Game equilibria as fixed-point semantics

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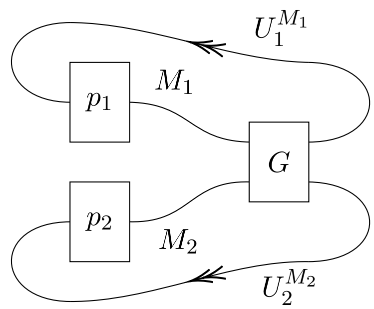
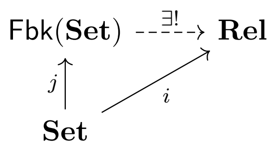
Consider a two-player game with a payoff function given by $g \colon M_1 \times M_2 \to U_1 \times U_2$, where M_1 and M_2 are the sets of possible moves for both players and U_1 and U_2 are the sets of utilities.

We define $G \colon M_1 \times M_2 \to U_1^{M_1} \times U_2^{M_2}$ to be given by the two partial evaluations of g; that is, $G(m_1, m_2) \coloneqq (g(-, m_2), g(m_1, -))$. In some sense, it outputs what any player *could do*, assuming the moves of the rest of the players are fixed. We also pick a pair of *selection functions* for the players, $p_1 \colon U_1^{M_1} \to M_1$ and $p_2 \colon U_2^{M_2} \to M_2$, which represent which move they would pick if they perfectly knew the utility they would extract from it. This idea follows previous notions of *selection function* [EO10, HOS+15]. We can now consider the following representation of the game in $\mathsf{Fbk}(\mathbf{Set})$, the free category with feedback over **Set**. [KSW02]



Every traced category is a category with feedback, and the semantics of a category with feedback in a traced category are known as *fixed-point semantics*. In particular, **Rel** is a category with feedback, and there exists a unique feedback-preserving functor making the following diagram commute. Here we call i to the inclusion of functions into relations and j to the inclusion into the free category with feedback.



Applying this functor we obtain the same diagram in **Rel**, the category of relations. The feedback loop is now given by the compact closed structure. When read in terms of regular logic, this diagram gives an equilibrium for the game.

$$\exists m_1 \in M_1, m_2 \in M_2.(p_1(f(-, m_2)) = m_1) \wedge (p_2(f(-, m_1)) = m_2).$$

When the selection function of the players is utility-maximizing and it does not need to be multivalued, this witnesses the existence of Nash equilibria. The moves can be copied to the output to obtain a subset of $M_1 \times M_2$ defining the equilibria.

References

[EO10] Martín Hötzel Escardó and Paulo Oliva. Selection functions, bar recursion and backward induction. *Math. Struct. Comput. Sci.*, 20(2):127–168, 2010.

[HOS+15] Jules Hedges, Paulo Oliva, Evguenia Sprits, Viktor Winschel, and Philipp Zahn. Higher-order game theory. *CoRR*, abs/1506.01002, 2015.

[KSW02] Piergiulio Katis, Nicoletta Sabadini, and Robert F. C. Walters. Feedback, trace and fixed-point semantics. *ITA*, 36(2):181–194, 2002.