Natural Structure and the Dialectic of Concept Formation

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To: categories@mta.ca
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There is another crucial dialectic making particulars (neither abstract nor concrete) give rise to an abstract general ... A mathematical model of it can be based on the hypothesis that a given set of particulars is somehow itself a category (or graph), i.e., that the appropriate ways of comparing the particulars are given but that their essence is not. Then their "natural structure" (analogous to cohomology operations) is an abstract general and the corresponding concrete general receives a Fourier-Gelfand-Dirac functor from the original particulars. That functor is usually not full because the real particulars are infinitely deep and the natural structure is computed with respect to some limited doctrine; the doctrine can be varied, or "screwed up or down" as James Clerk Maxwell put it, in order to see various phenomena.

(*Message to Categories Mailing List*)

Cohomology Operations as Natural Structure

Solving topological problems in an category of algebraic invariants.

e.g. $H : \mathsf{Top}^{\mathsf{op}} \to \mathsf{AbGrp}$

$$\begin{tikzcd} & X \arrow[dr, dashed] & \\ A \arrow[ur, hook, "i"] \arrow[rr, "f"'] & & Y \end{tikzcd} \qquad \longmapsto \qquad \begin{tikzcd} & H(X) \arrow[dl, "Hi"'] & \\ H(A) & & H(Y) \arrow[ul, dashed] \arrow[ll, "Hf"] \end{tikzcd}$$

No commuting $H(Y) \to H(X)$ implies no lift $X \to Y$.

If we do have a commuting $e : H(Y) \to H(X)$, now what?

We can look at natural transformations $\alpha : H^n \Rightarrow H$. If $\alpha_X \circ e \neq e \circ \alpha_Y$, then no lift.

These natural transformations are *cohomology operations*, and equip the groups $H(X)$ with more algebraic structure = more fine grained invariants.

Structure $\dashv$ Semantics, 1963[1]: Algebraic Structure

$\mathcal{T}$, category of algebraic (Lawvere) theories: full subcategory of $\mathcal{N}/\mathbf{Cat}$ on functors $\mathcal{N} \to \mathbb{T}$ bijective on objects and preserving finite coproducts.

$\mathcal{K}$, full subcategory of $\mathbf{Cat}/\mathrm{Set}$ on functors $X \xrightarrow{U} \mathrm{Set}$ for which all $[U^n, U]$ small.

Sem : $\mathcal{T}^{\mathrm{op}} \to \mathcal{K}$. Theory $\mapsto$ Category of models (with forgetful) $[\mathbb{T}^{\mathrm{op}}, \mathrm{Set}]_{fp} \xrightarrow{U} \mathrm{Set}$

Str : $\mathcal{K} \to \mathcal{T}^{\mathrm{op}}$. $U : X \to \mathrm{Set}$ gives unique finite coproduct preserving $\mathcal{N} \to \mathrm{Set}^X$ sending 1 to U. Take full image of this. In short: n-ary operations are $[U^n, U]$.

(1963, Theorem III.1.2) Str is left adjoint to Sem.

> *The whole semantical process has an adjoint which to almost any functor X [to S] ... assigns its unique "structure", the best approximation to X in the abstract world of S-sorted theories; the adjunction $X \to$ Sem(Str X) is a sort of closure: "the particular included in the induced concrete general".* (*Author's comments on TAC reprint of "Functorial Semantics..."*)

[1] Functorial Semantics of Algebraic Theories (1963).

What is this notion of algebraic structure? Refuting the idea that an algebraic theory can only arise syntactically, it simultaneously refutes the idea that algebraic structure constitutes a "new" or "alternative approach" or "categorical counterpart" to universal algebra. In fact, they constitute an essential feature that was long implicitly present, for example in the study of cohomology operations. ... Algebraic structure results simply from the application of the general notion of Natural Structure within the doctrine of algebraic theories.

(*Fifty Years of Functorial Semantics, Invited Lecture, 2013*)

In general the calculation of the structure of a non-representable functor seems to be hopelessly difficult, however the brilliant construction by Eilenberg and MacLane of [Eilenberg-MacLane spaces] not only paved the way for calculating cohomology operations but illustrated the importance of changing categories.

(*Author's comments on TAC reprint of "Functorial Semantics..."*)

Algebraic Structure of a Representable

For $U : X \to \mathsf{Set}$ representable $U \cong X(A, -)$, structure reduces to calculation in X:

$$\begin{aligned}
[U^n, U] \cong \ & [X(A,-)^n, X(A,-)] && \cong \quad \text{(u.p. coproduct)} \\
& [X(\sqcup_n A, -), X(A,-)] && \cong \quad \text{(Yoneda)} \\
& X(A, \sqcup_n A) \cong U(\sqcup_n A).
\end{aligned}$$

- U : Mon $\to$ Set has structure $=$ theory of monoids.
- $\mathcal{P}$: Set$^{\mathsf{op}}$ $\to$ Set has structure $=$ theory of Boolean algebras.[2]
- $\mathcal{G}_R$: Group $\to$ Set, sending a group to the underlying set of its *group ring* over R. Structure is theory of rings extended with “involution” and “trace”.[3]
- Allowing more general arities (Linton), $\mathcal{O}$: Top$^{\mathsf{op}}$ $\to$ Set has infinitary structure $=$ theory of locales, the “algebraic closure” of Top.

[2] This is (1963, Example III.1.4).

[3] This is (1963, Example III.1.3).

[Schanuel, 1982] provides an astonishing example of the concreteness of the algebraic structure of a non-representable functor, which however has not yet been exploited sufficiently in analysis ... the underlying-bornological-set functor on the category of finite dimensional noncommutative complex algebras has as its unary structure precisely the monoid of entire holomorphic functions. (*Author's comments on TAC reprint of "Functorial Semantics..."*)

Another example: $\mathcal{C}^{\mathrm{op}} \times \mathcal{C} \xrightarrow{\mathrm{Hom}} \mathrm{Set}$ for locally small $\mathcal{C}$.[4]

Note that in defining Algebraic Structure, we just use that the codomain (e.g. Set) is an "algebra of the doctrine of finite products". Can replace it with another D-algebra.

Related aside: another approach to structure[5], working in the 2-category of categories with finite products (where $I = \mathrm{FinSet}^{\mathrm{op}}$)

$$\dfrac{\dfrac{u : X \to B}{\hat{u} : I \to B^X}}{I \xrightarrow[\text{b.o.}]{} A(u) \xrightarrow[\text{f.f.}]{} B^X}$$

[4] This is (1963, Example III.1.1). Study of this is the topic of: G.M. Kelly, E. Faro, *On the canonical algebraic structure of a category* (2000).

[5] From Lawvere's Foreword to Adámek, Rosický, Vitale, *Algebraic Theories*.

Natural structure with respect to a doctrine

Ordinal Sums and Equational Doctrines (1969) gives a form of structure-semantics adjunction relative to a *doctrine* D and algebra $\mathcal{S}$ of that doctrine.

Classically, take D to be the doctrine of Lawvere theories, $\mathcal{S} = \mathsf{Set}$.

$$\mathsf{Str}_{D,\mathcal{S}} = \mathcal{S}^{(-)} : \mathsf{Cat} \to \mathsf{Alg}(D)^{\mathsf{op}}$$
$$\mathsf{Sem}_{D,\mathcal{S}} = \mathsf{Hom}_D(-, \mathcal{S}) : \mathsf{Alg}(D)^{\mathsf{op}} \to \mathsf{Cat}$$

$\mathsf{Str} \dashv \mathsf{Sem}$. Induced monad is the "dual doctrine of D", $X \mapsto \mathsf{Hom}_D(\mathcal{S}^X, \mathcal{S})$.

Unit $\eta_X : X \to \mathsf{Hom}_D(\mathcal{S}^X, \mathcal{S}) : x \mapsto \ulcorner \varphi \mapsto \varphi(x) \urcorner$.

For any functor $F : \mathcal{A} \to \mathcal{B}$, call $\mathsf{Str}_{D,\mathcal{S}}(F)$ the *natural structure with respect to* $D, \mathcal{S}$.

e.g. keeping $\mathcal{S} = \mathsf{Set}$, we can vary the doctrine: props, multicategories, etc.

Concept formation: particulars, concrete generals, abstract generals

$$\begin{tikzcd}
P \arrow[rr, "\eta_M"] \arrow[dr, "M"'] & & \mathrm{Hom}_D(\mathcal{S}^P, \mathcal{S}) \arrow[dl, "(\mathrm{Sem}\circ\mathrm{Str})M"] \\
& \mathcal{S} &
\end{tikzcd}$$

- "Given set of particulars" (P, a category or graph)
- M, measurement or observation of particulars valued in an D-algebra $\mathcal{S}$
- "Natural structure", $\mathrm{Str}(M)$ relative to D and $\mathcal{S}$, "is an abstract general"
- "Corresponding concrete general" $\mathrm{Hom}_D(\mathcal{S}^P, \mathcal{S})$
- "receives a Fourier-Gelfand-Dirac functor" η_M "from the original particulars"
- "That functor is usually not full because the real particulars are infinitely deep and the natural structure is computed with respect to some limited doctrine."[6]

[6] "Thus there is no natural place in this account for any "concrete particular"; that would be a "category mistake". (Correspondence quoted online)

Spectrum of an object via doctrine of group actions

$$\begin{tikzcd} P \arrow[rr, "\eta"] \arrow[dr, "M"'] & & \mathsf{Set}_f^{\mathsf{Aut}(M)} \arrow[dl, "U"] \\ & \mathsf{Set}_f & \end{tikzcd}$$

Let M be a "measurement". For each p we get a simple invariant, $|M(p)|$.

Structure of M w.r.t. doctrine of group actions $\Rightarrow$ abstract general $=$ the group of natural automorphisms $[M, M]$. For each p we get an action of $\mathsf{Aut}(M)$ on $M(p)$.

Class equation of a group action on a finite set: $|M(p)| = \sum_i |\mathsf{Aut}(M)|/|\mathsf{Aut}(M)_i|$. Leads to expression of each $|M(p)|$ as a "spectrum" of coefficients: a finer invariant.

Observing a domain of individuals that form a collective due to definite mutual relations, and recording these observations as structure that varies in a natural way with respect to those mutual relations, leads to the emergence of general concepts that are abstracted from all the individuals, but that may then be applicable to a larger population, and in terms of which a more precise analysis of the individuals becomes possible. (*Fifty Years of Functorial Semantics, Invited Lecture, 2013*)

Measurement of sets

The first example should be the topos of finite sets with V a three-element set. There the monad is indeed the identity, as can be seen by adapting results of Stone and Post.[7] (*Message to Categories Mailing List, 21 March 2000*)

Considering now sets and functions, $X \to V$.

V-generalized points of X: "functionals" $V^X \xrightarrow{\alpha} V$ such that $\alpha(\lambda \circ \varphi) = \lambda \circ \alpha(\varphi)$ for all $\lambda : V \to V$, $\varphi : X \to V$.

(Lawvere) When V has at least three elements, the set of V-generalized points of a *finite* X is isomorphic to X, i.e. the "Dirac" *function* $X \to \mathrm{Hom}_{V^V}(V^X, V)$ is an iso.

For V an infinite set, the isomorphism holds for all sets X only if there exist no Ulam ("measurable") cardinals (Isbell, 1960).

[7] Statements and proofs appear in (Blass, 2008) and (Leinster, 2013) (in the setting of codensity monads/ultrafilters). See also §8.4 Lawvere & Rosebrugh.

Final slide

- There is more to “structure” than reconstruction of algebraic theories from all models.
- How do isometry groups arise as abstract generals?

The leap from particular phenomenon to general concept, as in the leap from cohomology functors on spaces to the concept of cohomology operations, can be analyzed as a procedure meaningful in a great variety of contexts and involving functorality and naturality, a procedure actually determined as the adjoint to semantics and called extraction of “structure” (in the general rather than the particular sense of the word).
(Idea 6 of “Seven Ideas in Introduced the 1963 Thesis”, Author’s comments on TAC reprint)

The often repeated slander that mathematicians think “as if” they were “platonists” needs to be combatted rather than swallowed. What mathematicians and other scientists use is the objectively developed human instrument of general concepts. (The plan to misleadingly use that fact as a support for philosophical idealism may have been an honest mistake by Plato, or it may have been part of his job as disinformation officer for the Athenian CIA organization ...) *(Message to Categories Mailing List, 2001)*

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