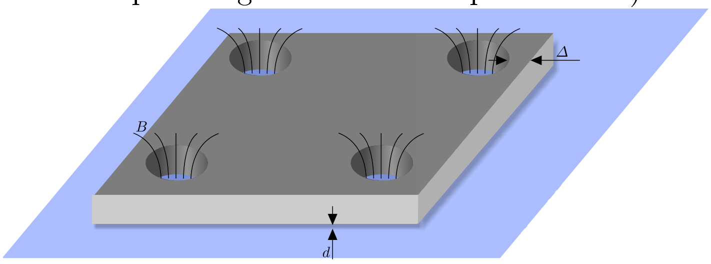
Problem 10: magnetic levitation

A rectangular superconducting plate of mass m has four identical circular holes, one near each corner, see figure. Each hole carries a certain magnetic flux (all the four fluxes are equal and of the same polarity). The plate is put on a horizontal surface which is also in a superconducting state. The magnetic push between the plate and the surface compensates the weight of the plate when the width of the air gap beneath the plate is d, which is much smaller than the distance between the plate's and holes' edges (denoted by Δ in figure); d is also much smaller than the radii of the holes.

When the plate levitates in such a way above the support, the frequency of its small vertical oscillations is ν_0. Next, a load of mass M is put on the plate, so that the load lays on the plate, and the plate levitates above the support. What is the new frequency ν of small vertical oscillations (when the load and plate together oscillate up and down)?



Hints after the first week. Due to the IPhO in Copenhagen, there are no hints this week (many contestants are attending IPhO and will not have internet access this week). Publication of the solutions of Problem 9 is delayed due to the same reason by one week.

Hints after two and half weeks. Study the solution of the Problem 1, Part C, question ii from the 43rd IPhO.

Hints after the third week. Find the repulsive force as a function of distance d: first calculate the magnetic field energy as a function of d, and further apply the virtual displacement method.

Hints after the fourth week. Let us notice that between the plates, B-field is horizontal, i.e. perpendicular to the vertical axis z (why?), and independent of the z-coordinate (why?). The value of the $B(x,y)$-field can be found using the Gauss law. In the neighbourhood of a circular hole, this is an easy task, because the contribution of the remote holes can be neglected (hence, the problem becomes cylindrically symmetric). The problem becomes more complicated in the region where the distance to all the holes has the same order of magnitude. Luckily, there is no need for us to solve that problem exactly, because the shape of the field-lines between the plates is independent of the distance between the plates (as long as $d \ll \Delta$); try to motivate this! Now, suppose that for a certain value of $d = d_0$, there is a certain dependence $B_0(x,y)$ (which we don't know). Then, using the Gauss law, we can express $B(x,y)$ for any value of d ($d \ll \Delta$) in terms of $B_0(x,y)$, d_0, and d. Hence, we can also express the magnetic field energy in terms of $B_0(x,y)$, d_0, and d, which makes it possible to find the force $F = F(d)$ using the virtual displacement method.

Correct results have been submitted by:

1. Oliver Edtmair (Austria), 2.594 pts
2. Dan-Cristian Andronic (Romania), 2.358 pts
3. **Szabó Attila** (Hungary), 2.752 pts
4. **Jakub Safin** (Slovakia), 2.502 pts
5. **Joao Victor De Oliveira Maldonado** (Brazil), 2.047pts; 2.047 pts
6. Madhivanan Elango (United Kingdom), 1.288 pts
7. Jordan Jordanov (Bolgaria), 1.171 pts
8. **Andres Põldaru** (Estonia) 1.367 pts
9. Kuo Pei-Cheng (Taiwan), 1.21 pts
10. Ly Nguyen (Vietnam), 0.704 pts
11. Jaemo Lim (Korea), 0.64 pts
12. Vũ Việt Linh (Vietnam), 0.513 pts
13. David Stein (Germany) 1 pt
14. Nguyen Ho Nam (Vietnam) 0.8 pts
15. Cyuan-Han Chang (Taiwan) 0.64 pts
16. Jan Ondras (Slovakia) 0.513 pts
17. Ng Fei Chong (Malaysia), 1 pt
18. Lars Dehlwes (Germany), 1.1 pts
19. Ismael Salvador Mendoza Serrano (Mexico), 0.792 pts
20. Péter Juhász (Hungary) 1 pts
21. Gema Ilham Baskara Darman (Indonesia). 1 pt

Bold font marks the recipients of the best solution award, $e^{1/4}$ each; these best solutions are attached below in the arrival order. Please note that the solution of Maldonado includes a simple arithmetic error at the very end which earned him also a penalty factor. The best-solution-awards were given to those solutions which made more or less convincing efforts showing that the result is independent on the specifics of the geometry (e.g. the exact positions of the holes). However, solutions submitted during the fifth week were not eligible because the last hints provided too detailed outline of a good solution.

Overall scores for the Problems 6-10:

Oliver Edtmair (Austria)	13.766
Szabó Attila (Hungary)	10.675
Dan-Cristian Andronic (Romania)	8.738
Lars Dehlwes (Germany)	6.742
Jakub Safin (Slovakia)	6.369
Madhivanan Elango (United Kingdom)	5.445
Jordan Jordanov (Bulgaria)	5.409
David Stein (Germany)	5.356
Kuo Pei-Cheng (Taiwan)	4.890
Kurenkov Mikhail (Russia)	4.849
Jaemo Lim (Korea)	4.776
Ly Nguyen (Vietnam)	4.672
Ismael Salvador Mendoza Serrano (Mexico)	4.480
Cyuan-Han Chang (Taiwan)	4.367
Andres Põldaru (Estonia)	3.498
Vũ Viêt Linh (Vietnam)	3.475
Joao Victor De Oliveira Maldonado (Brazil)	3,257
Jan Ondras (Slovakia)	3.124
Nguyen Ho Nam (Vietnam)	2.743
Hrishikesh Menon (India)	2.594
Ng Fei Chong (Malaysia)	2.464
Katerina Naydenova (Bulgaria)	2.146
Efim Mazhnik (Russia)	2.105
Jakub Mrožek (Poland)	2.000
Péter Juhász (Hungary)	2.000

Please note that these scores will be normalized before adding to the points for the first half of the Selection Round (to make the scores comparable).

Please let us know, which were your favourite Selection Round problems: send an e-mail to `wopho.selection@stkipsurya.ac.id` and provide an ordered list of your favourite problems (as long as you wish).

WoPhO Selection Round Problem 10
Physics Cup – Magnetic levitation
Attila Szabó, Grade 12
Leőwey Klára High School
Pécs, Hungary

Since both the support and the plate is superconducting, magnetic field lines cannot go through them, otherwise voltage would be induced in them: for the same reason, the flux in the holes is constant. Therefore, the field lines that get under the plate at the holes will go under it to the edges: this distance is much larger than the vertical room in which the field lines can exist, thus the lines under the plate will be essentially horizontal. It follows as well that the shape of the field lines above the plate will not vary as d varies, thus the magnetic energy in this region is constant.

Now we're going to calculate, how the magnetic energy stored between the plate and the support depends on d. We should consider that this field is almost entirely horizontal and that the same flux should be distributed in the varying room: these together yield that at a specified point, $B \sim 1/d$. Therefore, at all points the magnetic energy density is proportional to $B^2 \sim 1/d^2$, thus the average magnetic energy density is proportional to $1/d^2$ as well. As the volume of the space of interest is proportional to d, the magnetic energy stored under the plate is proportional to $1/d$.

The magnetic energy (omitting the constant part that is stored above the plate) is thus $W = K/d$ where K is constant. The repulsive force is then $F = -\mathrm{d}W/\mathrm{d}d = K/d^2$. When in the first case the system is balanced, $F = mg$, from this, $d = \sqrt{K/mg}$. The net force in the small oscillation approximation is $F(d+\mathrm{d}d) - mg = K\left(1/(d+\mathrm{d}d)^2 - 1/d^2\right) = -2K/d^3 \cdot \mathrm{d}d$: using this, the frequency of the oscillations is

$$\nu_0 = \frac{\sqrt{\frac{2K/d^3}{m}}}{2\pi} = \frac{g^{3/4}m^{1/4}}{\sqrt{2}\pi K^{1/4}}.$$

Consequently, the frequency is proportional to $m^{1/4}$ where m is the mass of the load. Therefore, the new frequency is

$$\frac{\nu}{\nu_0} = \left(\frac{M+m}{m}\right)^{1/4} \to \nu = \nu_0\left(\frac{M+m}{m}\right)^{1/4}.$$

Jakub Šafin (Xellos)
WoPhO Physics Cup, problem 10

As the plate is superconducting, the field lines of the magnetic field don't penetrate into it, so the magnetic flux that enters the plate through the 4 holes is equal to the one exiting the plate from the gap. And since the gap below the plate is very thin, any field line entering one of the 4 holes will curve in 90° below the plate almost immediately, so field lines between the plate and the surface are parallel to the surface.

Now, we don't know the magnetic field distribution in the air gap between the plate and the surface, but we can assume that its change when the plate moves (in such a way that $d \ll \Delta$) will be negligible - we can imagine it as "compressingör re-scaling the field below in vertical direction.

Let's denote the field at distance $Y = yd$ (d is an immediate value) above point X on the surface as $B(y, X)$; then, the invariance of this distribution means that for a fixed point (y, X) $(B(y, X) = B_0)$ and any point (y_1, X_1) below the plate,

$$\frac{B(y_1, X_1)}{B(y, X)} = b(y_1, X_1)$$

does not depend on d.

Also, the said magnetic flux through the holes doesn't change when the plate moves up or down - if it did, voltage would be induced in the superconducting material, which is impossible. That gives a border condition: the total flux from the borders of the air gap also remains constant.

How to calculate the flux? Let's just focus on one side of the square plate with side a (the result will be the same for all 4 sides, due to symmetry). On the surface right below that side, there's a segment of length a; let's consider point $X(x)$ as the point with distance x from one fixed end of this segment. Then, we integrate the flux over a rectangle and multiply by 4, accounting for all 4 sides:

$$\Phi = 4\int_{Y=0}^{d}\int_{x=0}^{a} B(y, X(x))\mathrm{d}Y\mathrm{d}x = B_0 d\int_{y=0}^{1}\int_{x=0}^{a} 4b(y, X(x))\mathrm{d}x\mathrm{d}y = B_0 d\alpha$$

We see that in the 2nd double integral, there are no variables that depend on d, so we denote that as a constant (with respect to d) α. And for Φ to stay constant, $B_0 d$ must stay constant as well, so the field is scaled as d^{-1}.

The magnetic field in the gap causes an upward force F_m to be exerted on the plate. We'll calculate this force from work it does when lifting the disk by $\mathrm{d}d$ - this work complements the change in energy of magnetic field E_m:

$$F_m = -\frac{\mathrm{d}E_m}{\mathrm{d}d}$$

Outside of the gap, changes in magnetic field caused by small changes in d are negligible. We already know the energy density in the gap: if we use a Cartesian system, in which $X = (x, z)$ and the area on the surface below the plate corresponds to $0 <= x <= a$, $0 <= z <= a$, then the energy density at point (y, x, z)

$$e_m(y,x,z) = \frac{B^2(y,x,z)}{2\mu_0} = \frac{B_0^2}{2\mu_0} b^2(y,x,z)$$

and total energy can be obtained as an integral

$$E_m = \int_{x=0}^{a}\int_{z=0}^{a}\int_{y=0}^{1} e_m(x,y,z)\mathrm{d}Y\mathrm{d}z\mathrm{d}x = \frac{B_0^2 d}{2\mu_0}\int_{x=0}^{a}\int_{z=0}^{a}\int_{y=0}^{1} b^2(y,x,z)\mathrm{d}y\mathrm{d}z\mathrm{d}x = \frac{B_0^2 d}{2\mu_0}\beta$$

where we, once again, replace the integral, which evaluates to a constant (with respect to d), with β.

Further substituting for B_0 gives

$$E_m = \frac{B_0^2 d}{2\mu_0}\beta = \frac{\Phi^2\beta}{2\mu_0 d\alpha^2} = \frac{\gamma}{d}$$

where the expression $\frac{\Phi^2\beta}{2\mu_0\alpha^2}$ is also independent on d, so we replace it by γ.

Now, the force is

$$F_m = -\frac{\mathrm{d}}{\mathrm{d}d}\frac{\gamma}{d} = \frac{\gamma}{d^2}$$

The next step is finding equilibrium $d = d_0$, for which the net force on the plate is zero. The only forces acting on the plate are downward force $F_g = mg$ of gravity and the mentioned F_m, so if they're balanced,

$$\frac{\gamma}{d_0^2} = F_m = F_g = mg$$

$$d_0 = \sqrt{\frac{\gamma}{mg}}$$

Finally, we'll find ν_0.

For air gap of width d, the (upward) acceleration on the plate is

$$a = \frac{F_m - F_g}{m} = \frac{\gamma}{d^2 m} - g$$

and for a harmonic oscillator, squared angular frequency of those oscillations is

$$\omega_0^2 = -\left.\frac{\mathrm{d}a}{\mathrm{d}d}\right|_{d=d_0} = 2\frac{\gamma}{d_0^3 m} = \frac{2g}{d_0} = 2g\sqrt{\frac{mg}{\gamma}}$$

so the frequency

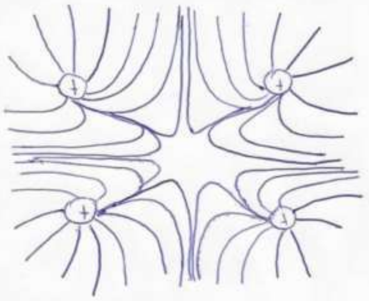
$$\nu_0 = \frac{\omega_0}{2\pi} = \frac{1}{2\pi}\sqrt{2g\sqrt{\frac{mg}{\gamma}}}$$

To find ν, we'll just imagine that the plate's mass increased to $M + m$, and subsitute m for $M + m$ in the above equation:

$$\nu = \frac{1}{2\pi}\sqrt{2g\sqrt{\frac{(m+M)g}{\gamma}}} = \nu_0\left(1 + \frac{M}{m}\right)^{\frac{1}{4}}$$

PROBLEM 10 Part 1

As $d << \Delta$, the field lines between the superconductors will be paralel to the superconductor plane.
It will be like if we put 4 magnetic monopoles and flatten the space for the field lines only to come out in 2 dimensions.
The configuration of the field lines will be the same independently of d. It will be analogous to 4 $+$ charges seen from above.



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Consider initially $d = d_1$, than we reduce d to d_2, $d_2 \approx d_1$
A force behind the plate upwards will make negative work

$$\int \frac{B^2}{2\mu_0} \cdot dV = F(d_1 - d_2)$$

But the Field behind the plate is dependent in the inverse of d, as the flux is constant at two equal positions when $d = d_1$ or d_2.

$B = \frac{\phi}{A}$ and $A \sim d$, so $B \sim \frac{1}{d}$

Also: $V \sim d$, so $\int \frac{B^2}{2\mu_0} dV \sim \left(\frac{1}{d_2} - \frac{1}{d_1}\right) \Rightarrow F \sim \frac{1}{d_1 d_2}$

PROBLEM 10

Part 2

As $d_1 \approx d_2$, $F \sim \frac{1}{d^2}$

The resultant force on the plate will be

$$F = \frac{A}{d^2} - B \xrightarrow{=} -\frac{2A(x-a)}{a^3}$$

where $B = mg$

The solution for that, when we consider $d \approx \sqrt{\frac{A}{B}} = a$

$y = x - a$; $y(0) = 0$ is

$$y = C_0 \sin\left(\sqrt{\frac{2A}{a^3 m}} \cdot t\right)$$

$\hookrightarrow$ frequency $= \frac{1}{2\pi}\sqrt{\frac{2A}{a^3 m}} = \nu$

$\nu^4 \sim \frac{A^2}{a^6 m^2} = \frac{B^3}{A m^2}$, $B = mg$ and $A =$ constant $\Rightarrow \nu^4 \sim \frac{1}{m}$

So $\nu^4 (M+m) = \nu_0^4 \cdot m \Rightarrow \boxed{\nu = \nu_0 \left(\frac{m}{M+m}\right)^{1/4}}$

The magnetic flux through the circular hole stays constant, due to the properties of a superconductor. Since the air gap is much smaller than the distance from the hole to the edge, I'll assume the magnetic field lines to be horizontal in the gap. I will also assume the shape of the magnetic field to be the same after the mass M has been put on, except of course that it is denser due to the decreased gap. I will also assume the magnetic field outside the gap to be the same in both cases.

Since the flux entering the gap is the same, the magnetic field density increases when decreasing the gap width and therefore also the energy increases. From the constant flux:

$$B_0 d_0 = Bd$$

where B is the magnetic field at some point in the gap. Since I assumed the shape of the field to be the same and that the field only compresses in the vertical direction, then the energy of the field, taking into account that the field strength is inversely proportional to d and volume is proportional to d, is:

$$E = \int \frac{B^2 dV}{2\mu_0} = \frac{k}{d}$$

where k is some constant. From the condition that the mass m is held at d_0:

$$-\frac{dE}{dd} = mg = \frac{k}{d_0^{\ 2}}$$

$$k = mgd_0^{\ 2}$$

$$F = -\frac{dE}{dd} = \frac{mgd_0^2}{d^2} = (M+m)g$$

$$\frac{1}{d} = \frac{1}{d_0}\sqrt{\frac{M+m}{m}}$$

Linearizing the force around the equilibrium point with the mass M on to get the frequency:

$$\omega^2 = 4\pi^2 f^2 = -\frac{dF}{dd}/(M+m) = \frac{2gd_0^2}{d^3}\frac{m}{M+m} = \frac{2g}{d_0}\sqrt{\frac{M+m}{m}}$$

$$\omega_0^2 = -\frac{dF}{dd}/m = \frac{2g}{d_0}$$

$$f = f_0\left(\frac{M+m}{m}\right)^{1/4}$$