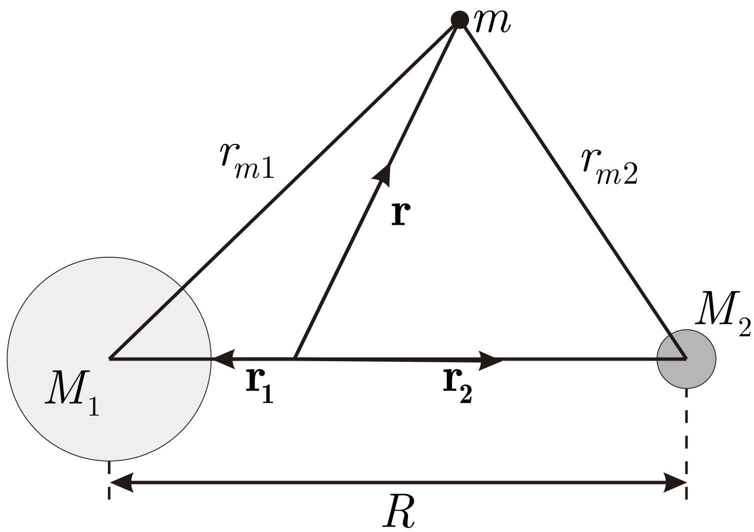
Lagrange Points
Stability

In a system that rotates with the Earth around the Sun, there are five equilibrium points (where the sum of the forces is zero). These 5 points are known as Lagrange Points (named after Joseph Lagrange, the first person to study this three-body system). Exact analysis of this system is very complicated and chaotic. In the following problem, the mass of the two bodies (M_1 and M_2) are taken to be much larger than that of the third body (m). The distance between M_1 and M_2 is taken to be R.



1. **Basic equations of the system**
   (a) Write down the vector of the total gravitational forces $\mathbf{F_g}$ on m.
   (b) By assuming $M_1, M_2 >> m$, determine the angular velocity of the M_1 and M_2 system (Ω).
   (c) In a frame that rotates with the system, there are fictitious forces on m. Write down the vector of the total forces on this mass ($\mathbf{F_\Omega}$) in this frame.
   (d) Choose a coordinate system where the three masses are in the xy-plane and the angular velocity Ω is in the positive z-axis. The center of the coordinate is set at the center of mass of M_1 and M_2 on the x-axis. Write the position of m as $\mathbf{r} = x(t)\hat{i} + y(t)\hat{j}$. In this rotating frame, write down the total forces on m in the x- and y-axis using parameter $\alpha = \frac{M_2}{M_1+M_2}$ and $\beta = \frac{M_1}{M_1+M_2}$ when the velocity of m is zero.

2. **Identifying Lagrange Points**
   There are 5 points with zero net forces in this rotating system. Three of them (call them L_1, L_2 and L_3) lie on the line connecting M_1 and M_2 (the x-axis) and the other two (call them L_4 and L_5) lie on the xy-plane on symmetric positions above and below the x-axis; that is, $y_4 = -y_5$.

(a) First consider the case of finding the position of L_1, L_2 and L_3. Use $x = (\nu - \alpha)R$, with ν the distance of m from M_1 in units of R. Write down the equation of force that must be satisfied to identify these points. Express this equation in terms of ν and α.

(b) The equation above gives rise to three cases (each for L_1, L_2 and L_3) to consider, $\nu < a$, $a < \nu < b$ and $b < \nu$. Determine the values of a and b.

From here on, we will also assume that α is small (in the Earth-Sun system, α is 3.0×10^{-6}). Use only the lowest order non-zero term in α, ignore all higher order terms in α. The following three questions will help you determine the three Lagrange points on the x-axis.

(c) For the first case, $\nu < a$, write $\nu = -1 + \delta_1$ with δ_1 a small positive number that depends on α. This value of ν will determine the position of the first Lagrange point at $x = -R(1 + \xi_1)$. Determine ξ_1 as a function of α.

(d) For the second case, $a < \nu < b$, write $\nu = 1 - \delta_2$ with δ_2 a small positive number that depends on α. This value of ν will determine the position of the second Lagrange point at $x = R(1 - \xi_2)$. Determine ξ_2 as a function of α.

(e) For the third case, $b < \nu$, write $\nu = 1 + \delta_3$ with δ_3 a small positive number that depends on α. This value of ν will determine the position of the third Lagrange point at $x = R(1 + \xi_3)$. Determine ξ_3 as a function of α.

Determining the fourth and fifth Lagrange points requires a more complicated method. First decompose the gravitational force on m into components parallel and perpendicular to the vector $\mathbf{r}$.

(f) Find the unit vector parallel to the vector $\mathbf{r}$, $\hat{\mathbf{e}}_{\parallel}$. Find also the unit vector perpendicular to the vector $\mathbf{r}$ on the xy-plane, $\hat{\mathbf{e}}_{\perp}$.

(g) Find the component of the force on m parallel to the vector $\mathbf{r}$, $F_{\Omega}^{\parallel}$, and find the component perpendicular to the vector $\mathbf{r}$, $F_{\Omega}^{\perp}$.

(h) Specify the condition that must be satisfied by the force component perpendicular to the vector $\mathbf{r}$ in order that mass m be in equilibrium. With this condition, determine the relation between r_{m1} and r_{m2}.

(i) Specify the condition that must be satisfied by the force component parallel to the vector $\mathbf{r}$ in order that mass m be in equilibrium. With this equation, determine the relation between r_{m1} and R.

(j) Now determine the position of the fourth Lagrange point (x_4, y_4) and the fifth Lagrange point (x_5, y_5).

3. **Lagrange Point Stability**
To test the stability of these Lagrange points, small perturbation are given to the mass m around its equilibrium points. Because the forces in this system depend on the

position (x, y) and the velocity (v_x, v_y) of the mass m, the restoring forces must be calculated for variations in position and velocity. Expand the total force as follows:

$$F_x(x_0+\delta x, y_0+\delta y, v_{x,0}+\delta v_x, v_{y,0}+\delta v_y) = \frac{\partial F_x}{\partial x}\delta x + \frac{\partial F_x}{\partial y}\delta y + \frac{\partial F_x}{\partial v_x}\delta v_x + \frac{\partial F_x}{\partial v_y}\delta v_y$$
$$F_y(x_0+\delta x, y_0+\delta y, v_{x,0}+\delta v_x, v_{y,0}+\delta v_y) = \frac{\partial F_y}{\partial x}\delta x + \frac{\partial F_y}{\partial y}\delta y + \frac{\partial F_y}{\partial v_x}\delta v_x + \frac{\partial F_y}{\partial v_y}\delta v_y.$$

This force has taken into account the contribution of the velocity of the mass m. All the partial derivatives are evaluated at the point $(x_0, y_0, v_{x,0}, v_{y,0})$.

(a) Write down the general form for $\frac{1}{m}\frac{\partial F_x}{\partial x}$, $\frac{1}{m}\frac{\partial F_x}{\partial y}$, $\frac{1}{m}\frac{\partial F_y}{\partial x}$, $\frac{1}{m}\frac{\partial F_y}{\partial y}$. Show that $\frac{\partial F_y}{\partial x} = \frac{\partial F_x}{\partial y}$.

(b) Calculate $\frac{1}{m}\frac{\partial F_x}{\partial v_x}$, $\frac{1}{m}\frac{\partial F_x}{\partial v_y}$, $\frac{1}{m}\frac{\partial F_y}{\partial v_x}$, $\frac{1}{m}\frac{\partial F_y}{\partial v_y}$.

These eight coefficients should act as a restoring constant (analog to the spring constant). Now we are ready to check the stability of the five Lagrange points. Consider only the lowest order term in α, ignore all higher order terms.

(c) The first Lagrange Point

   i. Show that $\frac{1}{m}\frac{\partial F_x}{\partial x} = c_1\Omega^2$. Determine c_1.

   ii. Show that $\frac{\partial F_y}{\partial x} = \frac{\partial F_x}{\partial y} = 0$.

   iii. Show that $\frac{1}{m}\frac{\partial F_y}{\partial y} = c_2\alpha\Omega^2$. Determine c_2.

   iv. By substituting $\delta x = Ae^{\lambda t}$ and $\delta y = Be^{\lambda t}$, with A and B nonzero, determine λ as a function of α and Ω only.

   v. There are four solutions to λ. Write down the condition that these solutions must satisfy in order that the first Lagrange point is stable and then determine the stability of this point.

   vi. For the Earth-Sun system α is 3.0×10^{-6} and Ω is 2π/year. If this point is stable, determine its period of oscillation (in days), if not, determine its time constant $1/\lambda$ (in days also).

(d) The second Lagrange Point

   i. Show that $\frac{1}{m}\frac{\partial F_x}{\partial x} = c_3\Omega^2$. Determine c_3.

   ii. Show that $\frac{\partial F_y}{\partial x} = \frac{\partial F_x}{\partial y} = 0$.

   iii. Show that $\frac{1}{m}\frac{\partial F_y}{\partial y} = c_4\Omega^2$. Determine c_4.

   iv. By substituting $\delta x = Ae^{\lambda t}$ and $\delta y = Be^{\lambda t}$, with A and B nonzero, determine λ as a function of α and Ω only.

   v. There are four solutions to λ. Write down the condition that these solutions must satisfy in order that the second Lagrange point is stable and then determine the stability of this point.

vi. For the Earth-Sun system: if this point is stable, determine its period of oscillation (in days), if not, determine its time constant $1/\lambda$ (in days also).

The third Lagrange point is similar to the second Lagrange point hence it need not be considered.

(e) The fourth Lagrange Point

i. Show that $\frac{1}{m}\frac{\partial F_x}{\partial x} = c_5\Omega^2$. Determine c_5.

ii. Show that $\frac{1}{m}\frac{\partial F_y}{\partial x} = \frac{1}{m}\frac{\partial F_x}{\partial y} = (c_6 + c_7\alpha)\Omega^2$. Determine c_6 and c_7.

iii. Show that $\frac{1}{m}\frac{\partial F_y}{\partial y} = c_8\Omega^2$. Determine c_8.

iv. By substituting $\delta x = Ae^{\lambda t}$ and $\delta y = Be^{\lambda t}$, with A and B nonzero, determine λ as a function of α and Ω only.

v. Define $M_1/M_2 = \xi$. Find the range of value of ξ for the fourth Lagrange point to be stable.

The fifth Lagrange point has the same behavior as the fourth Lagrange point, hence it need not be considered.