WoPhO A Model for Collisions Between Two Solid Objects

One way mechanical energy is lost during collision between two solid objects is by way of acoustic waves propagating inside both objects. Even though the real situation is quite complicated, in this problem it is modeled in a simple way. First treat each solid rod as a spring with unstretched length L_l and L_r. Spring constant times the length of the spring are then K_l and K_r, respectively. Mass density (mass per unit length) of the spring are given by ρ_l and ρ_r. Indexes l and r represent the left and right springs.

The left spring is moving with velocity $+v_0/2$, while the right spring is moving in the opposite direction with velocity $-v_0/2$. Each spring is initially relaxed. At $t = 0$, the springs collide at $x = 0$. The displacement of each point on the springs are described by a function $y(x, t)$ so that $x + y(x, t)$ represents the position of said point at time t; this point was at x when $t = 0$.

1. Derive the wave equation on the springs and write down the speed of the wave.

The general solution to the wave equation is given by $y(x, t) = \psi(ct - x) + \phi(ct + x)$, where c is the speed of the wave propagation. The form of ψ and ϕ are determined from the boundary conditions.

2. Write down the boundary conditions at $x = 0$, $x = -L_l$ and $x = L_r$.

3. Write down the function of $y(x, t)$ before the collision ($t \leq 0$), i.e. $y_{0,l}(x, t)$ and $y_{0,r}(x, t)$.

At $t = 0$, an acoustic wave starts to propagate in both springs away from the collision point $x = 0$. The dynamics of the system is analyzed using the space-time diagram as shown in Fig. 1. The horizontal axis represents time, and the vertical axis represents the position of points on the spring. Each line in the diagram represents an acoustic wave front that emerges each time a wave front arrives at the border.

For example, the line AB represents the position of the wave front emerging from the collision at point A ($x = 0$) as a function of time. Let functions $f_l(c_l t + x)$ and $f_r(c_r t - x)$ describe the waves emerging from the collision that propagates in the left and right springs, where c_l and c_r are the speed of the wave propagation in left and right spring, respectively. The space-time diagram indicates that $L_l/c_l > L_r/c_r$ in this problem. As the wave front of $f_r(c_r t - x)$ arrives at point B, a new reflected wave, $g_r(c_r t + x)$, emerges. The same event also occurs in the left spring at point C.

Now back in the right spring, when the wave front of $g_r(c_r t + x)$ arrives at the end of the spring ($x = 0$, at point D in the diagram), a new reflected wave $h_r(c_r t - x)$ and a new transmitted wave $h_l(c_l t + x)$ are generated. These phenomena always occur when a wave front arrives at the border; a new reflected wave or new reflected and transmitted waves are generated.

4. Write down the wave function $y(x, t)$ in the region I, II, III, IV, V, VI and VII in terms of y_0, f_r, f_l, g_r, h_r and h_l.

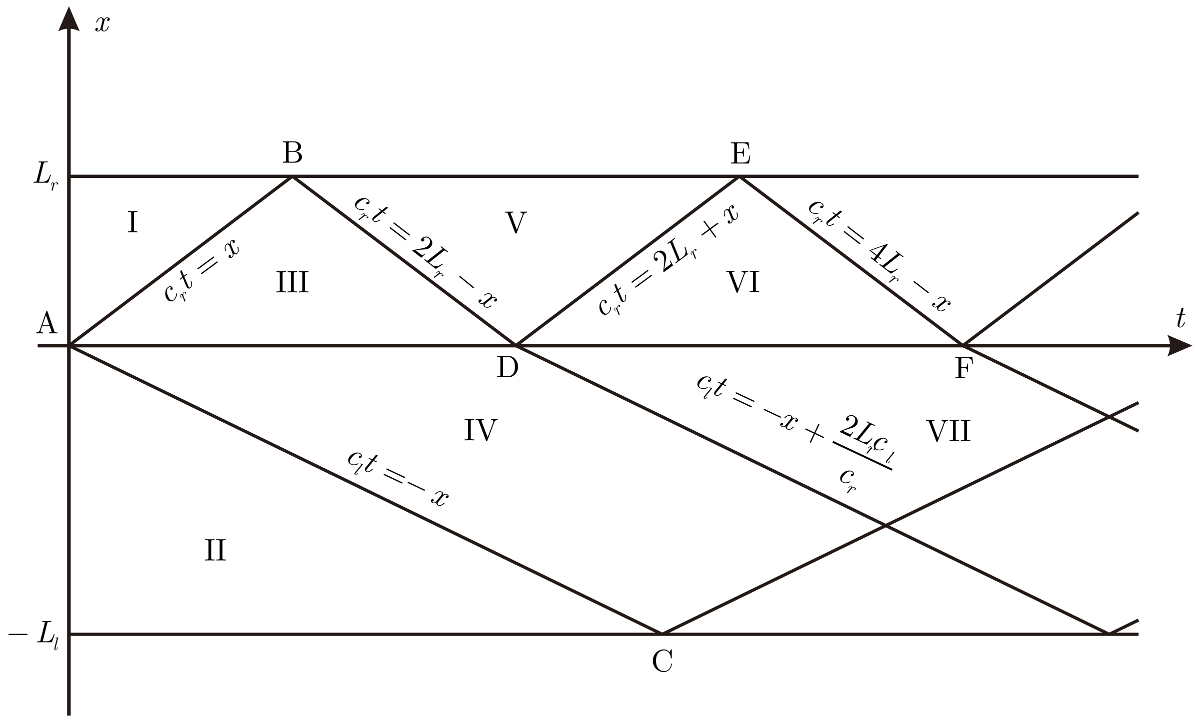

Figure 1: Space-time diagram

5. Using the boundary condition(s), determine the form of $f_r(c_r t - x)$ and $f_l(c_l t + x)$ in terms of the springs' properties and initial velocity.
6. Determine the velocity of the contact point ($x = 0$) immediately after the initial contact.
7. Using the boundary condition(s), determine the form of $g_r(c_r t + x)$ in terms of the springs' properties and initial velocities.

Now consider a case where both springs are identical except in its length. In this case, $\rho_l = \rho_r = \rho$, $K_l = K_r = K$. Take $L_r < L_l$.

8. Determine $y(x,t)$ in region III and IV. Draw a graph for $y(x)$ at $t = 0.4\frac{L}{c}$. For drawing the graph, you may use $L_r = 0.6L$, $L_l = L$ and $v_0 = 0.5c$.
9. Determine $y(x,t)$ in region V. Draw a graph for $y(x)$ at $t = 0.8\frac{L}{c}$, use the same L_r, L_l and v_0 as in the previous question.
10. When will the two springs separate? Draw a graph for $y(x)$, use the same L_r, L_l and v_0 as in the previous question.
11. Calculate the coefficient of restitution e between the springs.
12. Calculate the ratio of the translational kinetic energy of the springs after the collision to the kinetic energy before the collision.