Waves, instabilities and solitons

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I. LINEAR WAVES

A. Non-dispersive waves: gravity water waves on shallow water

Let $h = h(x)$ be the water depth, ρ — its density, and $\boldsymbol{v}$ — its velocity; let x be a vertical axis and y — horizontal (the origin is at the water bottom). We have two equations: continuity condition

$$\frac{\partial \rho}{\partial t} + \mathrm{div}\, \rho \boldsymbol{v} = 0, \tag{1}$$

and the Newton's second law

$$\rho \left(\frac{\partial}{\partial t} + \boldsymbol{v}\boldsymbol{\nabla} \right) \boldsymbol{v} + \boldsymbol{\nabla} p = 0. \tag{2}$$

For shallow water waves, velocity is horizontal and depends only on the horizontal coordinate: $\boldsymbol{v} = \hat{\boldsymbol{x}} v(x)$. Let us integrate these equations over y - the vertical coordinate and take into account that the water density ρ is constant:

$$\frac{\partial h}{\partial t} + \frac{\partial h v}{\partial x} = 0, \tag{3}$$

$$h \frac{\partial v}{\partial t} + h v \frac{\partial v}{\partial x} + \frac{\partial}{\partial x} \frac{g h^2}{2} = 0. \tag{4}$$

Here we have taken into account that $p = \rho g (h - y)$.

No we need to linearize this set of equations; to that end, let us assume that the uperturbed water depth is h_0, and $h = h_0 + \chi$, where $|\chi| \ll h_0$. We also assume that the water speed $v \ll c_w$, where c_w is the wave speed (yet to be found). Because of that, $\frac{\partial}{\partial t} \gg v \frac{\partial}{\partial x}$. So,

$$\frac{\partial \chi}{\partial t} + h_0 \frac{\partial v}{\partial x} = 0, \tag{5}$$

$$h_0 \frac{\partial v}{\partial t} + h_0 g \frac{\partial \chi}{\partial x} = 0. \tag{6}$$

From this system of equations, we can eliminate v:

$$\frac{\partial^2 \chi}{\partial t^2} = h_0 g \frac{\partial^2 \chi}{\partial x^2}. \tag{7}$$

This is a linear wave equation, the generic solution of which can be written as

$$\chi = f(x - \sqrt{h_0 g} t) + g(x + \sqrt{h_0 g} t), \tag{8}$$

where f and g are arbitrary functions. Let us notice that the wave speed $c_w = \sqrt{h_0 g}$ is independent of the wave vector k.

B. Dispersive waves: water waves on deep water

Upon taking divergence from Eq (2), neglecting the nonlinear small term $(\boldsymbol{v}\boldsymbol{\nabla})\boldsymbol{v}$, and taking into account that $\mathrm{div}\boldsymbol{v} = 0$, we obtain $\Delta p = 0$. In 2D geometry where x is the horizontal coordinate, and y — vertical, we obtain

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} = 0. \tag{9}$$

Since for the time being, we are going to derive a linearized equation, it will suffice if we derive it for a single Fourier component. Indeed, any function can be represented as a sum of sinusoidal signals, and owing to the superposition principle (which is valid for the solutions of any linear differential equation), any solution can be obtained as a superposition of sinusoidal solutions. Therefore we assume that $p = -\rho g y + p(y) e^{ikx}$: water surface is at $y = 0$, and the underwater region extends to $y < 0$. From Eq (9, $\frac{\partial^2 p(y,t)}{\partial y^2} = k^2 p(y, t)$, hence $p(y) = a e^{-ky} + p_0(t) e^{ky}$. The pressure oscillations need to decay for $y \to -\infty$, hence $a = 0$. At the free surface of water, $p = 0$, so that the pressure oscillations are caused by water level elevetion, $h = h_0(t) e^{ikx}$, where $h_0 = p_0 / \rho g$.

It is clear that the water velocity should decay according to the same exponential law ($\propto e^{ky}$) as the pressure oscillations (for a formal proof, see below). Hence we can integrate the continuity condition (1) over y to obtain

$$\frac{\partial \chi}{\partial t} + \frac{1}{k} \frac{\partial v}{\partial x} = 0. \tag{10}$$

For a water volumes near the water surface, the equation of motion can be written as

$$\frac{\partial v}{\partial t} + g \frac{\partial \chi}{\partial x} = 0; \tag{11}$$

Similarly to the shallow water waves, we eliminate here v to obtain

$$\frac{\partial^2 \chi}{\partial t^2} = \frac{g}{k} \frac{\partial^2 \chi}{\partial x^2}. \tag{12}$$

We can conlcude from here that the phase speed $u_p = \sqrt{\frac{g}{k}}$, and that the group speed

$$u_g = \frac{d\omega}{dk} = \frac{1}{2} \sqrt{\frac{g}{k}} = \frac{1}{2} u_g. \tag{13}$$

However, it is not a proper equation as it combines the elements of the coordinate space x, and the momentum space k. Let us rewrite it in the coordinate space by substituting $\frac{\partial}{\partial x} = ik$:

$$\frac{\partial^2 \chi}{\partial t^2} = ig\frac{\partial \chi}{\partial x}. \tag{14}$$

As usually with the complex-valued signals, we presume that the physically observed quantity is the real part of the complex-valued water elevation $\chi(x,t)$.

Now, let us derive formally that $v \propto e^{ky}$. Upon taking curl of Eq. (2), it can be rewritten as

$$\frac{\partial}{\partial t}\nabla \times \boldsymbol{v} = \boldsymbol{v} \times [\nabla \times \boldsymbol{v}] \tag{15}$$

which describes mathematically the fact that the flux of the vorticity $\nabla \times \boldsymbol{v}$ across material loops is conserved (is "frozen" into the fluid). In particular, if there was no vorticity initially, $\nabla \times \boldsymbol{v} = 0$, the flow will remain always potential: $\boldsymbol{v} = \nabla\varphi$; then, the incompressibility condition yields $\Delta\varphi = 0$, hence $\varphi = \varphi_0 e^{ikx+ky}$.

C. Dispersive wave propagation: ship wake

Now, let us analyze the implications of the dispersion relationship (13). For isntance, what will happen if a stone is thrown into water? Falling stone will cause a strongly localized initial perturbation, which can be considered to be a Dirack delta-function. In Fourier decomposition, this is a superposition of all the wavevectors $\boldsymbol{k}$. Let us ask, how long time do we need to wait at a distance r to be able to detect an arriving wave. For a wave propagating directly in the direction of an observer, the phase is given by

$$\varphi(k) = \omega t - kr = \sqrt{gk}t - kr \tag{16}$$

(all Fourier components start with a zero phase at zero distance from the falling site). For a fixed t, this is a function of k: strictly speaking, we cannot speak about a phase unless we fix the wave vector k, and the net water elevation needs to be expressed as a Fourier integral

$$\int e^{i(\sqrt{gk}t-kr)}dk.$$

This is an integral of an oscillating function, for which negative and positive contributions cancel mostly out. One can say that there is a destructive interference of different wavelengths. However, there is an exception of those wavelengths which arrive almost at the same phase and therefore add up constructively — this is a range of k-values near k_0 which realizes a local extremum of $\varphi(k)$, i.e. $k \in [k_0 - \delta k, k_0 + \delta k]$. Mathematically, one can arrive at the same conclusion upon integrating asymptotically $(t, r \to \infty)$ using the saddle point method.

The function (16) has a global maximum at

$$k = k_0 = gt^2/4r^2. \tag{17}$$

Thus, the signal observed is dominated by the waves of wavevector k_0 which arrive with the phase

$$\varphi_{\max} = \varphi(k_0) = \frac{gt^2}{4r}. \tag{18}$$

Now we can answer the posed question: the signal can be detected at the observation point if the phase is not too small, $\varphi \geq \varphi_0$, for instance with $\varphi_0 \approx 1$. According to Eq. (18), such a phase is achieved after the time interval $t = 2\sqrt{\varphi_0 r/g}$.

The above arguments can be applied to a more generic case of an arbitrary dependance $\omega(k)$: using $\varphi = \omega(k)t - kr$ and equating differential $d\varphi = td\omega - rdk$ to zero, we obtain

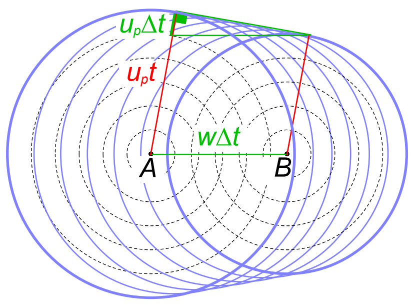
$$\frac{r}{t} = \frac{d\omega}{dk}, \tag{19}$$

an expression with which we recover a well-know formula for the group speed.

What about waves generated behind a ship? For a shallow water, the situation is simple: as soon as the ship speed w exceeds the wave speed $u = \sqrt{gh_0}$, a Mach cone is formed, and the direction θ of the shock wave propagation is defined by the Cherenkov resonance condition, $u\cos\theta = w$. In the case of dispersive waves (when the speed u depends on k), it is important to notice that the Cherenkov resonance condition is to be applied to the phase speed,

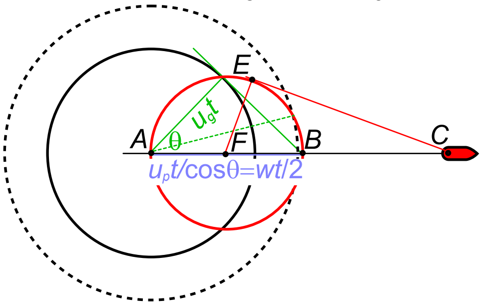
$$u_p \cos\theta = w. \tag{20}$$

Indeed, let us consider sinusoidal waves of a fixed wavelength generated by a moving boat. In the figure below, blue circles are the lines of equal phase of these waves; the envelope of these circles is the front where the wave superposition results in a constructive interference — the wave generated due to Cherenkov interference. A and B correspond to the positions of the boat in two moments of time, separated by a time interval Δt; then, Eq. (20) follows from the trigonometry of the green triangle in this figure.



Now, this Cherenkov resonance condition is satisfied for all the wavefronts parallel to the envelope (green oblique line in figure); however, there is also another condition for the waves to have a non-vanishing amplitude. Indeed, the wave packets (the energy) travel with the group speed, and thus only the waves at the correspondign distance can be seen.

So, for observable waves behind a boat, both Eq (20) and (19) need to be satisfied for a wave of arbitrary wavelength, generated by the boat at an arbitrary moment of time. So, if a wave was generated by the boat at a point A (see figure), the wave now needs to reside on the black circle of radius u_gt, and wave propagation (green oblique line) needs to correspond to the Cherenkov resonance condition, $\cos\theta = v_p/w$. This means that the tangent line to the wavefront intersects the boat trajectory at point B so that $AB = u_gt/\cos\theta = wt/2$. Therefore, for such a geometric configuration, the position of the point B is independent of the wave vector k; depending on the value of k, the radius of the black circle will vary (another circle is shown by a dashed line), but according to the Thales theorem, the position of the corresponding resonant wave package lies still on the red circle. The wave package E with largesr Mach cone angle forms a tangent EC with the red circle: the observable Mach cone angle $\alpha = \arcsin\frac{FE}{FC} = \arcsin\frac{1}{3} \approx 19°$.



Note that it is also possible to derive the shape of the lines of constant phase in the boat's frame of reference. Indeed, let us recall that according to Eqns. (18,17), the wave generated by a boat has a phase $\varphi = \frac{gt^2}{4r}$, and wavevector $k_0 = gt^2/4r^2$, where t is the time interval elapsed from the moment of generation, and r is the distance between the generation point and the observation point; the former is at the distance $-wt$ leftwards from the origin. Then,

$$x = r\cos\theta - wt \text{ and } y = r\sin\theta, \tag{21}$$

where the angle θ describes the wave propagation direction. The system of equations is closed by the condition for Cherenkov resonance, $w\cos\theta = \sqrt{g/k_0} = \frac{2r}{t}$, which leads us to $wt = 2r/\cos\theta$, and when combined with the expression $\varphi = \frac{gt^2}{4r}$, to $r = \frac{w^2\varphi}{g}\cos^2\theta$. When we substitute these expressions into Eq. (21), we end up with

$$x = -\frac{w^2\varphi}{g}\cos\theta(2-\cos^2\theta), \quad y = \frac{w^2\varphi}{g}\sin\theta\cos^2\theta. \tag{22}$$

These equations represent parametrically the lines of constant phase and are depicted in figure below.

Therefore, $y = \frac{1}{2}t\sqrt{g/k_0}\cos\theta - wt$ and $y = \frac{1}{2}t\sqrt{g/k_0}\sin\theta$ We can eliminate k and t from this system of four equations, to obtain a parametrical representation of the curves (angle θ serves as the parameter and the value of φ defines wich constant phase line we consider).

For more information about the waves generated by ships, see [1].

D. Instabilities: Rayleigh-Taylor instability

Thus far we have studied wave equations: the solutions will be waves which can travel with constant amplitude. However, linearized equations can also lead to exponantially decaying or exponentially growing solutions. Growing solutions means that there is an instability. A classical example of an instability which plays an important role in many fields of physics is the Rayleigh-Taylor instability when heavier liquid is put on top of a lighter liquid.

We proceed in the same way as in the case of deep water waves. Let us suppose that in the uperturbed state, region $y > 0$ is filled with liquid of density ρ_1, and $y < 0$ is filled with liquid of density $\rho_2 < \rho_1$. Then, according to the results of Section B, the velocity and pressure fluctuations depend on the coordinates proportionally $e^{ikx-k|y|}$. Our task is to tailor the solutions in $y < 0$ and $y > 0$ via the boundary condition at $y = 0$. To begin with, let us notice that the continuity condition (10) is still applicable, but only for the region $y < 0$; for $y > 0$, the '+' sign should be substituted with '−' (while for $y < 0$, a rising interface level $\chi(x,t)$ means that there needs to be an inflow of the liquid for filling up the extra volume below the raising interface, at $y > 0$ the liquid needs to flow away for emptying that volume). So, in these two regions, the velocities (and hence, the excess pressures) need to be in an opposite phase: in the region $y > 0$,

$$v = -v_0e^{ikx-k|y|}, \quad p = -p_0e^{ikx-k|y|}; \tag{23}$$

in the region $y < 0$,

$$v = v_0e^{ikx-k|y|}, \quad p = \frac{\rho_2}{\rho_1}p_0e^{ikx-k|y|}; \tag{24}$$

He we have also taken into account the following considerations. Due to the continuity law, the velocity amplitudes need to be equal. Due to the equation of motion, the accelerations (and hence, the velocities) are proportional to the excess pressure and inversely proportional to the density, hence the excess pressure in different regions needs to be proportional to the density.

The pressure difference across the interface $p_0(\frac{\rho_2}{\rho_1}+1)$ is caused by the hydrostatic excess pressure in the region $y<0$ due to the change of the interface height:

$$p_0(t)e^{ikx}\left(\frac{\rho_2}{\rho_1}+1\right)=(\rho_2-\rho_1)g\chi(x,t). \tag{25}$$

Now we can find the excess pressure for region $y<0$,

$$p=e^{-k|y|}\frac{\rho_2-\rho_1}{\rho_2+\rho_1}\rho_2 g\chi(x,t), \tag{26}$$

which leads us to the equation of motion near the level $y=0$

$$\frac{\partial v}{\partial t}+g\frac{\rho_2-\rho_1}{\rho_2+\rho_1}\frac{\partial\chi}{\partial x}=0; \tag{27}$$

Once we eliminate v from this equations, together with Eq. (10), we end up with the linearized equation

$$\frac{\partial^2\chi}{\partial t^2}=\frac{g}{k}\frac{\rho_2-\rho_1}{\rho_2+\rho_1}\frac{\partial^2\chi}{\partial x^2}. \tag{28}$$

Now, by substituting $\frac{\partial}{\partial t}=i\omega$ and $\frac{\partial}{\partial x}=ik$, we obtain immediately

$$\gamma\equiv i\omega=\pm\sqrt{gk\frac{\rho_1-\rho_2}{\rho_2+\rho_1}}, \tag{29}$$

which means that there is an exponentially growing solution $\propto e^{\gamma t}$. This means the presence of an instability: initially negligible perturbations develope eventually into significant distortions. As a rule, the most unstable natural modes (the sinusoidal perturbations with such a value of k which correspond to the heighest values of γ) will evolve (the other modes will not have enough time to develop). Here, the modes with large vakues of k are more unstable.

However, we have not yet taken into account cappilary effects. Across an interface of curvature radii R_1 and R_2, excess pressure $\delta p=\sigma(R_1^{-1}+R_2^{-1})$ is created, due to the surface tension σ. Here, R_1 and R_2 are the curvature radii of the interface intersections with two planes, perpendicular to each other and to the tangent of the interface. For a plane wave $\chi=\chi_0 eikx$, $R_2=\infty$, and $R_1^{-1}\approx\frac{\partial^2\chi}{\partial x^2}=-k^2\chi$; the approximate equality becomes exact at the limit of small wave amplitude.

This effect can be taken into account in Eq. (25) by adding the additional pressure term $\sigma k^2\chi$ to the right-hand-side. As a result, Eq. (28) is rewritten as

$$(\rho_2+\rho_1)\frac{\partial^2\chi}{\partial t^2}=\left[\frac{g}{k}(\rho_2-\rho_1)+k\sigma\right]\frac{\partial^2\chi}{\partial x^2}, \tag{30}$$

which means that

$$\gamma^2=\frac{(\rho_1-\rho_2)gk-\sigma k^3}{\rho_2+\rho_1}, \tag{31}$$

Now we can find the most unstable mode via equating $\frac{d\gamma^2}{dk}=$, i.e.

$$k_0=\sqrt{\frac{(\rho_1-\rho_2)g}{3\sigma}}. \tag{32}$$

Finally, note that Eq. (31) can be used to obtain a more general expression for the dispersion equation for the deep water waves. Indeed, the only thing we need to do is to put $\rho_1=0$ and return to $\omega\equiv-i\gamma$:

$$\omega^2=gk+\frac{\sigma}{\rho}k^3. \tag{33}$$

Task: consider a water cylinder of an infinite length and a radius r_0 in weightlessness (this can model water falling from a faucet). Find the instability exponent for the cylinder with respect to perturbations $r(x,t)=r_0+a(t)e^{ikx}$, where $r(x,t)$ is the perturbed radius of the cylinder as a function of the axial coordinate x and time t. You may assume that $k\gg 1/r_0$, so that the water motion is essentially one-dimensional.

E. Nonlinear waves: Korteweg-de Vries equation

Now we want to take into account nonlinearities. So, we return to Eqns. (3) and (4), which are precise under the assumption of shallowness. Without any additional assumptions, this would be a rather difficult task. Because of that, we separate left-propagating and right-propagating waves by assuming that the height is a function of velocity, $h=h(v)$. We can say that we are looking for specific solutions of the system (3) and (4) — such that the condition $h=h(v)$ is satisfied. Let us notice that for linear rightwards propagating waves, $h(x,t)=h(x-c_st)$ and $v(x,t)=v(x-c_st)$, hence $v=v(h)$; here, $c_s=\sqrt{gh_0}$. The same can ba applied to leftards propagating waves; however, if both propagation directions are present, the relationship $h=h(v)$ will fail.

Once we accpet the assumption $h=h(v)$, we can rewrite the system of equations (3) and (4) as

$$\frac{dh}{dv}\left(\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial x}\right)+h\frac{\partial v}{\partial x}=0, \tag{34}$$

$$\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial x}+g\frac{dh}{dv}\frac{\partial v}{\partial x}=0. \tag{35}$$

Once we multply the second one with $\frac{dh}{dv}$, and subtract from the first one, we obtain

$$\frac{dh}{dv}=\pm\sqrt{\frac{h}{g}}; \tag{36}$$

if we substitute this result into either of the equations, we end up with

$$\frac{\partial v}{\partial t}+(v\pm\sqrt{hg})\frac{\partial v}{\partial x}=0. \tag{37}$$

If the amplitude of the wave is not too large, we can approximate

$$\sqrt{h} \approx \sqrt{h_0} + \frac{1}{2} h_0^{-\frac{1}{2}} \frac{dh}{dv} v \approx \sqrt{h_0} \pm \frac{1}{2} g^{-\frac{1}{2} v}. \quad (38)$$

Using this expression, we can rewrite Eq. (37) as

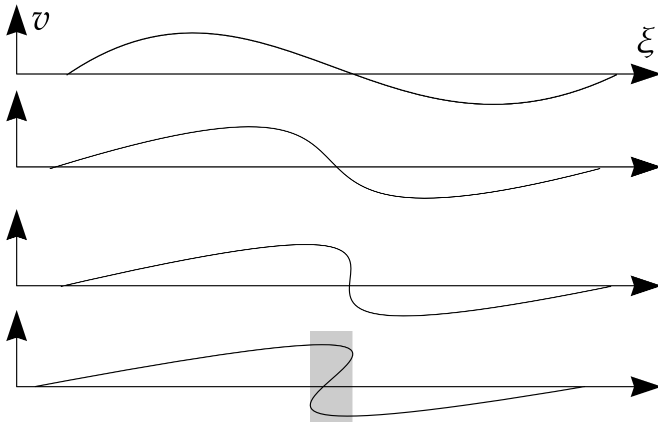
$$\frac{\partial v}{\partial t} + \left(\frac{3}{2} v \pm \sqrt{h_0 g} \right) \frac{\partial v}{\partial x} = 0. \quad (39)$$

This equation is most conveniently analyzed in a frame of reference, co-moving with linear waves; this corresponds to the change of variables $\xi = x \mp \sqrt{h_0 g} t$, $\tau = \frac{3}{2} t$, in which case we obtain

$$\frac{\partial v}{\partial \tau} + v \frac{\partial v}{\partial \xi} = 0. \quad (40)$$

Although this is a non-linear equation, it is easy to analyze, because it can be interpreted as the equation of motion for non-interacting particles: particles lay on the ξ-axis, and are assigned velocities as a function of their coordinate, $v = v(\xi)$. They satrt moving without acceleration, $\frac{dv}{dt} = 0$, which is equivalent to Eq (40).

Such a motion of non-interacting particles leads to wave-breaking: at a certain moment of time, faster particles "catch up" the slower ones, and after that, there are regions of particles with different speeds (marked in grey in figure below).



It should be emphasized that Eq. (40) is obtained for nonlinear waves in almost any media after the following steps: (a) separate waves which move in opposite diretions by assuming that all the parameters describing the state of the medium as a function of space- and time coordinates can be expressed as functions of each other; (b) assume that nonlinearity is not too large, so that only quadratic nonlinearities can be kept in power series expansions; (c) use reference frame which moves with the speed of linear waves; (d) renormalize time so as to get rid of the prefactor of the nonlinear term.

As we saw above, as long as Eq. (40) serves as a good approximation, the waves evolve towards wavebreaking. Quite often, however, this process is stopped by physical processes neglected thus far: dissipation and/or dispersion. Let us first consider the effect of dispersion. As we have seen previously, for linear waves in shallow water, $\omega = \sqrt{gh_0} k$, and in deep water, $\omega = \sqrt{gk}$. For an intermediate depth, we can also decompose the solution of linearized equation into Fourier components, yielding a certain dispersion relationship $\omega(k)$; if the depth is not too large, we can expect that $\omega(k) \approx \sqrt{gh_0} k$, and we can improve the approximation by using a power series expansion,

$$\omega = \sqrt{gh_0} k + \frac{1}{2} \frac{d^2\omega}{dk^2} k^2. \quad (41)$$

Being equipped with this dispersion relationship, we can revert back from the Fourier space to the physical space, using the correspondence $\omega \to -i\frac{\partial}{\partial t}$, $k \to i\frac{\partial}{\partial x}$:

$$\frac{\partial v}{\partial t} + \sqrt{gh_0} \frac{\partial v}{\partial x} + a \frac{\partial^3 v}{\partial x^3} = 0, \quad (42)$$

where $a = -\frac{1}{2} \frac{d^2\omega}{dk^2}$; we expect that for gravity water waves, $a > 0$ ($\omega(k) = \sqrt{gk}$ is a concave function, with a negative second derivative). For capillary waves, however, $a < 0$.

Note that mathematically, the transition from the Fourier space to the physical space can be done by multiplying the dispersion equation with $\hat{v}(k, \omega) e^{ikx - i\omega t}$, where $\hat{v}(k, \omega)$ is the Fourier transform of $v(x, t)$, and by integrating over ω and k (using integration by parts).

So, taking into account a weak dispersion adds an additional term $a\frac{\partial^3 v}{\partial x^3}$ to the differential equation of $v(x, t)$. The nonlinear version of the differential equation, Eq (40), can be also appropriately modeified, resulting in

$$\frac{\partial v}{\partial \tau} + v \frac{\partial v}{\partial \xi} + a \frac{\partial^3 v}{\partial \xi^3} = 0. \quad (43)$$

Note that with a proper choice of units, with ξ and τ being normalized to $\sqrt{a}$, we can get rid of the constant a, so that

$$\frac{\partial v}{\partial \tau} + v \frac{\partial v}{\partial \xi} + \frac{\partial^3 v}{\partial \xi^3} = 0. \quad (44)$$

This is called the Korteweg-de Vries (KdV) equation, the importance of which lies in the fact that it describes correctly any non-linear waves under the asymptotic limit of small nonlinearity (i.e. small amplitude) and long wavelength (assuming that the waves are dispersionless at the limit $k \to 0$, such as sound waves).

F. Solitons

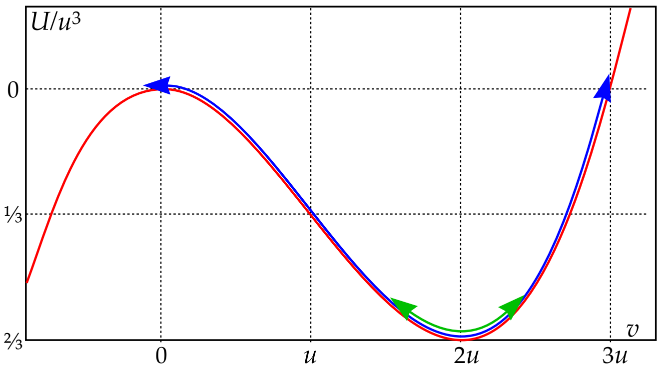
As we have seen abovem nonlinearity will lead to a wavebreaking. However, dispersion will oppose to it: wavebreaking means creating a large velocity gradient, which in Fourier space corresponds to creating high wavenumbers and hence, increasing the spectral width of the signal. Increased spectral width means stronger effect of dispersion: different wavelengths travel with different speeds and result in spreading the signal and decreasing the gradients. So, nonlinearity and dispersion have opposing effects, and the wave profile evolves towards a state when the two effects compensate each other.

It is quite easy to find if a differential equation admits solutions in the form of a solitary wave which can tarvel with constant speed with a stationary shape. To that end, a standard technique is used, by which stationary solutions are sought in a co-moving frame of reference, with $\zeta = \xi - u\tau$ and $\frac{\partial}{\partial\tau} = -u\frac{\partial}{\partial\zeta}$ (time derivative in the co-moving frame is dropped due to the assumption of stationarity). As a result, for KdV equation we obtain

$$(v-u)\frac{\partial v}{\partial\zeta} + \frac{\partial^3 v}{\partial\zeta^3} = 0. \quad (45)$$

This equation can be integrated once over ζ:

$$\frac{\partial^2 v}{\partial\zeta^2} = uv - \frac{1}{2}v^2 = -\frac{\partial U}{\partial v}, \quad U(v) = \frac{1}{6}v^3 - \frac{1}{2}uv^2. \quad (46)$$



This equation can be interpreted as the equation of motion for a fictitious particle with a unit mass, assuming that v is its coordinate, ζ is the time, and $U(v)$ is the potential energy as a function of coordinate, see figure below. Depending on the initial speed, the fictitious particle can perform different types of motions.

First, let us consider a periodic motion, depicted by green line: the fictitious particle performs nearly harmonic oscillations at the bottom of the potential minimum. In terms of KdV solutions, this corresponds to a nearly sinusoidal wave. Let us recall that v is the speed of the material particles; for a nearly harmonic wave, v oscillates around the mean value $v = 2u$: this is the speed of the medium as a whole. Thus, the (ξ,τ)-frame moves with the speed $\sqrt{gh_0}$, and there, long linear waves are at rest. Our wave propagates with the speed u (because we were looking for such solutions); in the laboratory system of reference, this gives us $\sqrt{gh_0} + u$, and relative to the medium, $\sqrt{gh_0} + u - 2u = \sqrt{gh_0} - u$. The reduction in speed by u is caused by dispersion: larger u corresponds to the higher oscillation frequency of the fictitious particle, and hence, to smaller wavelength and stronger effect of the dispersion.

Oscillation amplitude of the fictitious particle corresponds directly to the amplitude of the wave. By increasing the amplitude, the fictitious particle starts to "feel" a departure of the potential from a parabolic one; what we obtain are called cnoidal waves. For a really large amplitude, the fictitious particle spends considerable time in the vicinity of the plateau at the origin, hence the cnoidal waves have extended and flat bottom. At the limit case, the flat bottom becomes infinitely long; it is easy to see that this corresponds to the full energy of the fictitious particle being equal to zero. In that case, we have only a solitary wave, the shape of which can be found from the conservation of the fictitious particle's energy,

$$\frac{1}{2}\left(\frac{dv}{d\zeta}\right)^2 + \frac{1}{6}v^3 - \frac{1}{2}uv^2 = 0, \quad (47)$$

hence

$$\frac{d\tilde{v}}{\tilde{v}\sqrt{1-\tilde{v}}} = d\tilde{\zeta}, \quad \tilde{v} = \frac{v}{3u}, \quad \tilde{\zeta} = \zeta\sqrt{u}. \quad (48)$$

If we denote $\tilde{v} = \mathrm{ch}^{-2}w$, we obtain $d\tilde{v} = -2\frac{\mathrm{sh}w}{\mathrm{ch}^3 w}\cdot dw$, which leads us to $dw = d\tilde{\zeta}$, hence

$$v = 3u\,\mathrm{ch}^{-2}\zeta\sqrt{u}. \quad (49)$$

This solution appears to be very stable: even if it interacts with other perturbations, it always continues ultimately with the same shape and speed u (see below); such solutions of nonlinear wave equations are called solitons.

G. Lagrangian of the KdV equation

It appears that KdV equation has infinite number of integrals of motion. We are not going to write down all of these, but we'll do this at least to the first three ones, which follow from the symmetries of the equation. In order to be able to apply the Noether theorem, we need to write the KdV equation via a Lagrangian. Therefore we need to find such $L = L(v, \frac{\partial v}{\partial\tau})$ that KdV equation is the condition of minimality of the action $S = \int L d\tau$. It appears that such a function does, indeed, exist:

$$L = \frac{1}{2}\int\left[\frac{\partial\varphi}{\partial\tau}\frac{\partial\varphi}{\partial\zeta} + \frac{1}{3}\left(\frac{\partial\varphi}{\partial\zeta}\right)^3 - \left(\frac{\partial^2\varphi}{\partial\zeta^2}\right)^2\right]d\zeta, \quad (50)$$

where $\varphi = \int^\zeta v(\zeta')d\zeta'$ is the velocity potential. The variation of the contribution of the first term to the action S gives us

$$\frac{1}{2}\int dt\int d\zeta\left(\frac{\partial\delta\varphi}{\partial\tau}\frac{\partial\varphi}{\partial\zeta} + \frac{\partial\varphi}{\partial\tau}\frac{\partial\delta\varphi}{\partial\zeta}\right) =$$

$$\frac{1}{2}\int dt\int d\zeta\left(-\delta\varphi\cdot\frac{\partial}{\partial\tau}\frac{\partial\varphi}{\partial\zeta} - \frac{\partial}{\partial\zeta}\frac{\partial\varphi}{\partial\tau}\cdot\delta\varphi\right) =$$

$$-\int dt\int d\zeta\delta\varphi\cdot\frac{\partial v}{\partial\tau}.$$

The variation of the other terms can be obtained in a similar way; the overall result is

$$\delta S = -\int\int d\tau d\zeta \left(\frac{\partial v}{\partial \tau} + v\frac{\partial v}{\partial \xi} + \frac{\partial^3 v}{\partial \xi^3} \right) \delta\varphi, \qquad (51)$$

i.e. the KdV equation is, indeed, the condition of minimality for the given Lagrangian. This Lagrangian does not depend on time, so energy

$$E = \int d\zeta \frac{\delta L}{\delta \dot{\varphi}} \cdot \dot{\varphi} - L = \int d\zeta \left[-\frac{v^3}{3} + \left(\frac{\partial v}{\partial \zeta} \right)^2 \right] \qquad (52)$$

is conserved. Similarly, Lagrangian does not depend on ζ, so the momentum is also conserved,

$$P = -\int d\zeta \frac{\delta L}{\delta \dot{\varphi}} \cdot \frac{\partial \varphi}{\partial \zeta} = \int v^2 d\zeta. \qquad (53)$$

Finally, the Lagrangian does not depend on a shift $\varphi \to \varphi + \varphi_0$, which results in the conservation law

$$M = -\int d\zeta \frac{\delta L}{\delta \dot{\varphi}} = \int v d\zeta. \qquad (54)$$

Note that the conservation of these intergrals (E, P, and M) of motion can be verified without Lagrangian formalism, by taking time derivative and using the KdV equation.

H. Inverse scattering method: basic idea

REFERENCES

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