WoPhO Selection Round Problem 1
Induction Motor
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(a), (b) Let the time $t = 0$ be chosen such that then $\mathbf{n}$ is perpendicular to $\mathbf{B}$. Using this notation, the flux of the coil is $\Phi = -BAN \sin(\Omega - \omega)t$. According to Faraday's law, the induced voltage is $U = -\partial\Phi/\partial t = BAN(\Omega-\omega)\cos(\Omega-\omega)t$. To simplify the calculations, we change now to the exponential description of alternating current, that is, $\mathcal{U} = BAN(\Omega - \omega)\exp(i(\Omega - \omega)t)$. The impedance of the coil is (the angular frequency of the voltage is $\Omega - \omega$) $\mathcal{Z} = R + i(\Omega - \omega)L$, thus the current is

$$\mathcal{I} = \frac{\mathcal{U}}{\mathcal{Z}} = \frac{BAN(\Omega-\omega)}{\sqrt{R^2 + (\Omega-\omega)^2L^2}} \exp(i(\Omega-\omega)t - i\delta) \to I = \frac{BAN(\Omega-\omega)}{Z}\cos((\Omega-\omega)t - \delta),$$

where $Z = \sqrt{R^2 + (\Omega-\omega)^2L^2}$ is the magitude and $\delta = \arctan\frac{(\Omega-\omega)L}{R}$ is the argument of $\mathcal{Z}$ (in the end we have reverted back to the trigonometric form). The magnetic moment of the coil is $\mathbf{m} = IAN\mathbf{n}$, thus the torque exerted on it is

$$\tau = (\mathbf{m}\times\mathbf{B})_z = mB\cos(\Omega-\omega)t = NAB\cdot\frac{BAN(\Omega-\omega)}{Z}\cos((\Omega-\omega)t-\delta)\cos(\Omega-\omega)t =$$
$$= \frac{B^2A^2N^2(\Omega-\omega)}{Z}\left(\cos^2\alpha\cos\delta - \sin\alpha\cos\alpha\sin\delta\right),$$

where $\alpha(t) = (\Omega-\omega)t$. We should note that

$$\lim_{t\to\infty}\left(\frac{1}{t}\int_0^t \cos^2(x\cdot t^*)\mathrm{d}t^*\right) = \frac{1}{2} \text{ and } \lim_{t\to\infty}\left(\frac{1}{t}\int_0^t \sin(x\cdot t^*)\cos(x\cdot t^*)\mathrm{d}t^*\right) = 0$$

are the time averages of the above functions, thus the requested time average is

$$T = \frac{B^2A^2N^2(\Omega-\omega)}{Z}\frac{\cos\delta}{2} = \frac{B^2A^2N^2(\Omega-\omega)}{Z}\frac{R}{2Z} = \frac{B^2A^2N^2(\Omega-\omega)R}{2(R^2+(\Omega-\omega)^2L^2)} = \frac{B^2A^2N^2R\Omega s}{2(R^2+\Omega^2L^2s^2)};$$

in the last form we have introduced the expression of the slip. For the case of the small slip, we may obtain a simpler formula by calculating with s to the first order:

$$T_{(a)} = \frac{B^2A^2N^2R\Omega s}{2R^2} = \frac{B^2A^2N^2\Omega s}{2R}.$$

(c) By introducing $\beta = L\Omega/R$, we may simplify the expression for T:

$$T = \frac{B^2A^2N^2R\Omega s}{2(R^2 + R^2\beta^2s^2)} = \frac{B^2A^2N^2\Omega}{2R}\frac{s}{1+\beta^2s^2}.$$

Taking the numerical prefactor describing the motor equal to 1, we have to sketch the function $s/(1+\beta^2s^2)$ for $\beta = 10$. The plot for the reasonable interval is in Fig. 1.

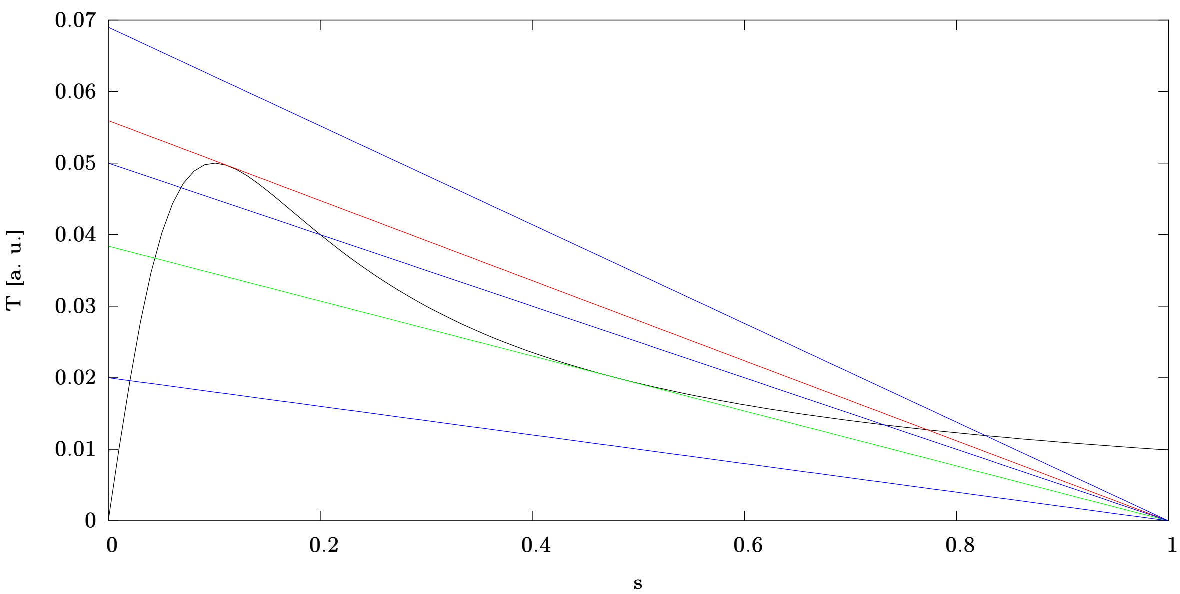

Figure 1. T (in the units of $B^2A^2N^2\Omega/2R$) as a function of s and some load characteristics.

(d) The maximum of T occurs where its derivative with respect to s (or equivalently, that of $s/(1+\beta^2 s^2)$) is 0. This derivative is $\frac{1-\beta^2 s^2}{(1+\beta^2 s^2)^2}$, the root of which is

$$s_0 = \frac{1}{\beta} = \frac{R}{L\Omega}.$$

In this case, the average torque is

$$T_{\max} = \frac{B^2 A^2 N^2 \Omega}{2R} \frac{1/\beta}{2} = \frac{B^2 A^2 N^2 \Omega}{4R} \frac{R}{L\Omega} = \frac{B^2 A^2 N^2}{4L}.$$

Note. If $\beta < 1$, i.e. the ohmic resistance is too large, s_0 would be greater than 1, which is nonsense. In this case, the $T(s)$ function is growing monotonously all over the reasonable $0 \le s \le 1$ range, thus the maximal torque belongs to $s_0 = 1$ and it is $T_{\max} = \frac{B^2 A^2 N^2 \Omega}{2R(1+L^2\Omega^2/R^2)}$.

(e) The energy loss in the system is the Joule heat dissipating in the coil. In average this is given by

$$\overline{P_{\text{loss}}} = I_{\text{eff}}^2 R = \frac{1}{2} I_{\text{peak}}^2 R = \frac{R}{2}\left(\frac{BAN(\Omega-\omega)}{Z}\right)^2.$$

The useful power is the mechanical power of the motor, which is given for rotations by $P_{\text{mech}} = \tau\omega$, or in average,

$$\overline{P_{\text{mech}}} = T\omega = \frac{B^2 A^2 N^2 (\Omega-\omega)\omega R}{2Z^2}.$$

The efficiency of the motor is thus

$$\eta = \frac{P_{\text{mech}}}{P_{\text{mech}} + P_{\text{loss}}} = \frac{\omega}{\Omega} = 1 - s.$$

(f) Let s_0 be the root of the equation $T(s) = K\omega = K\Omega(1-s)$: at this slip, the motor is working without accelerating or decelerating the load. Imagine that the slip gets perturbated, so it value becomes $s_0 + \delta s$. The system is stable when the total torque acting on it acts in order to reduce δs, that is, if $\delta s > 0$, thus $\delta\omega < 0$, the torque accelerates the system, thus $\Sigma T > 0$. This total torque is given by

$$\Sigma T = T(s_0 + \delta_s) - K\Omega(1 - s_0 - \delta s) = \left(\left.\frac{\partial T}{\partial s}\right|_{s=s_0} + K\Omega\right)\delta s \;\rightarrow\; \frac{\Sigma T}{\delta s} = \left.\frac{\partial T}{\partial s}\right|_{s=s_0} + K\Omega,$$

which must be greater than 0 to have the system stable. On the monotonously growing section of $T(s)$, the expression is clearly greater than 0, thus this section is stable, but the first stable section exceeds slightly this part of the graph: the critical state is the one where $\Sigma T/\delta s$ is exactly 0. The related equations are

$$\frac{B^2 A^2 N^2 \Omega}{2R} \frac{s_0}{1+\beta^2 s_0^2} = K\Omega(1-s_0); \qquad \frac{B^2 A^2 N^2 \Omega}{2R} \frac{1-\beta^2 s_0^2}{(1+\beta^2 s_0^2)^2} + K\Omega = 0.$$

Combining the equations give the irreducible cubic equation $s_0(1+\beta^2 s_0^2) = (1-s_0)(\beta^2 s_0^2 - 1)$. The positive numerical solutions for $\beta = 10$ are $s_1 = 0.1138$ and $s_2 = 0.4781$. We have already found that for $s < s_1$ the system is stable, consequently, for $s_1 < s < s_2$ the system is unstable and for $s_2 < s$ it is again stable, as at the roots of the above equation the continuous function $\Sigma T/\delta s$ changes sign.

Lines for some interesting K are plotted in Fig. 1. as well. Let $K_2 < K_1$ be the two values of K which constitute $s_0 = s_1$ and $s_0 = s_2$, respectively (green and red lines). We can find that for $K < K_2$ there is a single stable operating point with $s < s_1$ and for $K > K_1$ there is a single stable operating point with $s > s_2$. When $K_2 < K < K_1$, we can find three operating points: for one of them, $s < s_1$, for one $s > s_2$ and for one $s_1 < s < s_2$: the first two are thus stable, the last is unstable. If the system gets perturbated at the latter, the slip will begin to oscillate about and converge to one of the stable operating points.

(g) In the case of a negative slip, the motor is braking, but the efficiency rises over 1: this means, that rotating the rotor over an angular velocity of Ω takes mechanical energy, but it is gained back as electrical power. In practice, this is the principle of induction (or asynchronous) generators.