rigid body. Similar argumentation works not only for a rigid body, but for any constraints fixing relative positions of the parts of a system (and thereby reducing the number of degrees of freedom).

From the discussions of the previous paragraph we can derive also an important rule: if we write Lagrangian using generalized coordinates for a system with intrinsic constraints, the number of coordinates should be as small as possible. For instance, if we have a rigid body, we should use six coordinates and not seven, because the value of the seventh coordinate can be derived from the first six (with the seventh coordinate, we would need to add additional term to the Lagrangian fixing the value of the seventh coordinate).

So, we have now two alternative options: we can use the Lagrangian equation (16), and we can use the method 6 in which case we derive the equation of motion from the energy conservation law. These two approaches are fairly similar: in both cases we need to express the kinetic and potential energies in terms of generalized coordinates and time derivatives of these. However, there are also certain differences: in one case, we derive the equation of motion directly from the energy conservation law; in the other case we consider the difference of these two energies and apply a formula which we can either consider to be postulated, or derived from the Newton's laws in a fairly complicated way.

Which way is better? To begin with, it should be emphasized that while Eq. (16) can be always used, the method 6 based on energy conservation law can be applied only in those cases when there is a single degree of freedom, i.e. the state of the system can be described with only one generalized coordinate. Indeed, upon taking time derivative of the energy conservation law, we obtain one differential equation, but we need as many equations as there are unknown functions (degrees of freedom). However, for a majority of Olympiad problems, this condition is satisfied (keep in mind that each additional conserved quantity, e.g. momentum, reduces the effective number of degrees of freedom by one).

So, let us compare these two methods when we have one generalized coordinate q, and let us assume that the energies do not depend explicitly on time. In the case of Newtonian mechanics, kinetic energy is proportional to squared speed, so we may assume that $T = \frac{1}{2}\mathcal{M}(q)\dot{q}^2$. Then, the energy conservation law states that $\frac{1}{2}\mathcal{M}(q)\dot{q}^2 + V(q) = E$, hence $\frac{1}{2}\mathcal{M}'(q)\dot{q}^3 + \mathcal{M}(q)\dot{q}\ddot{q} + V'(q)\dot{q} = 0$ and

$$\mathcal{M}(q)\ddot{q} = -\frac{1}{2}\mathcal{M}'(q)\dot{q}^2 - V'(q).$$

Meanwhile, the Lagrangian is expressed as $\mathcal{L} = \frac{1}{2}\mathcal{M}(q)\dot{q}^2 -$

[34] From which we dropped the mathematical piece using variational analysis

$V(q)$; then, with $\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot{q}} = \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{M}(q)\dot{q} = M'(q)\dot{q}^2 + M(q)\ddot{q}$, Eq. (16) is rewritten as

$$M'(q)\dot{q}^2 + M(q)\ddot{q} = \frac{1}{2}\mathcal{M}'(q)\dot{q}^2 - V'(q).$$

It is easy to see that we obtained in both cases the same equation, and that mathematically, difficulty level was almost the same. However, we needed to memorize Eq. (16), which makes the method based on the energy conservation law slightly easier.

Before we make any final conclusion, let us consider a system of two balls of mass m, connected with a spring of length a and stiffness k, rotating with angular momentum $\vec{L}$ (which is perpendicular to the spring) in weightlessness. Here, it seems that we have two degrees of freedom (the angle and length of the spring), but an additional (to the energy) conservation law (of angular momentum) reduces the effective number of degrees of freedom down to one. Let us use the deformation x of the spring as the generalized coordinate. Then,

$$T = \frac{m\dot{x}^2}{4} + \frac{L^2}{m(a+x)^2}, \quad \Pi = \frac{1}{2}kx^2.$$

The remarkable thing here is that the kinetic energy depends now not only on $\dot{x}$, but also on x; in effect, the second term of the kinetic energy behaves as a potential one, and can be combined into an effective potential energy in the expression for the full energy. Following the method 6, we obtain

$$\frac{1}{2}m\ddot{x}\dot{x} - \frac{2L^2}{m(a+x)^3}\dot{x} + kx\dot{x} = 0 \Rightarrow \ddot{x} = \frac{4L^2}{m^2(a+x)^3} - 2\frac{k}{m}x.$$

Further, let us try to obtain the same result using the Lagrangian (NB! This will be wrong!):

$$\mathcal{L} = \frac{m\dot{x}^2}{4} + \frac{L^2}{m(a+x)^2} - \frac{1}{2}kx^2,$$

hence

$$\frac{1}{2}m\ddot{x} = -\frac{2L^2}{m(a+x)^3} - kx \;\Rightarrow\; \ddot{x} = -\frac{4L^2}{m^2(a+x)^3} - 2\frac{k}{m}x.$$

This is not the same result as before — the first term in right hand side has a different sign! So, what went wrong? The first result is clearly the correct one as the total energy is clearly conserved here. What went wrong is that by making use of the angular momentum conservation law to reduce the number of coordinates we changed the starting and ending points in the configuration space. As we proved above, the least action principle [and hence, Eq. (16)] is valid if we don't use conservation laws to reduce the number of degrees of freedom, and all the conservation laws themselves are to be considered as the consequence of Eq. (16). In this case, the original number of degrees of freedom was two: we can use the deformation x and the rotation angle φ of the spring to describe fully the state of the system. If we use these two coordinates with the corresponding Lagrangian then everything will be correct: the action

$$\int_{t_1}^{t_2}\left[\frac{m\dot{x}^2}{4} + \frac{m\dot{\varphi}^2(a+x)^2}{4} - \frac{1}{2}kx^2\right]\mathrm{d}t$$

is minimized by the true trajectory if we compare the trajectories connecting the initial state x_1, φ_1 and the final state x_2, φ_2. Now, however, we have dropped the variable φ, and if we drop the condition for the initial and final angles, many more trajectories will connect the initial state x_1 with the final state x_2: the true trajectory does no longer need to be the one with the smallest action. An important lesson from this analysis is that **don't use Eq. (16) if you reduce the number of degrees of freedom by making use of a constraint (a conservation law) which involves time derivatives of the coordinates**, because by fixing the value of a time derivative we do not fix the value of the coordinate itself. If you have such conservation laws and manage to find so many constraints that you can bring the number of coordinates down to one, go ahead and use the method 6; otherwise keep the original number of coordinates and apply Eq. (16).

Finally, let us also emphasize that the Lagrangian is given by the difference of kinetic and potential energies only in the case of classical mechanics; in other cases, the first task is to figure out the expression for the Lagrangian. How to do it? Basically there are two options. Assuming we know already the equation of motion in Euclidian coordinates x_i, we can do it by trial and error finding such $\mathcal{L}(x_i, \dot{x}_i, t)$ that Eq. (16) becomes identical to the equation of motion. Note that the original equation of motion does not need to have origins in physics. However, once we have found the corresponding Lagrangian, we can interpret it physically: for instance, if the Lagrangian obeys translational symmetry, we can use the Noether's theorem to find a conserved quantity and call it momentum[35]. In electromagnetism, we'll use this method to derive generalized momentum of a charged particle in magnetic field.

The second option works if we study a system which can be considered quantum-mechanically; let us illustrate this by considering a relativistic point mass. We know that the least action principle in mechanics corresponds to the Huygens principle [see Eq. (17)] and hence, the action must be the phase of the quantum-mechanical probability wave, multiplied by $\hbar$ — in that case the classical action would be the small-speed-limit of the relativistic one. So, with m denoting the relativistic mass and m_0 the rest mass of a particle,

$$\mathcal{S} = \int(\vec{p}\cdot\vec{v} - E)\mathrm{d}t = \int(mv^2 - T - V)\mathrm{d}t = \int[m(v^2 - c^2) - V]\mathrm{d}t,$$

hence

$$\mathcal{L} = m(v^2 - c^2) - V = -m_0\sqrt{1 - \frac{v^2}{c^2}} - V.$$

It is easy to verify that if we put this Lagrangian into Eq. (16), by keeping in mind that $V = V(x, y, z)$ and $\vec{v} = (\dot{x}, \dot{y}, \dot{z})$, we obtain relativistic Newton's 2^{nd} law. As we can see, there is no kinetic energy now included into the Lagrangian.

6 HINTS

1. Write out the balance of torques for the contact point O of the hoop and the shaft. What is the angle that the tangent to the shaft at point O forms with the horizon (given that the wire slips on the shaft)?

2. Write down the equation for the torques for the cylinder & block system with respect to the contact point of the cylinder and the inclined plane. What angle with respect to the horizon is formed by the tangent to the cylinder constructed at the position of the little block?

[35] As a matter of fact, in the case of translational symmetry we don't even need to use the Noether's theorem: $\frac{\partial\mathcal{L}}{\partial q_i} = 0$, hence $\frac{\mathrm{d}}{\mathrm{d}t}p_i = 0$, where $p_i = \frac{\partial\mathcal{L}}{\partial\dot{q}_i}$ is the i-th component of the momentum.

3. According to the idea 4, consider the system “rod CD + the mass m” as a whole; there are four forces acting on it: $m\vec{g}$, $\vec{F}$, and the tension forces of the rods, $\vec{T}_{AC}$ and $\vec{T}_{BD}$. The tension forces are the ones which we don't know and don't want to know. According to the idea 2, these will drop out from the balance of torques acting on the rod CD with respect to the intersection point of AC and BD. Indeed, due to the fact 20, the tension force in the rod AC is parallel to AC; the same applies to the rod BD. Now, what must be the torque of force F? For what direction of the force will this torque be achieved with the minimum magnitude?

4. The vector sum of the forces $\vec{F}$ and $m\vec{g}$ has to compensate the sum of the friction and the normal force $\vec{f} = \vec{N} + \vec{F}_h$, i.e. has to be at an angle $\arctan \mu$ with respect to the normal to the plane. Let us draw the force triangle $m\vec{g} + \vec{f} + \vec{F} = 0$: the vector $m\vec{g}$ can be drawn immediately (its direction and magnitude are known), the direction of $\vec{f}$ can be noted by a straight line passing through the terminal point of $m\vec{g}$. $\vec{F}$ has to connect that straight line to the initial point of $m\vec{g}$. For which direction is its magnitude minimal?

5. Go to the reference frame of the inclined surface (invoke Ideas 7 and 8) and use the same method as for problem 4 ($\vec{a} + \vec{g}$ functions as the effective gravity $\vec{g}_e$).

6. Use a rotating reference frame associated with the cylinder (where the block is at rest, and the centrifugal force $\vec{f}_t$ is constant and pointing downwards). *(a)* The terminal point of the net force of gravity and centrifugal force is moving on a circle and has to be equal to the net force $\vec{f}$ of the normal and frictional forces. What is the maximum allowed angle between the vectors $\vec{f}_t$ and $\vec{f}$ so that there be no slipping? For which direction of $m\vec{g}$ is the angle between the vectors $\vec{f}_t$ and $\vec{f}$ maximal? *(b)* There are still only three forces; as long as there is an equilibrium, these three vectors must form a triangle and hence, must lay on the same plane. According to the idea K-11, we'll depict the force balance in this plane, i.e. in the plane defined by the vectors $\vec{g}$ and $\vec{f}_t$. The approach used in part (a) can still be used, but the terminal point of $\vec{f}_t + m\vec{g}$ draws only an arc of a full circle. Determine the central angle of that arc. Depending on the arc length, it may happen that the maximal angle between the surface normal (= the direction of $\vec{f}_t$) and $\vec{f}$ is achieved at one of the endpoints of the arc.

7. Notice that while rolling at constant speed, the centre of mass of the whole cart moves also with a constant speed, i.e. there should be no horizontal forces acting on the cart. Also, each of the cylinders rotates with a constant angular speed, hence there should be no torque acting on it, hence the friction force must be zero. Use the rotating frame of a wheel; apply the ideas 12 and11 to substitute one asymmetric body (the cylinder with a hole) with two symmetric bodies, a holeless cylinder, and a superimposed cylinder of negative density; further use ideas 9 and 10 to draw the gravity and centrifugal forces; keep in mind that the rod can provide any horizontal force, but cannot exert any vertical force.

8. Based on the idea 14, on which line does the intersection point of the frictional forces have to lie? What can be said about the two angles formed by the frictional force vectors and the thread's direction. Given the Idea no. 1 (the axis is perpendicular with the tension in the thread)? Now combine the two conclusions above. Where is the intersection point of the friction force vectors? What is the direction of the cylinder's velocity vectors at the points where the cylinder rests on the rough band? Where is the cylinder's instantaneous rotation axis (see how to find it in the kinematics brochure)? What is the velocity vector of the cylinder's centre point? *(b)* Will the equilibrium condition found above be violated if the surface is uniformly rough?

9. Draw a circle whose diameter is the straight line connecting the points of support. Use Fact no. 22: which curve can the ball move along? Where is the bottom-most point of this curve?

10. Consider the torques acting on the rod with respect to the hinge. For which angle α will the net force of the normal and frictional forces push the rod harder against the board?

11. By how much will the block descend if the thread is extended by δ?

12. Let's assume that the horizontal component of the tension in the rope is T_x. What is the vertical component of the tension next to the ceiling? Next to the weight? Write down the condition for the balance of the forces acting on a) the weight and b) the system of weight & rope (cf. Idea no. 4).

13. Seeing as $H \ll L$, clearly the curvature of the rope is small, and the angle between the tangent to the rope and horizon remains everywhere small. From the horizontal force balance for the rope, express the horizontal component of the tension force T_x as a function of the length l (note that while T_x remains constant over the entire hanging segment of the rope, we'll need its value at the point P separating the hanging and lying segments). Write down the balance of torques acting on the hanging piece of the rope with respect to the holding hand (according to what has been mentioned above, the arm of the gravity force can be approximated as $l/2$). As a result, you should obtain a quadratic equation for the length l.

14. Use Idea 9: change into the reference frame of the rotating hinge. *a)* Following the idea 19, write down the condition of torque balance with respect to the hinge (Idea no. 2) for a small deviation angle φ. Which generates a bigger torque, $m\vec{g}$ or the centrifugal force? (Note that alternatively, the idea 21 can be also used to approach this problem). *b)* Following the idea 21, express the net potential energy for the small deviation angles φ_1 and φ_2 using the energy of the centrifugal force (which resembles elastic force!) and the gravitational force; according to the idea 20, keep only the quadratic terms. You should obtain a quadratic polynomial of two variables, φ_1 and φ_2. The equilibrium $\varphi_1 = \varphi_2 = 0$ is stable if it corresponds to the potential energy minimum, i,e, if the polynomial yields positive values for any departure from the equilibrium point; this condition leads to two inequalities. First, upon considering $\varphi_2 = 0$ (with $\varphi_2 \neq 0$) we conclude that the multiplier of φ_1^2 has to be positive. Second, for any $\varphi_2 \neq 0$, the polynomial should be strictly positive, i.e. if we equate this expression to zero and consider it as a quadratic equation for φ_1, there should be no real-valued roots, which means that the discriminant should be negative.

15. Apply the ideas 19 and 22 for such a angular position of the beam, for which the magnitude of the buoyant force doesn't change (i.e. by assuming a balance of vertical forces). From idea no. 2, take the centre of mass for the pivot point. While computing the torque of the buoyant force, use Ideas 11, 12: if a certain region has no displaced water, the displaced water density is zero, but it can be represented as overlapping negative and positive mass densities: $0 = \rho_w + (-\rho_w)$. The cross-section of the underwater part of the beam could be represented as a superposition of a rectangle and two symmetrically positioned narrow triangles (one of them of negative mass).

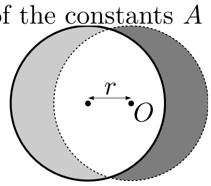
16. The container & water system is affected by the gravity and the normal reaction force of the horizontal surface on the liquid. Since we know the pressure of the liquid at the base of the container, we can express the mass of the container from the vertical condition for equilibrium.

17. To compute the first correction using the perturbation method we use the Fact 51 and the reference system of the block sliding down uniformly and rectilinearly: knowing the magnitude and the direction of the frictional force we can find its component in $\vec{w}$ and $\vec{u}$ direction. The sign of the latter flips after half a period, and so it cancels out upon averaging.

18. Let us choose the origin of the vertical x-axis to be a point on the surface of the ocean very far from the iron deposit. For the zero reference point of the Earth's gravitational potential we shall choose $x = 0$ (i.e. $\varphi_{\text{earth}} = gx$), for that of the iron deposit we shall take a point at infinity. Then, for the points on the ocean's surface very far from the iron deposit, the gravitational potential is zero. It remains to find an expression for the potential above the iron deposit as a function of x (using the principle of superposition) and equate it to zero.

19. Let us employ the reference frame of the platform. Let us the consider the balance of torques with respect to the axis of the small disk (then the lever arm of the force exerted by that axis is zero). Let us divide the disk into little pieces of equal size. The frictional forces acting on the pieces are equal by magnitude and are directed along the linear velocities of the points of the disk (in the chosen reference frame). Since the motion of the disk can be represented as a rotation around an instantaneous axis, then concentric circles of frictional force vectors are formed (centred at the instantaneous rotation axis). Clearly, the net torque of these vectors with respect to the disk's axis is the smaller, the smaller is the circles' curvature (i.e. the farther the instantaneous rotation axis is): the torque is zero when the instantaneous rotation axis is at infinity and the concentric circles become parallel straight lines. An instantaneous rotation axis at infinity means that the motion is translational, $\omega_3 = 0$ (since the linear velocity $v = \omega_3 r$ of a given point is finite, but $r = \infty$).

20. The instantaneous axis of rotation is at a distance $r = v/\omega$ from the disk's axis. Let's use the same imaginary slicing as in the previous problem. Now compute the component of the net force in the direction of motion. Notice that the frictional forces on the points that are symmetrical with respect to the instantaneous rotation axis balance each other across a whole circular region of radius $R - r$. The non-balanced region is unfortunately shaped for calculation. Let us imagine extending the "balanced" region up to R (the dashed circle in the figure). The part of this extended balanced region, where there is no actual rotating disk underneath (the dark gray crescent in the figure), could be represented as a superposition of the two disks, one rotating clockwise and the other – anticlockwise. In that case the clockwise component partakes in the balancing, whereas the anticlockwise component remains unbalanced. To sum up, two thin crescent-shaped regions remain unbalanced: one corresponds to the the real disk (light gray in the figure), the other — to a disk rotating anticlockwise (dark gray); normal to $\vec{v}$, the width of these regions is everywhere equal to r. The net force is the easiest to find by integrating across the crescent-shaped regions using the polar coordinate φ: $|\mathrm{d}\vec{F}| = A \cdot dS$, where dS is the area of the surface element; $dF_x = A\cos\varphi dS = B\cos^2\varphi d\varphi$, $F_x = \int dF_x = B\int_0^{2\pi}\cos^2\varphi d\varphi$. What are the values of the constants A and B?



21. Consider the unit vector $\vec{\tau}$ directed along the infinitesimal displacement vector of the centre of the mass at the instant when the pencil begins moving. Let's express its coordinates in the Cartesian axes (x, y, z), where x is parallel to the pencil and the (x, y)-plane is parallel to the inclined slope. Using the spatial rotations formulae we represent it in the new coordinates (x', y', z), which are rotated with respect to (x, y, z) around the z-axis by an angle φ (so that the axis x' is horizontal). Using the spatial rotations formulae we express the vector's $\vec{\tau}$ vertical coordinate z' in the (x', y', z') coordinate axes, which is obtained from the axes (x', y', z) by rotating about the x' by the angle α.

22. The string connects the two points with the shortest distance along the cylinder's side; when unfolded, the cylinder is a rectangle. Consider the vertical plane touching the surface of the cylinder that includes the hanging portion of the string. This plane and the cylinder touch along a straight line s. If you imagine unfolding the cylinder, the angle between the string and the straight line s is equal to the cylinder's inclination angle α. Given this, l is easy to find. When the weight oscillates, the trace of the string still stays straight on the unfolded cylinder. Therefore the length of the hanging string (and thus the weight's potential energy) do not depend in any oscillatory state on whether the surface of the cylinder is truly cylindrical or is unfolded into a planar vertical surface (as long as the spatial orientation of the axis s is preserved).

23. Write down the two equations describing the balance of force and torques, and then another one that describes the linear relation between the elongations of the string: $T_1 - T_2 = T_2 - T_3$.

24. Initially only the vertical forces affect the hanging block, therefore the initial displacement vector is also vertical. If the acceleration of the large block is a_1, that of the block on top of it — a_2 and that of the hanging block — a_3, then $a_1 + a_2 = a_3$ holds. Now we can write down Newton's 2$^{\text{nd}}$ law for each body. The fourth and the final unknown is the tension in the string.

25. Go to the reference frame of the wedge-block. In the borderline case, the force of inertia's and gravity's net force on the ball m is normal to the left slope (so that the ball stay at rest there). Consider the net forces acting on the balls. Their components normal to the surface they rest on are $\vec{F}_{\perp 1}$ and $\vec{F}_{\perp 2}$. These are equal to the normal forces $\vec{N}_1$ and $\vec{N}_2$ acting on the balls and therefore have to have equal magnitudes ($F_{\perp 1} = F_{\perp 2}$) to ensure that the force balance is achieved horizontally for the wedge-block.

26. Let's take the displacement ξ of the wedge as coordinate describing the system's position. If the wedge moves by ξ, then the block moves the same amount with respect to the wedge, because the rope is unstretchable, and the kinetic energy changes by $\Pi = mg\xi \sin\alpha$. The velocity of the wedge is $\dot{\xi}$ and that of the block is $2\dot{\xi}\sin\frac{\alpha}{2}$ (found by adding velocities, where the two vectors $\dot{\xi}$ are at an angle α), therefore the net kinetic energy $K = \frac{1}{2}\dot{\xi}^2(M + 4m\sin^2\frac{\alpha}{2})$. Then we find $\Pi'(\xi) = mg\sin\alpha$ and $\mathcal{M} = M + 4m\sin^2\frac{\alpha}{2}$; their sum gives the answer.

27. Again, let's take the wedge's displacement as the coordinate ξ; if the displacement of the block along the surface of the wedge is η, then the centre of mass being at rest gives $\eta(m_1\cos\alpha_1 + m_2\cos\alpha_2) = (M + m_1 + m_2)\xi$. From here one can extract η as a function of ξ, but to keep the formulae brief it's better not to substitute this expression everywhere. The kinetic energies of the block can be found as sums of horizontal $[\frac{1}{2}m_i(\dot{\xi} - \dot{\eta}\cos\alpha_i)^2]$ and vertical $[\frac{1}{2}m_i(\dot{\eta}\sin\alpha_i)^2]$ energies.

28. When writing down energy conservation, note that the block's velocity is twice the cylinder's velocity horizontal component and that the latter is equal to the vertical component, too (why?). Project Newton's 2$^{\text{nd}}$ law onto the axis that passes through the top corner of the step and the cylinder's centre: this axis is perpendicular both to the normal force between the block and the cylinder and to the cylinder's tangential acceleration. Second question: the ratio of two normal forces is constant (why? what is it equal to? Hint: compare the horizontal accelerations of the cylinder and the block and remember Newton's 2$^{\text{nd}}$ law), therefore they will be equal to zero at the same instant.

29. By projecting Newton's 2$^{\text{nd}}$ law on the axis in the direction of the normal force we see that the normal force is the smallest at the bottommost point of the trajectory's arch-shaped part. (There, the centripetal acceleration is the largest, gravitational force's component along the axis is the smallest).

30. The energy of the "pellet & block" system is always conserved; momentum will only start to be conserved once the pellet passes the bottommost point. When it arrives there for the second time, the block's velocity is maximal (why?).

31. Let's apply Idea no. 46 for $\vec{P}$: the system's net momentum is $P = \omega lm + 2\omega lM$, net force $F = (m + M)g - T$. The same using rotational considerations: with respect to the leftmost ball's initial position, the angular momentum is $l(2\omega l)M$ (velocity is $2\omega l$, the velocity's lever arm — l); net torque is $(T + Mg)l$. Now, for the formula given in Idea no. 46 we need the angular acceleration $\varepsilon = \dot{\omega}$. Let's find it using Method no. 6: $\Pi = l\varphi(m + 2M)$, $K = \frac{1}{2}\dot{\varphi}^2 l^2(m + 4M)$. Another solution route: the ratio of accelerations is 1:2; there are four unknowns (two normal forces, acceleration and string tension); equations: three force balances (for either ball and the rod) and one torque balance (wrt the left endpoint of the rod).

32. Method no. 6: for the generalized coordinate ξ we can use the displacement of the thread's endpoint. Ideas no. 34,12: the change of the system's CM y-coordinate is $\xi\rho h/M$ (h — the difference in the heights of the thread's endpoints, M — the net mass of the system; assume that $\xi \ll h$). For the x-coordinate it's $2\xi\rho R/M$.

33. $\langle T(1 + \cos\alpha)\rangle = 2mg$, $T = \langle T\rangle + \tilde{T}$, where $|\tilde{T}| \ll T$. Based on the Idea no. 20 we ignore the tiniest term $\langle \tilde{T}\alpha^2\rangle$ and note that $\langle \alpha^2\rangle > 0$.

34. We have to consider two options: either all the bodies move together, or the rightmost large block moves separately. Why cannot the situations occur where *(a)* all three components move separately, or *(b)* the left large block moves separately?

35. After the collision the ball's trajectories are orthogonal crossing straight lines; the angle with respect to the initial trajectory is determined by how much the collision was off-centre.

36. For slightly non-central motion: what will be the direction of momentum of the ball that was first to be hit? Now apply the Idea no. 52 again. Central motion: express the velocities after the collision via the horizontal component of the momentum p_x that has been transferred to one of the balls. What is the transferred vertical component p_y? Energy conservation provides us an equation to find p_y (it is convenient to express the energy as $p^2/2m$).

37. The graph looks like n intersecting straight lines; the intersection point of a pair of straight lines corresponds to a collision of two balls (the graph of either ball's motion is a jagged line; at a collision point the angles of the two jagged lines touch one another so that it looks as if the two straight lines intersect).

38. Initial velocities in the centre of mass: $\frac{mv}{m+M}$, $\frac{Mv}{m+M}$, final velocities are zero; friction does work: μmgL.

39. Based on the figure we immediately obtain (to within a multiplicative constant) the magnitudes and directions of the momenta, but not which momentum is which ball's. It is necessary to find out where the ball marked with an arrow will proceed after the collision. Fact no. 27 will help choose from the three options.

40. Energy: in time dt the distribution of the liquid will change: there is still some water at the centre, but a certain mass dm has been displaced from above to the level of the tap (and then through the tap), so the change in the system's potential energy is $gH \cdot dm$. Momentum: the water in the barrel obtains the total momentum $\rho gHS \cdot dt$ from the walls. This momentum is passed on to the stream of water with the mass $\rho Sv \cdot dt$.

41. Energy is not conserved: the grains of sand slip and experience friction. In time dt the sand landing on the conveyor belt receives momentum $dp = v \cdot dm = v\mu \cdot dt$ from the belt: the force between the freshly fallen sand and the belt is $F_1 = dp/dt$. The sand already lying on the belt experiences the gravitational force mg which is compensated by the component of the friction parallel to the belt, $F_2 = mg\cos\alpha$, where $m = \sigma L$ is the mass of the sand on the belt and $\sigma v = \mu$. The minimization has to be done over v.

42. During the collision $\Delta p_\perp = \sqrt{2gh}$.

43. Consider a short section of the path along the hill with length dl. In addition to the change in the potential energy work is done to overcome friction, $dA_h = \mu mg \tan\alpha \cdot dl$. WE find $dA_h = C \cdot \mathrm{d}x$, where C is a constant. Summing over all such little path increments dl we find $A_h = C\Delta x$.

44. The kinetic energy $K = \frac{m}{2}\dot{x}^2 + M\dot{x}^2$, where x is the displacement along the slanted surface; $\Pi = (M+m)\sin\alpha$. Having found the acceleration a we change into a reference frame (of the cylinder) moving with acceleration a (Ideas no. 7 and 8), where the block is being displaced along the effective acceleration due to gravity — as low as possible.

45. According to the Ideas no. 61 and 62, the angular momentum of the rod before the collision is $L_0 = Mlv - \frac{1}{3}Ml^2\omega$; after the collision $L_1 = Mlv' - \frac{1}{3}Ml^2\omega'$; $L_1 = L_2$. The expression for energy is $K = \frac{1}{2}Mv^2 + \frac{1}{6}Ml^2\omega^2$. The condition for being at the end: $v' + l\omega' = 0$ (we consider ω to be positive if the rotation is in the direction marked in the figure).

46. The angular momentum with respect to the impact point before the collision: $mv(x - \frac{l}{2}) - I_0\omega$, where $v = \omega\frac{l}{2}$ and $I_0 = \frac{1}{12}ml^2$.

47. The instantaneous rotation axis passes the contact point of the cylinder and the floor; its distance from the centre of mass does not change, so we can use Idea no. 65; $I = \frac{3}{2}mR^2$.

48. Let us direct the z axis upward (this will fix the signs of the angular momenta). The final moment of inertia with respect to the x-axis is $-\frac{7}{5}mv_yR - muR$ and with respect to the y-axis is $\frac{7}{5}mv_xR$.

49. Immediately after the first collision the centres of masses of both dumbbells are at rest, the velocities of the colliding balls reverse direction, the non-colliding balls' velocities don't change. Both dumbbells act like pendula and complete half an oscillation period, after which the second collision occurs – analogous to the first one.

50. The grains of sand perform harmonic oscillations in the plane perpendicular to the cylinder's axis — like a mathematical pendulum of length $l = R$ in the gravitational field $g\cos\alpha$; along the axis there is uniform acceleration ($a = g\sin\alpha$). Focussing occurs if the time to cross the trough along its axis is an integer multiple of the oscillation's half-period.

51. Observing the equilibrium position we conclude that the centre of mass lies on the symmetry axis of the hanger. The three suspension points must be located on the two concentric circles mentioned by Idea no. 69. Therefore one of the circles must accommodate at least two points out of the three, while the circles' centre (the hanger's centre of mass) must lie inside the region bounded by the hanger's wires on its symmetry axis. There is only one pair of circles that satisfies all these conditions. Computing the radii l_1 and l_2 of the circles using trigonometry we determine the reduced length of the pendulum $l_1 + l_2$ and, using that, the oscillation period.

52. The effective mass of the moving water can be found using the acceleration of the falling ball. For the rising bubble the effective mass is exactly the same, the mass of the gas, compared to that, is negligibly small.

53. The water stream could be mentally divided into two parts: the leftmost stream will turn to the left upon touching the trough, the rightmost — to the right. Thus, two imaginary 'water tubes' form. In either tube the static pressure is equal to the external pressure (since there is the liquid's outer surface in the vicinity): according to Bernoulli's law, the velocity of the liquid cannot change. Based on the conservation of momentum horizontally, the momentum flows of the left- and right-flowing streams have to add up to the original stream's momentum flow's horizontal component. Note that due to continuity, $\mu = \mu_v + \mu_p$.

54. Due to continuity $(u+v)(H+h) = Hu$ Const, where $h = h(x)$ is the height of the water at point x and $v = v(x)$ is the velocity. We can write down Bernoulli's law for an imaginary 'tube' near the surface (the region between the free surface and the stream lines not far from the surface): $\frac{1}{2}\rho(u+v)^2 + \rho g(H+h) = \frac{1}{2}\rho u^2 + \rho gH = \text{Const}$. We can ignore that small second order terms (which include the factors v^2 or vh)

55. The phase trajectory is a horizontal rectangle with sides L and $2mv$, where L is the distance from the block to the wall; the adiabatic invariant is thus $4Lmv$.

56. Consider the balance of torques. For the net force vectors of the normal and frictional forces, when you extend them, their crossing point must be above the centre of mass.

57. Let's write down Newton's 2^{nd} law for rotational motion with respect to the crossing point of the normal forces: the angular momentum of the bug is $L = mvl\sin\alpha\cos\alpha$, the speed of change of this angular momentum will be equal to the torque due to gravity acting on the bug (the other forces' lever arms are zero). When computing the period, note that the acceleration is negative and proportional to the distance from the bottom endpoint, i.e. we are dealing with harmonic oscillations.

58. The blocking occurs if the net force of normal and frictional forces pulls the rod downwards.

59. Once the blocking occurs we can ignore all the forces apart from normal and frictional ones. Suppose it has occurred. Then the net frictional and normal forces acting from the left and from the right have to balance each other both as forces and torques, i.e. lie on the same straight line and have equal magnitudes. Thus we obtain the angle between the surface normal and the net force of friction and normal force.

60. Consider the direction of the torque acting on the plank with respect to the point of contact, when the plank has turned by an angle φ: the contact point shifts by $R\varphi$, the horizontal coordinate of the centre of mass shifts by the distance $\frac{h}{2}\varphi$ from the original position of the contact point.

61. The only force from the surface on the system vessel & water is equal to the hydrostatic pressure $\rho gh\pi R^2$; it balances the gravitational force $(m+\rho V)g$. Note that $H = R - h$.

62. The gravitational potential of the centrifugal force is $\frac{1}{2}\omega^2r^2$, where r is the distance from the rotation axis.

63. Assume the reference frame of the large block (which moves with acceleration a). Where does the effective gravity (the net force of the gravity and the force of inertia) have to be directed? What is a? With which acceleration does the little block fall in this reference frame? What is the tension T of the thread? Having answers to these questions we can write down the equilibrium condition for the large block $ma = T(1-\sin\alpha)$.

64. Let us use the displacement of the sphere (down the inclined surface) as the generalized coordinate ξ. What is the displacement of the sphere (up the other inclined surface)? Evidently $\Pi = (m - M)g\xi \sin\alpha$. The normal force between the two bodies can be found by projecting Newton's 2$^{\text{nd}}$ law onto the inclined surface's direction.

65. Let the displacement of the large cylinder be ξ, the horizontal displacement of the middle and the leftmost cylinder, respectively, x and y. What is the relationship between them given that the centre of mass is at rest? What is the relationship between them given that the length of the rods does not change? From the two equations thus obtained we can express x and y via ξ. If we assume the displacement to be tiny, what is the relationship between the vertical displacement z of the middle cylinder and the horizontal projection of the rod's length, $\xi - x$? Knowing these results, applying Method no. 6 is straightforward.

66. Where is the small displacement ξ of the sphere directed (see Idea no. 31)? What is the displacement of the ring expressed via ξ? Use Method no. 6.

67. Use Idea no. 40 along with energy conservation by projecting the force and the acceleration in the Newton's 2$^{\text{nd}}$ law radially.

68. Let us use some ideas from kinematics to find the acceleration of the sphere (K1, K29 and K2: by changing into the reference frame moving with velocity v we find the component of the sphere's acceleration along the rod and by noticing that the horizontal acceleration of the sphere is zero, we obtain, using trigonometry, the magnitude of the acceleration). Now use Newton's 2$^{\text{nd}}$ law.

69. Using the velocity v of the sphere we can express the velocity of the block at the moment being investigated (bearing in mind that their horizontal velocities are equal). Using Idea no. 40 we find that the block's (and thus the sphere's) horizontal acceleration is zero; by using Newton's 2$^{\text{nd}}$ law for the sphere and the horizontal direction we conclude that the tension in the rod is also zero. From the energy conservation law we express v^2 and from Newton's 2$^{\text{nd}}$ law for the sphere and the axis directed along the rod we obtain an equation wherein hides the solution.

70. Using Newton's 2$^{\text{nd}}$ law investigate whither the system's centre of mass will move — to the left or to the right (if the centre of mass had not move, then the both events would have happened at the same time).

71. To answer the first part: show that the force perpendicular to velocity is zero (use Method no. 3 and Idea no. 27). To answer the second part use Method no. 3 and idea 56.

72. Due to the length of the thread there are no horizontal forces, i.e. the horizontal component of momentum is conserved, and so is the energy. From the two corresponding equation the limiting velocity $v = v_0$ can be found, for which the bottom sphere ascends exactly to the height of the top one. Note that at that point its vertical velocity is zero, cf. Idea no. 44.

73. Use Idea no. 51. Options: all block keep together; everything slides; the top one slides and and the bottom two stay together (why is it not possible that the top two keep together and the bottom one slides?).

74. Which conservation law acts when the two boys collide (during a limited time of collision) — do we consider the collision absolutely elastic or inelastic (can momentum be lost and where? If it is inelastic, where does the energy go?), see Idea no. 58? After the collision: the common acceleration of the two boys is constant, knowing the initial and final velocities finding the distance becomes an easy kinematics problem.

75. Prove that for a vertical thread the velocity v is maximal (by applying Idea no. 44 for the rotation angle of the rod show that its angular velocity is zero in that position; use Idea no. 61). Then it only remains to apply energy conservation (remember that $\omega = 0$). For the acceleration a, let us use idea 44 and notice that horizontal acceleration of the centre must be zero; this follows from the Newton's 2$^{\text{nd}}$ law for the horizontal motion (there are no horizontal forces at that moment). Further, notice that the vertical coordinate of the centre of mass is arithmetic average of the coordinates of the endpoints, $x_O = \frac{1}{2}(x_A + x_B)$; upon taking time derivative we obtain $\dot{x}_O = \frac{\dot{x}_A}{2}$ and $\ddot{x}_O \equiv a = \frac{\ddot{x}_A}{2}$ (keep in mind that x_B is constant). Hence, the acceleration of O can be found as half of the vertical acceleration of the rod's upper end A; this is the radial, i.e. centripetal component of the acceleration of point A on its circular motion around the hanging point. Finally, for the tension force T we have now one equation $T + N - mg = ma$, but we need still another one. To obtain that, we need to consider angular motion of the rod in the frame of O. At the given moment of time, our new frame moves translationally to right with speed v, and with upwards acceleration a. The torque with respect to the centre of mass is caused only by T and N (gravity and inertial forces have zero arm). So we can relate $T - N$ to the angular acceleration of the rod via Newton's 2$^{\text{nd}}$ for angular motion. To find angular acceleration, let us notice that in the laboratory frame, the speeds of A and O are equal; indeed, both velocities are horizontal (vertical velocity of A is zero because it is the lowest point of its trajectory, and $\dot{x}_O = \frac{\dot{x}_A}{2}$), and using the kinematics idea 35 (projections of the velocities to the direction of the rod must be equal) we can conclude that the velocities must be also equal. Thus, in the moving frame, the speed of point A is zero and so is its centripetal acceleration. Hence, the acceleration must be perpendicular to the rod; we know the vertical component $2a$ of this acceleration and using trigonometry can deduce its modulus. With this acceleration, we can find the angular acceleration of the rod.

76. Find the instantaneous rotation axis (make sure that its distance from the centre of mass is $\frac{l}{2}$). Prove that the centre of mass moves along a circle centred at the corned of the wall and the floor, whereas the polar coordinate of the centre of mass on that circle is the same as the angle φ between the wall and the stick. Express the kinetic energy as a function of the derivative $\dot{\varphi}$ of the generalized coordinate φ using the parallel-axis

(Steiner's) theorem and express the energy conservation law as $\omega^2 = f(\varphi)$; using Method no. 6 we obtain $\varepsilon = \dot{\omega} = \frac{1}{2}f'(\varphi)$. When the normal force against the wall reaches zero, the acceleration of the centre of mass is vertical: present this condition using the tangential and radial accelerations of the centre of mass on its circular orbit ($\frac{l}{2}\varepsilon$ and $\frac{l}{2}\omega^2$ respectively) and use it as an equation to find φ.

77. Based on Idea no. 64 we find that $\omega = 6v/l$. Using energy and momentum conservation we eliminate the puck's velocity after the collision and express the mass ratio.

78. The forces along the normal to the surface are elastic forces, so the energy in vertical direction is conserved during the collision: after the collision the corresponding velocity component is the same as before. To find the other two unknowns, the horizontal and angular velocities, we can obtain one equation using Idea no. 64. The second equation arises from *(a)* the condition that the velocity of the ball's surface is zero at the contact point (no sliding; *(b)* the equation arising from 60).

79. Using the idea 51 we investigate the sliding and rolling regimes. In the latter case the quickest way to find the answer is to use Idea no. 65.

80. The velocity can be found from the conservation laws for energy and momentum (note that the hoop is moving translationally). To find the acceleration it is convenient to use the non-inertial reference frame of the hoop, where the centripetal acceleration of the block is easily found. The condition for the radial balance of the block gives the normal force between the block and the hoop (don't forget the force of inertia!); the horizontal balance condition for the hoop provides an equation for finding the acceleration.

81. Let us assume the block's velocity to be approximately constant. For a certain time t_l the base slides to the left with respect to the block and the momentum imparted by the frictional force at that time is also directed to the left. During the remaining time t_r the base slides to the right with respective momentum directed to the right as well. The equilibrium condition is that the two momenta have equal magnitudes; hence we ding the equilibrium value of t_l/t_r. From the graph we find the velocity for which that ratio has the needed value.

82. As the water flows against the paddles it obtain the same vertical velocity u as the paddles themselves. You should also keep in mind that in the paddle's frame of reference, the speed of water is decreased $\frac{v-u}{v}$ times as compared with speed in the lab frame; the mass flow rate is decreased by the same factor (in lab frame, some water is "lost", i.e. remains between two paddles and never hits a paddle). This allows to compute the momentum imparted to the paddle per unit time (i.e. the force), which ends up being proportional to the difference: $F \propto (v-u)^2$. From there, it is not very hard to find the maximum of the power Fu.

83. In the reference frame of the board the problem is equivalent to the problem no. 52.

84. Go into the (accelerated) reference frame of the wagon, where the effective gravity $\sqrt{a^2+g^2}$ is at a small angle with respect to the vertical. The load will oscillate yet remain motionless at the end if the cable is vertical at the stopping moment and the load's velocity is zero. It is possible when the corresponding position is the maximal deviation during the oscillation. Therefore the oscillation amplitude has to be the same both during the acceleration and deceleration, so that even when the deceleration begins the cable has to be vertical. In that case, how are the acceleration time and the oscillation period related?

85. If the shockwave is at the point where the intersection area of its wavefront and the considered body is S, then what is the force acting on the body? Let us assume that the body stays (almost) at the same place as the shockwave passes it. Then the momentum imparted during the time dt can be found using the cross-sectional area S and the distance $dx = c_s \cdot dt$ covered by the wavefront. Note that $S \cdot dx$ is the volume element. Finally we sum over all imparted momenta.

86. The rod will act like a spring (since the rod is thin and made out of steel, while steel is elastic). After the left sphere has collided with the stationary sphere, the latter will acquire velocity v_0 and the former will stay at rest. Then the dumbbell, as a system of spheres and springs, will begin oscillating around its centre of mass. What is the velocity of the centre of mass? Convince yourself that after half a period the single sphere is already far enough that the left sphere is not going to collide with it again. The oscillations of the dumbbell will decay little by little — so some energy will be lost there.

7 ANSWERS

1. $\arcsin \frac{R\mu}{(R+l)\sqrt{\mu^2+1}}$.

2. $\arcsin \frac{m}{M+m}\frac{\mu}{\sqrt{\mu^2+1}}$.

3. $mg/2$.

4. a) $\mu mg/\sqrt{1+\mu^2}$; b) $mg\sin(\arctan\mu - \alpha)$.

5. $\mu \geq \frac{|g\sin\alpha - a\cos\alpha|}{g\cos\alpha + a\sin\alpha}$, if $g + a\tan\alpha > 0$.

6. a) $\omega^2 R \geq g\sqrt{1+\mu^{-2}}$;
b) $\omega^2 R \geq g\sqrt{1+\mu^{-2}}$, if $\mu < \cot\alpha$ and $\omega^2 R \geq g(\cos\alpha + \mu^{-1}\sin\alpha)$ if $\mu > \cot\alpha$

7. $v = 3\sqrt{gR}$

8. $v/2$.

9. $\tan 2\alpha = h/a$

10. $\mu_1 \geq \sqrt{l^2-h^2}/h$

11. $3mg$

12. $2\arctan[(1+\frac{m}{M})\cot\alpha]$

13. $\sqrt{2HL\mu + \mu^2H^2} - \mu H \approx \sqrt{2HL\mu} - \mu H \approx 7.2\,\mathrm{m}$.

14. a) $\omega^2 < g/l$; b) $\omega^2 < (2-\sqrt{2})g/l$

15. $\frac{1}{2}(1-3^{-1/2})\rho_v \approx 211\,\mathrm{kg/m^3}$

16. $\frac{\pi}{3}\rho R^3$

17. $v/\sqrt{\mu^2\cot^2\alpha - 1}$

18. $\frac{4}{3}\pi G r^3 \Delta\rho/g(r+h) \approx 0.95\,\mathrm{cm}$

19. $-\omega$

20. $\mu mgv/\omega R$

21. $\cos\varphi\tan\alpha < \tan 30°$

22. $L - \pi R/2\cos\alpha$; $2\pi\sqrt{L/g}$

23. $\frac{1}{12}mg$, $\frac{1}{3}mg$, $\frac{7}{12}mg$

24. $mg/(2M+m)$

25. $m < M\cos 2\alpha$.

26. $mg\sin\alpha/[M + 2m(1-\cos\alpha)] =$
$mg\sin\alpha/[M + 4m\sin^2\frac{\alpha}{2}]$.

27. $g\frac{(m_1\sin\alpha_1 - m_2\sin\alpha_2)(m_1\cos\alpha_1 + m_2\cos\alpha_2)}{(m_1+m_2+M)(m_1+m_2)-(m_1\cos\alpha_1+m_2\cos\alpha_2)^2}$.

28. $mg(5\sqrt{2}-4)/6)$; Simultaneously.

29. $\cos\alpha \geq \frac{1}{3}(2 + v^2/gR)$

30. $2\frac{m}{M+m}\sqrt{2gR}$

31. $mMg/(m+4M)$

32. $F_x = 2Ra\rho$, $F_y = (m+\rho L)g - \rho(L - \pi R - 2l)a$, where $a = \rho g(L - \pi R - 2l)/(m + \rho L)$.

33. The one that had not been pushed.

34. If $F \leq 2\mu mg\frac{m+M}{2m+M}$: $a_1 = a_2 = \frac{1}{2}\frac{F}{M+m}$; otherwise $a_1 = \frac{F}{M} - \mu g\frac{m}{M}$, $a_2 = \mu g\frac{m}{2m+M}$.

35. On a half-circle.

36. *(a)* $v/5$; *(b)* $v/4$.

37. $n(n-1)/2$

38. $\sqrt{2\mu gL(1+\frac{m}{M})}$

39. 3.5; was coming from below right.

40. A: $\sqrt{2gh}$; $\sqrt{gh}$.

41. $2R\mu\sqrt{gl\sin\alpha}$, $\sqrt{gl\sin\alpha}$.

42. $u - \mu\sqrt{2gh}$.

43. $mg(h + \mu a)$.

44. $\arctan\frac{2}{5} \approx 21°48'$.

45. *(a)* $(3v - \omega l)/4$; *(b)* $(v - \omega l)/2$.

46. At a distance $2l/3$ from the holding hand, where l is the length of the bat.

47. $\frac{2}{3}\frac{F}{M}\frac{a}{R}$

48. $(v_{x0}, v_{y0} - \frac{5}{7}u)$

49. $L/v_0 + \pi\sqrt{m/2k}$

50. $\frac{1}{2}\pi^2(n+\frac{1}{2})^2 R\tan\alpha$

51. $1.03\,\mathrm{s}$

52. $2.0\,\mathrm{g}$

53. $v_1 = v_2 = v$; $\cot^2\frac{\alpha}{2}$

54. $\sqrt{gH}$.

55. $5\,\mathrm{m/s}$.

56. *(a)* $\tan \leq 2\mu$; *(a)* impossible.

57. $g(1-\frac{x}{l})\sin^{-1}\alpha$; $\frac{\pi}{2}\sqrt{l\sin\alpha/g}$

58. $\mu < \cot\alpha$.

59. $\mu_1 < \tan\frac{\alpha}{2}$ and $\mu_2 < \tan\frac{\alpha}{2}$.

60. $R > h/2$

61. $\sqrt[3]{3m/\pi\rho}$

62. $\omega^2R^2/2g$

63. $M/m = \frac{(1-\sin\alpha)^2}{\sin\alpha}$.

64. $\frac{2mM}{M+m}g\tan\alpha$

65. $g/9$.

66. $g\frac{m+M}{m+M\sin^2\alpha}\sin^2\alpha$.

67. $2/3R$

68. $m[g - v^2(2l-x)/\sqrt{2}l^2]$

69. $M/m = 4$, $u = \sqrt{gl/8}$.

70. The first one arrives first

71. A straight line; if $\omega \neq 0$

72. $\sqrt{2gl(1+m/M)}$

73. $\frac{F}{3m}$, if $\frac{F}{m\mu g} < 6$; $\frac{F}{4m} + \frac{1}{2}\mu g$, if $6 < \frac{F}{m\mu g} < 10$; $3\mu g$, if $\frac{F}{m\mu g} > 10$

74. $m^2v^2/2(M^2 - m^2)\mu g$

75. $v = \sqrt{(l-\frac{H}{2})g}$, $a = \frac{g}{2}(1-\frac{H}{2l})$, $T = \frac{mg}{4}(3 - \frac{H}{2l} + \frac{l}{6H})$.

76. $\arccos\frac{2}{3} \approx 48°12'$

77. $M/m = 4$.

78. *(a)* $\omega = 5v_0/7R$, $v_x = 5v_0/7$, $v_y = \sqrt{2gh}$;
(b) $v_y = \sqrt{2gh}$, $v_x = v_0 - 2\mu v_y$,
$\omega = 5\sqrt{2gh}\mu/R$.

79. $\frac{5}{7}g\sin\alpha$, if $\mu > \frac{2}{7}\tan\alpha$, otherwise $g\sin\alpha - \mu g\cos\alpha$

80. $\sqrt{\frac{2gr}{m+M}\frac{1+\cos\varphi}{m\sin^2\varphi+M}}m\cos\varphi$;
$\frac{gm\sin 2\varphi}{m\sin^2\varphi+M}[\frac{1}{2} + \frac{m^2\cos\varphi(1+\cos\varphi)}{(m\sin^2\varphi+M)(m+M)}]$

81. $0.6\,\mathrm{m/s}$

82. $\frac{4}{27}\mu v^2$

83. $v/\cos\alpha$

84. $n^{-2}Lg/4\pi^2 l$, $n = 1, 2, \ldots$

85. *(a),(b)* $(p_1 - p_0)V/mc_s$.

86. $\frac{1}{2}v_0$; no, a fraction goes into the longitudinal oscillations of the rod and then (as the oscillations die) into heat