ITERATIVE PROCESSES

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1. Iterative processes

Monoidal categories are an algebraic structure that provides operations for parallel and sequential composition. Their morphisms are conventionally understood to be processes that flow transforming an input into an output: the hom-profunctor $\mathrm{hom}\colon \mathbf{C}^{op} \times \mathbf{C} \to \mathbf{Set}$ returns the set of processes from Let a given input to a given output.

This *flow* from input to output is the standard but not the only possible one. We can use the graphical calculus of the monoidal bicategory of profunctors to depict process flows that are different from the standard one; and we can obtain their corresponding profunctors. The basic building blocks of this graphical calculus come in adjoint pairs. They are input and output sockets, joints and forks, sources and sinks. Joints and forks form pseudomonoids with sources and sinks, respectively. [2]

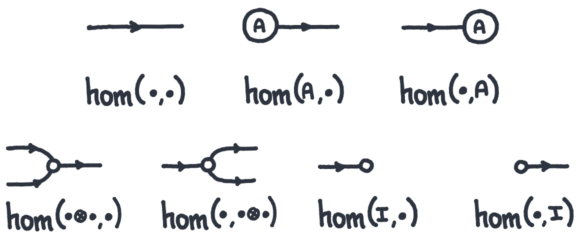

Figure 1. Blocks of the graphical calculus of the monoidal bicategory of profunctors.

For instance, a process from input to output that can access a external source of type X and produce an output of type Y is described by the following profunctor.

Figure 2. The input-ouptut profunctor φ_{XY}.

Our motivation is to describe processe that repeatedly take inputs of type X and produce outputs of type Y. We could obtain these by computing the final coalgebra of the profunctor φ_{XY}.

1.1. Definitions of combs.

Let us first introduce some definitions on repeated processes. The motivation for combs is to capture processes that go over n stages of taking inputs to outputs.

Definition 1 (Combs)**.** Let $(\mathbf{C}, \otimes, I)$ be a monoidal category. An n-comb is graphically defined with the following repeated diagram.

Alternatively, it can be defined as $\varphi_{XY}^{\diamond n}(I, I)$, using n-fold profunctor composition of φ_{XY}. Explicitly, it is given by the following coend.

$$\mathsf{Comb}^n_{XY} \coloneqq \int^{M_0,\dots,M_{n-1}} \prod_{i=0}^{n} \hom(X \otimes M_{i-1}, Y \otimes M_n), \text{ with } M_{-1}, M_n \coloneqq I$$

This n-comb represents a process with n stages. In order to describe an infinitely repeating process, we would like to take a limit that makes n go to infinity. However, a problem of this definition is that it does not allow us to construct a map $\mathsf{Comb}^{n+1}_{XY} \to \mathsf{Comb}^n_{XY}$. We would need a variant of the definition that allows for the discarding of some part of the comb.

Definition 2 (Open combs)**.** Let $(\mathbf{C}, \otimes, I)$ be a monoidal category. An *open n-comb* is graphically defined with the following repeated diagram.

Alternatively, it can be defined as $(\varphi_{XY}^{\diamond n} \diamond \mathbf{1})(I)$. Explicitly, it is given by the following coend.

$$\mathsf{oComb}^n_{XY} \coloneqq \int^{M_0,\dots,M_n} \prod_{i=0}^{n} \hom(X \otimes M_{i-1}, Y \otimes M_n), \text{ with } M_{-1} \coloneqq I.$$

The main difference between combs and open combs resides in the bounding of the last object, M_n. For open combs, it is unbounded in its covariant part, which is to say that we apply the terminal profunctor to it; for ordinary combs, we just conventionally define $M_n \coloneqq I$. This difference does not matter when the category is semicartesian.

Proposition 1 (Combs and open combs coincide in semicartesian categories)**.** *In a semicartesian category, combs and open combs coincide.*

$$\mathsf{Comb}^n_{XY} \cong \mathsf{oComb}^n_{XY}.$$

1.2. **Iterative processes as a final coalgebra.**

Definition 3. Let $(\mathbf{C}, \otimes, I)$ be a monoidal category and let X, Y be two objects. An *iterative process* from X to Y is defined as an element of the following limit.

$$\mathsf{Iter}(X, Y) \coloneqq \varprojlim_n \mathsf{oComb}^n_{XY}.$$

A better justification for the definition of an iterative process is to exhibit it as greatest fixpoint of precomposing with the input-output functor φ_{XY}. In other words, we will compute it as the final coalgebra of the functor $(\varphi_{XY} \diamond \bullet) \colon \mathsf{Prof}(\mathbf{C}, \mathbf{1}) \to \mathsf{Prof}(\mathbf{C}, \mathbf{1})$ using Adámek's theorem.

Theorem 1 (Adámek [1])**.** *Let* $\mathbf{D}$ *be a category with an initial object* 0 *and* ω*-colimits. Let* $F \colon \mathbf{D} \to \mathbf{D}$ *be an endofunctor preserving all* ω*-colimits. The initial F-algebra is the colimit of*

$$0 \xrightarrow{i} F0 \xrightarrow{Fi} F^2 0 \longrightarrow \cdots \longrightarrow F^n 0 \xrightarrow{F^n i} \cdots$$

We know that $(\varphi_{XY} \diamond \bullet) \colon \mathsf{Prof}(\mathbf{C}, \mathbf{1}) \to \mathsf{Prof}(\mathbf{C}, \mathbf{1})$ is a functor on a category with a terminal object (the terminal profunctor) that has all colimits (because it is a presheaf category) and we can also check that $(\varphi_{XY} \diamond \bullet)$ preserves them. We are in the conditions for Adámek's theorem, in its dual form.

Proposition 2. *The final* φ_{XY}*-coalgebra is the limit of*

$$\mathbf{1} \xleftarrow[Fi]{} \varphi_{XY} \diamond \mathbf{1} \longleftarrow \varphi_{XY}^{\diamond 2} \diamond \mathbf{1} \longleftarrow \cdots \longleftarrow \varphi_{XY}^{\diamond n} \diamond \mathbf{1} \longleftarrow \cdots$$

and because $\mathsf{oComb}^n_{XY} \cong (\varphi_{XY}^{\diamond n} \diamond \mathbf{1})(I)$*, this is precisely the definition of an iterative process.*

References

[1] Jiří Adámek. Free algebras and automata realizations in the language of categories. *Commentationes Mathematicae Universitatis Carolinae*, 015(4):589–602, 1974.

[2] Mario Román. Open diagrams via coend calculus. *Electronic Proceedings in Theoretical Computer Science*, 333:65–78, Feb 2021.