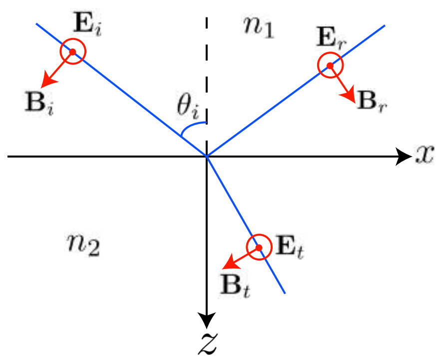
Dielectric Slab Waveguide

1 Total Internal Reflection

The electric field of a polarized monochromatic plane wave can be generally represented as $\mathbf{E}(\mathbf{r}, t) = \mathbf{E} \exp i(\mathbf{k}.\mathbf{r} - \omega t)$, where $\mathbf{E}$ is the amplitude of the wave, k the wavenumber, and ω the frequency. Suppose that a monochromatic plane wave with frequency ω travels in the medium of refractive index n_1, and is incident on the boundary of another medium of refractive index n_2. The incoming wave forms an angle θ_i with respect to the normal of the boundary. Throughout this problem, we only consider transverse electric (TE) polarized wave where the electric field is perpendicular to the plane of incidence and all media are non-magnetic.

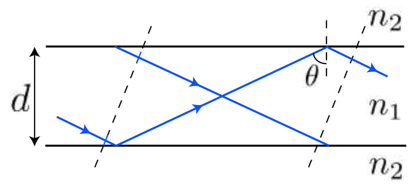


1. In the case of $n_1 > n_2$, there exists a critical angle θ_c where the incoming wave will be totally reflected for $\theta_i > \theta_c$ (total internal reflection or TIR). The phase of the reflected wave lags by δ with respect to the incident wave. Derive δ and state it in terms of n_1, n_2, and θ_i.

2. Using the necessary boundary conditions, derive the reflectance R for the case of TIR. Show that the wave is perfectly reflected for all $\theta_i > \theta_c$.

2 Constructive Phase Matching

The most simple dielectric waveguide is a planar slab with thickness d and refractive index n_1 located in a homogeneous background medium with refractive index n_2 ($n_2 < n_1$). In the case of TIR, the slab can be used to guide waves without loss, with the additional condition that the waves interfere constructively. In other words, the wavefronts should be preserved as the waves travel inside the waveguide. The wavenumbers for the vacuum, medium n_1, and medium n_2 are taken to be k_0, k_1, and k_2, respectively.



1. Find the necessary condition for the constructive phase matching.

2. The wave can only be guided without loss for certain values of θ. Show that in these cases, θ must satisfy the equations:

$$k_1 d \cos\theta - \delta = m\pi; \qquad m = 0, 1, 2, 3, ... \tag{1}$$

Verify that the equations above can also be written as:

$$\sqrt{u^2 + v^2} = \frac{k_0 d}{2}\sqrt{n_1^2 - n_2^2}, \tag{2}$$

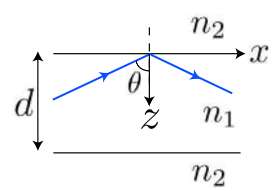
$$u\ \tan u = v \qquad \text{or} \qquad -u\ \cot u = v, \tag{3}$$

with $u = \frac{k_1 d}{2}\cos\theta$ and $v = \frac{d}{2}\sqrt{k_1^2\ \sin^2\theta - k_2^2}$.

3 Maxwell's Equations

The Maxwell wave equation for the electric field in a dielectric medium of relative permittivity ε is

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\mathbf{E}(\mathbf{r}, t) = \mu_0 \varepsilon \varepsilon_0 \frac{\partial^2 \mathbf{E}(\mathbf{r}, t)}{\partial t^2}. \tag{4}$$



In the case of the slab waveguide shown in the figure above, $\varepsilon = n_1^2$ for $0 < z < d$, and $\varepsilon = n_2^2$ for $z < 0$ or $z > d$. Taking the system coordinates such that the wave travels in the xz-plane, the electric field can be generally written as

$$\mathbf{E}(\mathbf{r}, t) = \mathbf{E}(x, z, t) = \mathbf{E}(z) \exp i(\beta x - \omega t), \tag{5}$$

where β is the effective propagation constant along the waveguide due to the translational symmetry of the structure in the x-direction. In the case of waveguiding the TE polarized wave ($\mathbf{E}(z) = E(z)\hat{\mathbf{y}}$), $\mathbf{E}(\mathbf{r}, t)$ should be simple harmonic inside the slab and decay exponentially outside.

1. What is the relation of β to k_1 and θ?

2. From the boundary conditions at $z = 0$ and $z = d$, derive from the Maxwell equations the condition for waveguiding as found in Part 2.

4 Mode Solutions

The waveguide mode solutions are solutions of θ where waveguiding occurs inside the slab. The solution for $m = 0$ (see Part 2) is commonly called the fundamental mode (the lowest mode or the first mode), the $m = 1$ mode is called as the second mode, and so on.

1. Sketch curves in (u, v) coordinates that represent Eqs. (2)-(3). Determine the necessary condition for only one mode solution to exist.

2. Show that the maximum number of modes supported by the dielectric slab is

$$M = \left\lceil \frac{k_0 d}{\pi} \sqrt{n_1^2 - n_2^2} \right\rceil, \tag{6}$$

where the $\lceil\rceil$ symbol denotes the ceiling function for which the expression inside is increased to the nearest integer.

3. Verify that the number of mode solutions is incremented by one for every increase of frequency:

$$\Delta\omega = \frac{\pi c}{d\sqrt{n_1^2 - n_2^2}}. \tag{7}$$

4. From eq.1, show that the group velocity $(d\omega/d\beta)$ of each supported mode solution is

$$v_g = \frac{d \tan\theta + \frac{\partial\delta}{\partial\beta}}{\frac{n_1 d}{c\cos\theta} - \frac{\partial\delta}{\partial\omega}}. \tag{8}$$

5. Show that the maximum time disparity for different modes in the dielectric slab waveguide to travel a distance L is

$$\tau = \frac{L}{c}(n_1 - n_2). \tag{9}$$

6. For $n_1 = 1.7$, $n_2 = 1.5$, $\lambda = 800$ nm (in vacuum), and $d = 1$ μm, find all the mode solutions for θ (with $\theta > \theta_c$). Plot the electric field $E(z)$ for these solutions.