WoPhO Selection Round Problem 5
Motion in the electrostatic field of a dipole
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Part 1. Motion of the dipole

1 The torque acting on an electric dipole is given in the vector form by $\boldsymbol{\tau} = \mathbf{p} \times \mathbf{E}$, where $\mathbf{p}$ is the moment of the dipole, $\mathbf{p} = q\mathbf{d}$ and $\mathbf{E}$ is the electric field at the centre of the dipole. Let θ be the angle between the line joining the point charge to the centre of the dipole and $\mathbf{p}$. Using this notation, the magnitude of torque becomes $\tau = -Eqd\sin\theta = -\frac{1}{4\pi\epsilon_0}\frac{Q}{L^2}qd\sin\theta$ which equals to $I\ddot{\theta}$, where $I = 2m(d/2)^2 = md^2/2$ is the moment of inertia. Using the standard small-angle approximations the equation of rotation becomes

$$-\frac{1}{4\pi\epsilon_0}\frac{Qqd}{L^2}\theta = \frac{md^2}{2}\ddot{\theta}$$

$$\ddot{\theta} = -\frac{1}{2\pi\epsilon_0}\frac{Qq}{mdL^2}\theta.$$

This is a harmonic equation, the solution of which is a harmonic oscillation with angular frequency $\omega = \sqrt{\frac{1}{2\pi\epsilon_0}\frac{Qq}{mdL^2}}$, the period is therefore

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{2\pi\epsilon_0\frac{L^2md}{Qq}}.$$

2 As the force acting on each point charges are in the line joining them to the charge Q, there is no torque exerted on the dipole with respect to that charge. Consequently, the angular momentum with respect to the fixed point charge is constant, I'll denote it by N as L is used already. Initially, $N = 2m\cdot Lu$, at a distance r, $N = 2m\cdot rv_{t1}$: equating them gives

$$v_{t1} = \frac{Lu}{r}.$$

The electric potential energy of the dipole is given by (using first-order approximations in d)

$$W(r) = \frac{1}{4\pi\epsilon_0}\left(\frac{Qq}{r+d/2} - \frac{Qq}{r-d/2}\right) = -\frac{Qq}{4\pi\epsilon_0}\frac{d}{r^2}.$$

The conservation of energy between the initial and the examined state:

$$-\frac{Qq}{4\pi\epsilon_0}\frac{d}{L^2} + \frac{1}{2}\cdot 2mu^2 = -\frac{Qq}{4\pi\epsilon_0}\frac{d}{r^2} + \frac{1}{2}\cdot 2mv_{t1}^2 + \frac{1}{2}\cdot 2mv_{n1}^2$$

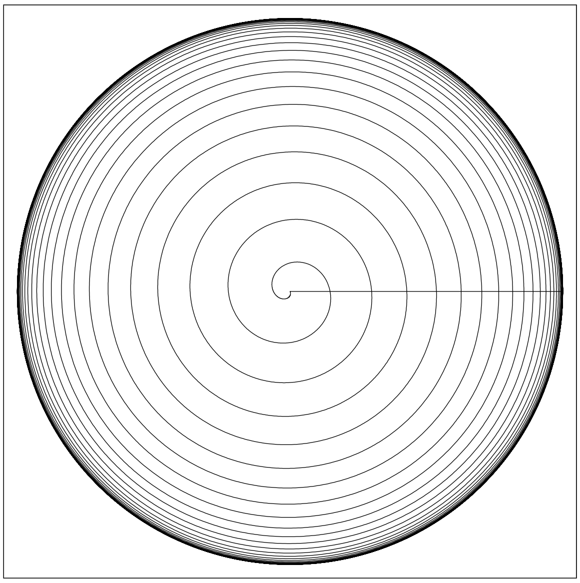
$$\frac{mL^2u^2 - \frac{Qqd}{4\pi\epsilon_0}}{L^2} = \frac{mL^2u^2 - \frac{Qqd}{4\pi\epsilon_0}}{r^2} + mv_{n1}^2$$

$$v_{n1} = \sqrt{\left(L^2u^2 - \frac{Qqd}{4\pi\epsilon_0 m}\right)\left(\frac{1}{L^2} - \frac{1}{r^2}\right)}.$$

3 If $r < L$, the second factor in the above expression is negative; as the square root must real, the first factor must be negative as well, that is, $L^2u^2 - \frac{Qqd}{4\pi\epsilon_0 m} < 0$. The critical value of u is then

$$v_{\text{cr}} = \sqrt{\frac{Qqd}{4\pi\epsilon_0 mL^2}}.$$

4 Initially, the distance won't change, but as radiation effects start to take place, the energy of the system will decrease: we may interpret it as u becoming lower, lower than v_{cr}. This means, that the dipole diverts from the circular orbit and as the radial velocity becomes larger with r becoming smaller, the distance will converge to 0 with an increasing speed: this is shown in the sketch.



5 As in the case $u < v_{\text{cr}}$ r must be lower than L, the radial velocity will point towards the fixed charge, i.e. $v_{n1} = -\dot{r}$. Substituting the expression for v_{n1} into this differential equation:

$$\frac{\mathrm{d}r}{\mathrm{d}t} = -\sqrt{L^2(v_{\text{cr}}^2 - u^2)\left(\frac{1}{r^2} - \frac{1}{L^2}\right)} = \sqrt{(v_{\text{cr}}^2 - u^2)\frac{L^2 - r^2}{r^2}}$$

$$-\frac{r\,\mathrm{d}r}{\sqrt{L^2 - r^2}} = \sqrt{v_{\text{cr}}^2 - u^2}\,\mathrm{d}t$$

Integrating both sides:

$$\left[\sqrt{L^2 - r^2}\right]_{r=L}^{r_1} = t\sqrt{v_{\text{cr}}^2 - u^2}$$

$$t_1 = \sqrt{\frac{L^2 - r_1^2}{v_{\text{cr}}^2 - u^2}} = \frac{\sqrt{3}}{2}\frac{L}{\sqrt{v_{\text{cr}}^2 - u^2}}.$$

(I have used the given $r_1 = L/2$ value in the last step. Note that the dipole will get to the point charge in a finite time, $L/\sqrt{v_{\text{cr}}^2 - u^2}$, this reasons that in the previous sketch the dipole eventually reaches the fixed charge.)

Part 2. Motion about the fixed dipole

1 The distance between the charge and the respective charges of the dipole using first-order approximations in d:

$$r_- = \sqrt{r^2 + \frac{d^2}{4} - dr\cos\theta} = r - \frac{d\cos\theta}{2}; \qquad r_+ = \sqrt{r^2 + \frac{d^2}{4} + dr\cos\theta} = r + \frac{d\cos\theta}{2};$$

using these expressions we obtain the formula of the electric potential:

$$\varphi = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{r + d\cos\theta/2} - \frac{1}{r - d\cos\theta/2}\right) = -\frac{q}{4\pi\epsilon_0}\frac{d\cos\theta}{r^2}.$$

2 $\mathbf{E} = -\nabla\varphi$; using the cylindrical expression of the gradient:

$$E_n = -\frac{\partial\varphi}{\partial r} = -\frac{q}{2\pi\epsilon_0}\frac{d\cos\theta}{r^3}; \qquad E_t = -\frac{1}{r}\frac{\partial\varphi}{\partial\theta} = -\frac{q}{4\pi\epsilon_0}\frac{d\sin\theta}{r^3}.$$

3

$$\tau = rF_t = rQE_t = -\frac{Qqd}{4\pi\epsilon_0}\frac{\sin\theta}{r^2}.$$

4 The angular momentum of the charge with respect to the centre of the dipole is $N = 2mrv_{t2} = 2mr^2\omega = 2mr^2\dot{\theta}$. The time derivative of $N^2/2$ is

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}N^2\right) = N\dot{N} = N\tau = -2mr^2\dot{\theta}\cdot\frac{Qqd}{4\pi\epsilon_0}\frac{\sin\theta}{r^2} = -\frac{Qqdm}{2\pi\epsilon_0}\sin\theta\cdot\dot{\theta} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{Qqdm}{2\pi\epsilon_0}\cos\theta\right).$$

Integrating both sides with respect to time gives

$$\frac{1}{2}N^2 = \frac{Qqdm}{2\pi\epsilon_0}\cos\theta + C.$$

Using the condition $N = 2m \cdot Lu$ at $\theta = 0$ we get $C = 2m^2L^2u^2 - \frac{Qqdm}{2\pi\epsilon_0}$ and $N^2/2 = 2m^2r^2v_{t2}^2$ gives

$$2mr^2v_{t2}^2 = 2m^2L^2u^2 + \frac{Qqdm}{2\pi\epsilon_0}(\cos\theta - 1)$$

$$v_{t2} = \sqrt{\frac{L^2u^2}{r^2} + \frac{Qqd}{4\pi\epsilon_0 mr^2}(\cos\theta - 1)}$$

5 Using the conservation of energy:

$$\frac{1}{2}\cdot 2mu^2 - \frac{Qqd}{4\pi\epsilon_0 L^2} = -\frac{Qqd}{4\pi\epsilon_0}\frac{\cos\theta}{r^2} + \frac{1}{2}\cdot 2m\left(\frac{L^2u^2}{r^2} + \frac{Qqd}{4\pi\epsilon_0 mr^2}(\cos\theta - 1)\right) + \frac{1}{2}\cdot 2mv_{n2}^2$$

$$mu^2 - \frac{Qqd}{4\pi\epsilon_0 L^2} = -\frac{Qqd\cos\theta}{4\pi\epsilon_0 r^2} + \frac{mL^2u^2}{r^2} + \frac{Qqd\cos\theta}{4\pi\epsilon_0 r^2} - \frac{Qqd}{4\pi\epsilon_0 r^2} + \frac{1}{2}\cdot 2mv_{n2}^2$$

$$\frac{mL^2u^2 - \frac{Qqd}{4\pi\epsilon_0}}{L^2} = \frac{mL^2u^2 - \frac{Qqd}{4\pi\epsilon_0}}{L^2} + \frac{1}{2}\cdot 2mv_{n2}^2$$

$$v_{n2} = \sqrt{\left(L^2u^2 - \frac{Qqd}{4\pi\epsilon_0 m}\right)\left(\frac{1}{L^2} - \frac{1}{r^2}\right)} = v_{n1}.$$

6 As I have noted at the end of the previous part, $v_{n1} = v_{n2}$, thus the differential equation to be written down is the same in the two cases as well, like the solutions. Therefore,

$$t_2 = t_1 = \frac{\sqrt{3}}{2}\frac{L}{\sqrt{v_{\mathrm{cr}}^2 - u^2}},$$

where $v_{\mathrm{cr}} = \sqrt{\frac{Qqd}{4\pi\epsilon_0 mL^2}}$ as in Part 1.

Part 3. Circular motion

1 We may use the formula for v_{t2} derived in Part 2 as the rod doesn't exert any torque upon the point charge. Taking the restriction $r = L$ into account gives for the tangential (and thus total) velocity

$$v(\theta) = \sqrt{u^2 + \frac{Qqd}{4\pi\epsilon_0 mL^2}(\cos\theta - 1)}.$$

It's easy to see that the greater is $\cos\theta$ the greater is v, therefore the maximal velocity belongs to $\theta = 0$ and is $v_{\max} = u$. The minimal velocity is bit more difficult to find. If $v(\pi) = \sqrt{u^2 - \frac{Qqd}{2\pi\epsilon_0 mL^2}}$ is real then the charge will reach this position with minimal $\cos\theta$ and this expression gives the minimal velocity. However, if the above expression is imaginary (i.e. $u^2 - \frac{Qqd}{2\pi\epsilon_0 mL^2}$ is negative) then the charge won't reach the mentioned state but turns back halfway. In this case, $v_{\min}$ is 0. With a small mathematical trick we can get a unified expression for $v_{\min}$:

$$v_{\min} = \Re\left(\sqrt{u^2 - \frac{Qqd}{2\pi\epsilon_0 mL^2}}\right).$$

2 The centripetal acceleration of the point charge is $a = -v^2/L$: this acceleration must be provided by the normal electric force and the rod force:

$$-2m\frac{v^2}{L} = QE_n + N$$

$$N = -2m\frac{u^2 + \frac{Qqd}{4\pi\epsilon_0 mL^2}(\cos\theta - 1)}{L} + \frac{Qq}{2\pi\epsilon_0}\frac{d\cos\theta}{L^3} = \frac{Qqd}{2\pi\epsilon_0 L^3} - \frac{2mu^2}{L},$$

which is independent of θ.

3 If the force needed to hold the charge on course is 0, then it will be able to move along it without the rod. The condition is thus $N = 0$, from which,

$$u_c = \sqrt{\frac{Qqd}{4\pi\epsilon_0 mL^2}} = v_{\mathrm{cr}},$$

being not too surprising knowing the found similarities of the motion.

We need to consider that $u_c^2 - \frac{Qqd}{2\pi\epsilon_0 mL^2} = -\frac{Qqd}{4\pi\epsilon_0 mL^2} < 0$, thus the charge cannot go along a full circle. Solving the equation

$$v(\theta) = \sqrt{u_c^2 + \frac{Qqd}{4\pi\epsilon_0 mL^2}(\cos\theta - 1)} = 0$$

gives $\cos\theta = 0$, thus $\theta = 90°$. Therefore, the point charge will initially bounce back and forth between $\theta = \pm 90°$. Due to radiation, the object will lose energy, this can be taken into account by slightly decreasing u: in the equation

$$v(\theta)r = \sqrt{L^2 u_c^2 + \frac{Qqd}{4\pi\epsilon_0 m}(\cos\theta - 1)} = 0$$

this will result in a slight decreasing of $(\cos\theta - 1)$ and so that of θ. As u gets smaller than u_c, the charge is going to approach the dipole faster and faster (compare with Part 2.6). These specialities of the motion (the decreasing of θ may be exaggerated) are indicated in the sketch.