WoPhO Selection Round Problem 4
Atmospheric evaporation by Jeans escape
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Part 1. Exobase height

(a) According to the ideal gas law: $pV = NkT \to n_V(h) = N/V = p(h)/kT$. The mean free path is consequently $\lambda(h) = kT/\sigma p(h)$, specially, for the height of 250 km, $p(h) = 21\ \mu\text{Pa}$ and $T = 1000$ K, thus $\lambda = 3.288$ km.

(b) By definition, $\lambda(h_{\text{EB}}) = H$; according to the above expression, $kT/\sigma p(h_{\text{EB}}) = H$, so $p(h_{\text{EB}}) = kT/\sigma H = 1.151\ \mu\text{Pa}$. It follows from the provided pressure function that $p_{\text{ref}}/p(h_{\text{EB}}) = \exp\left(\frac{h_{\text{EB}}-h_{\text{ref}}}{H}\right)$, thus $h_{\text{EB}} = h_{\text{ref}} + H\ln\left(p_{\text{ref}}/p(h_{\text{EB}})\right) = 424.2$ km.

Part 2. Atmospheric escape flux

(a) By integration with respect to φ between $-\pi$ and π and with respect to θ between 0 and π (these are the domains of the respective angular variables) we get the following distribution for the modulus of the velocity:

$$w(v)\mathrm{d}v = \left(\frac{m}{2\pi kT}\right)^{3/2} \exp\left(-\frac{mv^2}{2kT}\right) v^2 \mathrm{d}v \cdot 2\pi \cdot 2 = \sqrt{\frac{2m^3}{\pi k^3 T^3}} \exp\left(-\frac{mv^2}{2kT}\right) v^2 \mathrm{d}v.$$

The probability of the velocity being more than a given value V is the integral $\int_{v=V}^{\infty} w(v)\mathrm{d}v$. Consequently, the requested probability is

$$P = \sqrt{\frac{2m^3}{\pi k^3 T^3}} \int_{v=v_{\text{esc}}}^{\infty} \exp\left(-\frac{mv^2}{2kT}\right) v^2 \mathrm{d}v = 2.589 \cdot 10^{-3},$$

using the numeric values of $v_{\text{esc}} = \sqrt{2GM_{\text{Earth}}/(R_{\text{Earth}} + h_{\text{EB}})} = 1.084 \cdot 10^4$ m/s, $m = 1.008$ u, $k = 1.381 \cdot 10^{-23}$ J/K and $T = 1000$ K.

(b) Consider a surface element of $\mathrm{d}A$ the normal vector of which is radial. It is easy to see that if a gas particle has a velocity greater than the escape velocity, it will escape, regardless of its direction, having it has a radial component pointing upwards. This is true since in the exosphere we may ignore any collision that would turn the particle back.

Now we calculate the volume out of which a particle with velocity $\mathbf{v}$ can go through $\mathrm{d}A$ is a short time $\mathrm{d}t$. It is clearly a prism, the vector of the generator of which is $\mathbf{v}\mathrm{d}t$. The altitude of such a prism is $v\mathrm{d}t \cdot \cos\theta$ if θ is the angle of $\mathbf{v}$ to the radial upward direction (i.e. we choose the $+z$ direction to point radially upwards), therefore the volume of this prism is $\mathrm{d}^2V = v\cos\theta \cdot \mathrm{d}A\mathrm{d}t$. The particles of velocity $\mathbf{v}$ will go through the chosen area segment during $\mathrm{d}t$ out of such a volume.

The number of hydrogen atoms in this volume is $\mathrm{d}^2N = n_{\text{H}} v\cos\theta \cdot \mathrm{d}A\mathrm{d}t$ and according to the Maxwellian distribution, the number of atoms having a velocity vector $\mathbf{v}$ among them is:

$$\mathrm{d}^5N_{\mathbf{v}} = \mathrm{d}^2N \cdot f(\mathbf{v})\mathrm{d}^3\mathbf{v} = \left(\frac{m}{2\pi kT}\right)^{3/2} \exp\left(-\frac{mv^2}{2kT}\right) v^2 \sin\theta \mathrm{d}v\mathrm{d}\theta\mathrm{d}\varphi \cdot n_{\text{H}} v\cos\theta \mathrm{d}A\mathrm{d}t =$$

$$= n_{\text{H}} \left(\frac{m}{2\pi kT}\right)^{3/2} \exp\left(-\frac{mv^2}{2kT}\right) v^3 \mathrm{d}v \cdot \sin\theta\cos\theta\mathrm{d}\theta \cdot \mathrm{d}\varphi \cdot \mathrm{d}A\mathrm{d}t.$$

Integration with respect to θ between 0 and $\pi/2$ (since only the upward moving particles are interesting now) and with respect to φ between $-\pi$ and π gives the following number-of-atoms differential:

$$\mathrm{d}^3N_v = n_{\text{H}} \left(\frac{m}{2\pi kT}\right)^{3/2} \exp\left(-\frac{mv^2}{2kT}\right) v^3 \mathrm{d}v \cdot \frac{1}{2} \cdot 2\pi \cdot \mathrm{d}A\mathrm{d}t = n_{\text{H}} \sqrt{\frac{m^3}{8\pi k^3 T^3}} \exp\left(-\frac{mv^2}{2kT}\right) v^3 \mathrm{d}v\mathrm{d}A\mathrm{d}t.$$

So many atoms of velocity between v and $v + \mathrm{d}v$ will leave through the area $\mathrm{d}A$ in time $\mathrm{d}t$. Therefore, the differential of the flux, which is the number of atoms going through unit surface in unit time is

$$\mathrm{d}\Phi = \frac{\mathrm{d}^3N_v}{\mathrm{d}A\mathrm{d}t} = n_{\text{H}} \sqrt{\frac{m^3}{8\pi k^3 T^3}} \exp\left(-\frac{mv^2}{2kT}\right) v^3 \mathrm{d}v.$$

The flux of escaping atoms is the integral of this flux differential between v_{esc} and ∞:

$$\Phi = n_{\text{H}} \sqrt{\frac{m^3}{8\pi k^3 T^3}} \int_{v=v_{\text{esc}}}^{\infty} \exp\left(-\frac{mv^2}{2kT}\right) v^3 \mathrm{d}v = 7.504 \cdot 10^{11}\ \frac{1}{\text{m}^2\text{s}}$$

with the same numeric values as above.

Part 3. Evaporation of the atmosphere

(a) The force acting on the surface of the Earth by the atmosphere is $F = P_0 \cdot A = P_0 \cdot 4\pi R_{\text{Earth}}^2$ which obviously comes from the gravity acting on the atmosphere: $F = mg$, so $m = 4\pi R_{\text{Earth}}^2 P_0/g$. The number of molecules in the atmosphere is therefore $N = N_{\text{A}} \cdot m/M_{\text{air}} = 1.115 \cdot 10^{44}$, so the number of hydrogen molecules is $N_{\text{H2}} = \chi_{\text{H}} N$, thus the number of hydrogen atoms is

$$N_{\text{H}} = 2\chi_{\text{H}} \cdot N = 1.226 \cdot 10^{38}.$$

(b) As the concentration of hydrogen atoms won't change at the exobase, the flux of escaping atoms is constant as well; for the total surface of Earth, the number of escaping atoms in unit time is $I_{\text{H}} = \Phi \cdot 4\pi(R_{\text{Earth}} + h_{\text{EB}})^2 = 4.35 \cdot 10^{26}$ 1/s, so the time required for the evaporation of half of the hydrogen atmosphere is

$$\tau_{\text{H}} = \frac{N_{\text{H}}/2}{I_{\text{H}}} = 1.41 \cdot 10^{11} \text{ s} = 4465 \text{ years}.$$

(c) Doing the integral as in Part 2b, but with the data $n_{\text{He}} = 2.5 \cdot 10^{12}$ m^{-3} and $m_{\text{He}} = 4.0026$ u we find $\Phi_{\text{He}} = 2.190 \cdot 10^4 \ \frac{1}{\text{m}^2\text{s}}$, thus $I_{\text{He}} = \Phi_{\text{He}} \cdot 4\pi(R_{\text{Earth}} + h_{\text{EB}})^2 = 1.27 \cdot 10^{19}$ 1/s. The number of helium atoms is $N_{\text{He}} = N\chi_{\text{He}} = 5.574 \cdot 10^{38}$. The atmospheric half-life of helium is thus

$$\tau_{\text{He}} = \frac{N_{\text{He}}/2}{I_{\text{He}}} = 2.19 \cdot 10^{19} \text{ s} = 6.95 \cdot 10^{11} \text{ years}.$$

Remark. The assumption that the concentration of atoms remains constant over the disappearing of a large part of those atoms is really weird. It's more realistic to assume that the hydrogen concentration is proportional to the total number of hydrogen atoms. In this case, if the number of atoms at a moment is N, the intensity of evaporation is $I = N\frac{I_{\text{H}}}{N_{\text{H}}}$: it is easy to see, that this intensity is the negative time derivative of N, so we get the following differential equation:

$$\dot{N} = -N\frac{I_{\text{H}}}{N_{\text{H}}} \to N(t) = N(0)\exp\left(-\frac{I_{\text{H}}}{N_{\text{H}}}t\right),$$

thus the half-life is

$$\tau_{\text{H}} = \ln 2\frac{N_{\text{H}}}{I_{\text{H}}} = 1.95 \cdot 10^{10} \text{ s} = 6190 \text{ years for hydrogen and}$$

$$\tau_{\text{He}} = \ln 2\frac{N_{\text{He}}}{I_{\text{He}}} = 3.04 \cdot 10^{19} \text{ s} = 9.63 \cdot 10^{11} \text{ years for helium.}$$

We can see that the half-life in this case differs by a factor of $2\ln 2 = 1.38$, which is totally acceptable since with the roughly given input data we can only give an order-of-magnitude estimate and the sought effect is of 9 orders of magnitude, way larger than this factor.