OPEN DIAGRAMS VIA COEND CALCULUS

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Abstract. Morphisms in a monoidal category are usually interpreted as *processes*, and graphically depicted as square boxes. In practice, we are faced with the problem of interpreting what *non-square boxes* ought to represent in terms of the monoidal category and, more importantly, how should they be composed. Examples of this situation include *lenses* or *learners*. We propose a description of these non-square boxes, which we call *open diagrams*, using the monoidal bicategory of profunctors. A graphical coend calculus can then be used to reason about open diagrams and their compositions.

1. Introduction

1.1. **Open Diagrams.** Morphisms in monoidal categories are interpreted as processes with inputs and outputs and generally represented by square boxes. This interpretation, however, raises the question of how to represent a process that does not consume all the inputs at the same time or a process that does not produce all the outputs at the same time. For instance, consider a process that consumes an input, produces an output, then consumes a second input and ends producing an output. Graphically, we have a clear idea of how this process should be represented, even if it is not a morphism in the category.

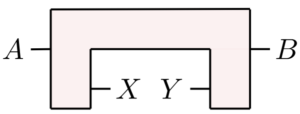

Figure 1. A process with a non-standard shape. The input A is taken at the beginning, then the output X is produced, strictly after that, the input Y is taken; finally, the output B is produced.

Reasoning graphically, it seems clear, for instance, that we should be able to *plug* a morphism connecting the first output to the second input inside this process and get back an actual morphism of the category.

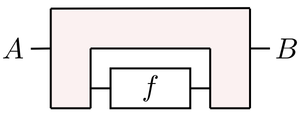

Figure 2. It is possible to plug a morphism $f\colon X \to Y$ inside the previous process (Figure 1), and, importantly, get back a morphism $A \to B$.

The particular shape depicted above has been extensively studied by [Ril18] under the name of (monoidal) *optic*; it can be also called a *monoidal lens*; and it has applications in bidirectional data accessing [PGW17, BG18, Kme18] or compositional game theory [GHWZ18]. A multi-legged generalization has appeared also in quantum circuit design

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[CDP08] and quantum causality [KU17] as a notational convention, see [Rom20]. It can be shown that boxes of that particular shape should correspond to elements of a suitable *coend* (Figure 3, see also §1.2 and [Mil17, Ril18]). The intuition for this coend representation is to first consider a tuple of morphisms, and then quotient out by the equivalence relation generated by sliding morphisms along connected wires.

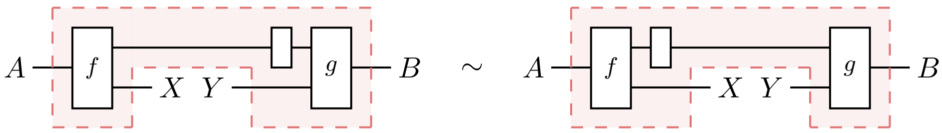


FIGURE 3. A box of this shape is meant to represent a pair of morphisms in a monoidal category quotiented out by "sliding a morphism" over the upper wire.

It has remained unclear, however, how this process should be carried in full generality and if it was in solid ground. Are we being formal when we use these *open* or *incomplete* diagrams? What happens with all the other possible shapes that one would want to consider in a monoidal category? In principle, they are not usual squares. For instance, the second of the shapes in Figure 4 has three inputs and two outputs, but the first input cannot affect the last output; and the last input cannot affect the first output.[1]

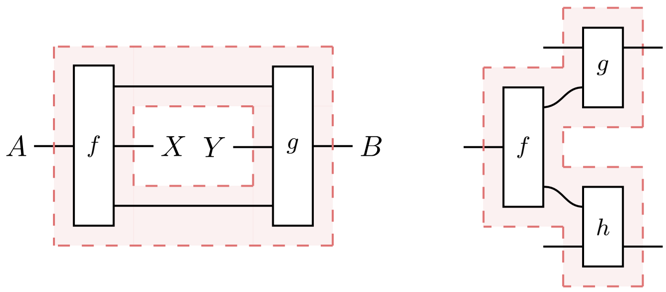


FIGURE 4. Some other shapes for boxes in a monoidal category.

This text presents the idea that incomplete diagrams should be interpreted as valid diagrams in the monoidal bicategory of profunctors; and that compositions of incomplete diagrams correspond to reductions that employ the monoidal bicategory structure. At the same time, this gives a graphical presentation of *coend calculus*.

1.2. **Coend calculus.** Coends are particular cases of colimits and *coend calculus* is a practical formalism that uses Yoneda reductions to describe isomorphisms between them. Their dual counterparts are *ends*, and formalisms for both interact nicely in a *(Co)End calculus* [Lor19].

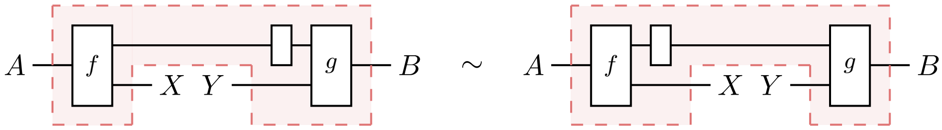
Definition 1.1. The **coend** $\int^{X \in \mathbf{C}} P(X,X)$ of a profunctor $P \colon \mathbf{C}^{op} \times \mathbf{C} \to \mathbf{Set}$ is the coequalizer of the action of morphisms on both arguments of the profunctor.

$$\int^{X \in \mathbf{C}} P(X,X) \cong \mathrm{coeq}\left(\bigsqcup_{f \colon B \to A} P(A,B) \rightrightarrows \bigsqcup_{X \in \mathbf{C}} P(X,X) \right).$$

An element of the coend is an equivalence class of pairs $[X, x \in P(X,X)]$ under the equivalence relation generated by $[X, P(f,-)(z)] \sim [Y, P(-,f)(z)]$ for each $f : Y \to X$.

[1]This particular shape comes from a question by Nathaniel Virgo on `categorytheory.zulipchat.com`.

Our main idea is to use these equivalence relations to deal with the quotienting arising in non-square monoidal boxes.



$$\int^{M} \mathbf{C}(A, M \otimes X) \times \mathbf{C}(M \otimes Y, B).$$

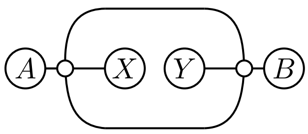
FIGURE 5. We can go back to the previous example (Figure 3) to check how it coincides with the quotienting arising from the dinaturality of a coend.

1.3. **Contributions.** Our first contribution is a graphical calculus of *shapes* of open diagrams (§2), with semantics on the monoidal bicategory of profunctors, and with an emphasis on representing monoidal structures. We show how to compose and simplify shapes (§3). Our second contribution is a graphical calculus with *open diagrams,* in terms of the category of pointed profunctors, and hinting at a pseudofunctorial analogue of *functor boxes* [Mel06] (§4).

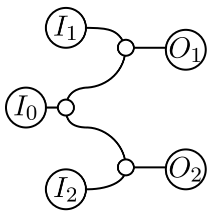
As examples, we recast the multiple ways of composing *monoidal lenses* and other coend constructions on the literature on optics (§2.3). We study categories with feedback (§2.4) and *learners* (§8.4).

2. Shapes of Open Diagrams

In the same sense that morphisms sharing the same domain and codomain are collected into an hom-set; open diagrams sharing the same *shape* will be collected into a set. Our first step is to provide a graphical calculus for these shapes and, at the same time, an interpretation that assigns a set to each shape (Figure 6).



$$\cong \int^{M,N} \mathbf{C}(A, M \otimes X \otimes N) \times \mathbf{C}(M \otimes Y \otimes N, B),$$



$$\cong \int^{M,N} \mathbf{C}(I_0, M \otimes N) \times \mathbf{C}(I_1 \otimes M, O_1) \times \mathbf{C}(N \otimes I_2, O_2).$$

FIGURE 6. The shapes of Figure 4, interpreted as sets.

2.1. **Inputs, outputs, junctions and forks.** Shapes will be interpreted in **Prof**, the monoidal bicategory of profunctors. Its 0-cells are small categories $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \dots)$; its 1-cells from $\mathbf{A}$ to $\mathbf{B}$ are profunctors $\mathbf{A}^{op} \times \mathbf{B} \to \mathbf{Set}$; and its 2-cells are natural transformations (see [Lor19, §5]). Two profunctors $P \colon \mathbf{A}^{op} \times \mathbf{B} \to \mathbf{Set}$ and $Q \colon \mathbf{B}^{op} \times \mathbf{C} \to \mathbf{Set}$ compose into a profunctor $(P \diamond Q) \colon \mathbf{A}^{op} \times \mathbf{C} \to \mathbf{Set}$ given by

$$(P \diamond Q)(A, C) \coloneqq \int^{B \in \mathbf{B}} P(A, B) \times Q(B, C).$$

The monoidal product is the cartesian product of categories, two profunctors $P_1 \colon \mathbf{A}_1^{op} \times \mathbf{B}_1 \to \mathbf{Set}$ and $P_2 \colon \mathbf{A}_2^{op} \times \mathbf{B}_2 \to \mathbf{Set}$ can be joint into the profunctor $(P_1 \otimes P_2) \colon (\mathbf{A}_1 \times \mathbf{A}_2)^{op} \times (\mathbf{B}_1 \times \mathbf{B}_2) \to \mathbf{Set}$ defined by

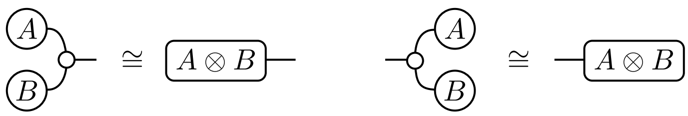
$$(P_1 \otimes P_2)(A_1, A_2, B_1, B_2) \coloneqq P_1(A_1, B_1) \times P_2(A_2, B_2).$$

The string diagrammatic calculus for monoidal bicategories has been studied by Bartlett [Bar14] expanding on a strictification result by Schommer-Pries [SP11]. It is similar to the graphical calculus of monoidal categories, with the caveat that deformations correspond to invertible 2-cells instead of equalities. For instance, arrows between diagrams on this text will denote natural transformations, which are 2-cells of the bicategorical structure of profunctors.

Definition 2.1 (Input and output ports)**.** Every object $A \in \mathbf{C}$ determines two profunctors (Ⓐ–) $\coloneqq \mathbf{C}(A, -) \colon \mathbf{1}^{op} \times \mathbf{C} \to \mathbf{Set}$ and (–Ⓐ) $\coloneqq \mathbf{C}(-, A) \colon \mathbf{C}^{op} \times \mathbf{1} \to \mathbf{Set}$ via its contravariant and covariant Yoneda embeddings.

Definition 2.2 (Junctions and forks)**.** Every monoidal category $\mathbf{C}$ has a canonical pseudomonoid structure on the monoidal bicategory **Prof** given by (⋗○–) $\coloneqq \mathbf{C}(- \otimes -, -)$ and (○–) $\coloneqq \mathbf{C}(I, -)$, and also a canonical pseudocomonoid structure given by (–○⋖) $\coloneqq \mathbf{C}(-, - \otimes -)$ and (–○) $\coloneqq \mathbf{C}(-, I)$.

Proposition 2.3. *By definition,* (Ⓘ–) $\cong$ (○–) *and* (–Ⓘ) $\cong$ (–○)*; moreover,*



In general, Yoneda embeddings are pseudofunctorial (see Proposition 8.3).

2.2. **Copying and discarding.** Shapes define sets in terms of coends, making them less practical for direct manipulation. However, shapes can be reduced to more familiar descriptions in some particular cases. For instance, if $\mathbf{C}$ is cartesian monoidal, the shape of Figure 7 reduces to a pair of morphisms $\mathbf{C}(I_0 \times I_1, O_1)$ and $\mathbf{C}(I_0 \times I_2, O_2)$. This justifies our previous intuition, back in Figure 4, that the input I_1 should not be able to affect O_2, while the input I_2 should not be able to affect O_1.

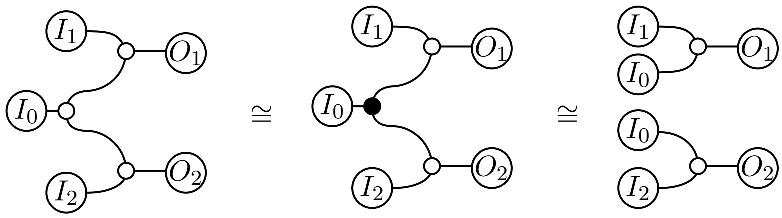


FIGURE 7. Simplifying a diagram.

Our second step is to justify some reductions like these in the cases of cartesian, cocartesian and symmetric monoidal categories. Every object of the category of profunctors has already a canonical pseudocomonoid structure lifted from **Cat** which is given by (–●⋖) $\coloneqq \mathbf{C}(-^0, -^1) \times \mathbf{C}(-^0, -^2)$ and (–●) $\coloneqq 1$, and also a pseudomonoid structure given by (⋗●–) $\coloneqq \mathbf{C}(-^1, -^0) \times \mathbf{C}(-^2, -^0)$, and (●–) $\coloneqq 1$. These two structures "copy and discard" representable and corepresentable functors, respectively (see Proposition 8.5).

Proposition 2.4 (Cartesian and cocartesian)**.** *A monoidal category is cartesian if and only if* (–○) $\cong$ (–●) *and* (–○⋖) $\cong$ (–●⋖)*, i.e. the monoidal structure coincides with the canonical one. Dually, a monoidal category is cocartesian if and only if* (⋗●–) $\cong$ (⋗○–) *and* (○–) $\cong$ (●–)*.*

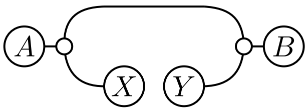
Proof. The natural isomorphism $\mathbf{C}(X, Y \otimes Z) \cong \mathbf{C}(X, Y) \times \mathbf{C}(X, Z)$ is precisely the universal property of the product; a similar reasoning holds for initial objects, terminal objects and coproducts. □

Proposition 2.5 (Symmetric monoidal)**.** *If a monoidal category* **C** *is* symmetric *then its symmetric pseudomonoid structure can be lifted from* **Cat** *to* **Prof***. We have* $\sigma\colon$ (⤫⊸) ≅ (⊸) *and* $\sigma^*\colon$ (⟜⤫) ≅ (⟜)*, dual 2-cells in the bicategory* **Prof** *that commute with unitors and associators (see also Proposition 8.4).*

2.3. Example: Lenses.

Profunctor optics and lenses have been extensively studied in functional programming [Kme18, Mil17, PGW17, BG18] for bidirectional data accessing. The theory of optics uses coend calculus both to describe how optics compose and how to reduce them in sufficiently well-behaved cases to tuples of morphisms. Categories of monoidal optics and the informal interpretation of optics as *diagrams with holes* have been studied in depth [Ril18]. We will study lenses from the perspective of the graphical calculus of **Prof**. This presents a new way of describing reductions with coend calculus that also formalizes the intuition of lenses as *diagrams with holes.*

Definition 2.6. A monoidal lens [Mil17, PGW17, Ril18, "Optic" in Definition 2.0.1] from $A, B \in \mathbf{C}$ to $X, Y \in \mathbf{C}$ is an element of the following set.

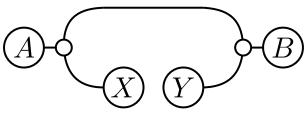


$$= \int^{M} \mathbf{C}(A, M \otimes X) \times \mathbf{C}(M \otimes Y, B)$$

For applications [FJ19, GHWZ18], the most popular case of monoidal lenses is that of cartesian lenses.

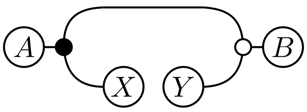
Proposition 2.7. *In a cartesian category* **C***, a lens* $(A, B) \to (X, Y)$ *is given by a pair of morphisms* $\mathbf{C}(A, X)$ *and* $\mathbf{C}(A \times Y, B)$*. In a cocartesian category, lenses are called* prisms *[Kme18] and they are given by a pair of morphisms* $\mathbf{C}(S, A + T)$ *and* $\mathbf{C}(B, T)$*.*

Proof. We write the proof for lenses, the proof for prisms is dual and can be obtained by mirroring the diagrams. The coend derivation can be found, for instance, in [Mil17].



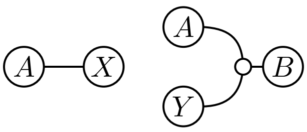
$$\int^{M} \mathbf{C}(A, M \times X) \times \mathbf{C}(M \times Y, B)$$

≅ {(⟜) ≅ (•⟜)} ≅ {Universal property of the product}



$$\int^{M} \mathbf{C}(A, M) \times \mathbf{C}(A, X) \times \mathbf{C}(M \times Y, B)$$

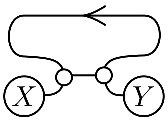
≅ {Copy} ≅ {Yoneda lemma}



$$\mathbf{C}(A, X) \times \mathbf{C}(A \times Y, B)$$ □

2.4. Example: Feedback.

Shapes do not need to be limited to a single category. For instance, we can make use of the opposite category to introduce feedback, in the sense of the *categories with feedback* of [KSW02]. Wires in the opposite category will be marked with an arrow to distinguish them.



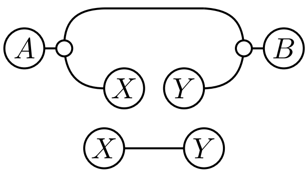
$$= \int^{M \in \mathbf{C}} \mathbf{C}(M \otimes X, M \otimes Y).$$

FIGURE 8. A shape with feedback, interpreted as a set.

Proposition 2.8 (see [Sta13])**.** *Profunctors form a compact closed bicategory. The dual of a category is its opposite category.*

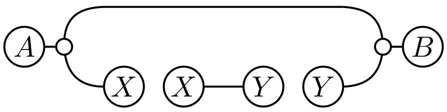
3. Composing and Reducing Shapes

We have been focusing on the invertible transformations between shapes, but arguably the most interesting case is that of non-invertible transformations. Our next step is to describe rules for composing and reducing diagrams that translate to valid coend calculus reductions. For instance, as we saw in the introduction (Figure 2), a lens $(A, B) \to (X, Y)$ can be composed with a morphism $X \to Y$ to obtain a morphism $A \to B$.



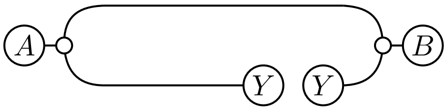
$$\left(\int^{M} \mathbf{C}(A, M \otimes X) \times \mathbf{C}(M \otimes Y, B)\right) \times \mathbf{C}(X, Y)$$

$\cong$ {Isotopy} $\qquad \cong$ {Continuity}



$$\int^{M} \mathbf{C}(A, M \otimes X) \times \mathbf{C}(X, Y) \times \mathbf{C}(M \otimes Y, B)$$

$\to$ $\{\varepsilon_X\}$ $\qquad \to$ {Composition along X}

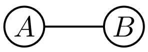


$$\int^{M} \mathbf{C}(A, M \otimes Y) \times \mathbf{C}(M \otimes Y, B)$$

$\to$ $\{\varepsilon_Y\}$ $\qquad \to$ {Composition along Y}

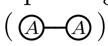
$$\int^{M,N} \mathbf{C}(A, M \otimes N) \times \mathbf{C}(M \otimes N, B)$$

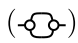
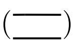
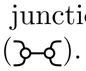
$\to$ $\{\varepsilon_\otimes\}$ $\qquad \to$ {Composition along $M \otimes N$}



$$\mathbf{C}(A, B)$$

FIGURE 9. Composing a lens with a morphism, formalizing Figure 2.

Definition 3.1 (Joining and splitting wires)**.** Identities and composition define natural transformations $\eta_A \colon (\quad) \to$  and $\varepsilon_A \colon$  $\to$ (——). They determine an adjunction, as the following transformations are identities.

In the same vein, junctions and forks have natural transformations $\varepsilon_\otimes \colon$  $\to$ (——) and $\eta_\otimes \colon$  $\to$ . They determine an adjunction, as the following transformations are identities.

3.1. **Example: Categories of Optics.** Two lenses of types $(A, B) \to (X, Y)$ and $(X, Y) \to (U, V)$ can be composed with each other to form a category of optics [Ril18]. There is, however, another way of composing two lenses. When the base category is symmetric, a lens $(A, Y) \to (X, V)$ can be composed with a lens $(X, B) \to (U, Y)$ into a lens $(A, B) \to (U, Y)$. We will observe that, even if **Prof** is symmetric, the reduction explicitly uses symmetry on the base category **C**.

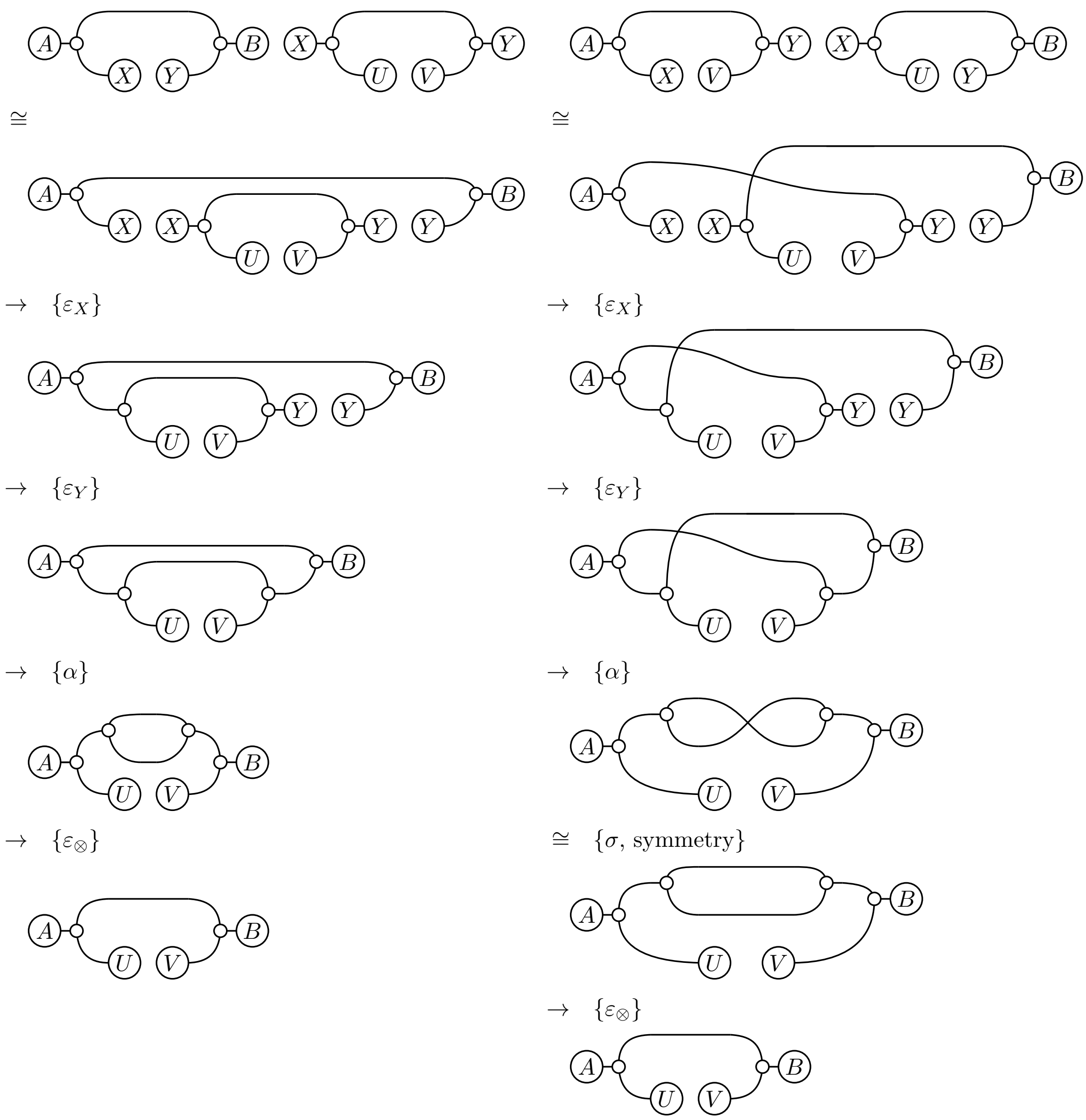
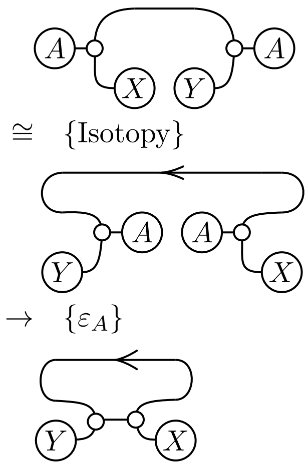


FIGURE 10. In parallel, two possible compositions of optics.

3.2. **Example: from Lenses to Dynamical Systems.** In [SSV16, Definition 2.3.1], a discrete dynamical system, a Moore machine, is characterized to have the same data as a lens $(A, A) \to (X, Y)$. The following derivation is a conceptual justification of this coincidence: a lens with suitable types can be made into a morphism of the free category with feedback [KSW02], subsuming particular cases such as *Moore machines*.



$$\int^{M} \mathbf{C}(A, M \otimes X) \times \mathbf{C}(M \otimes Y, A)$$

$\cong$ {Commutativity of $(\times)$}

$$\int^{M} \mathbf{C}(M \otimes Y, A) \times \mathbf{C}(A, M \otimes X)$$

$\to$ {Composition along A}

$$\int^{M} \mathbf{C}(M \otimes Y, M \otimes X)$$

FIGURE 11. From lenses to dynamical systems.

4. Open Diagrams

Our final step is to justify how to obtain the diagrams that originally motivated this text (*open diagrams*) by "looking inside" the shapes. So far, the element of a set described by a shape could be only expressed as a derivation of the shape from the empty diagram. In this section, we show diagrams that summarize these derivations and that represent specific elements of the shape.

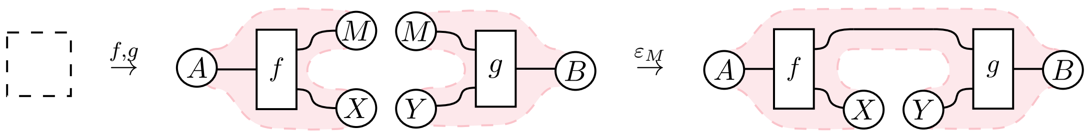


FIGURE 12. Open diagrams represent specific elements.

4.1. Open Diagrams.

Open diagrams will be interpreted in $\mathbf{Prof}_*$, the symmetric monoidal bicategory of pointed profunctors. Its 0-cells are categories with a chosen object; its 1-cells from $(\mathbf{A}, X)$ to $(\mathbf{B}, Y)$ are profunctors $P \colon \mathbf{A}^{op} \times \mathbf{B} \to \mathbf{Set}$ with a chosen point $p \in P(X, Y)$; and its 2-cells are natural transformations preserving that chosen point.

Proposition 4.1. *Reductions on shapes can be lifted to reductions on open diagrams.*

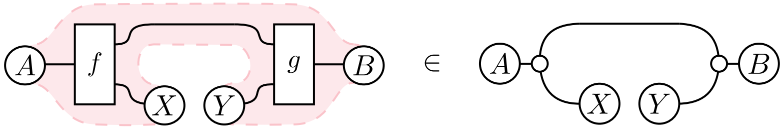
Proof. There exists a pseudofunctor $U \colon \mathbf{Prof}_* \to \mathbf{Prof}$ that forgets about the specific point. It holds that $a \in A$ for every element $(A, a) \in \mathbf{Prof}_*((\mathbf{1}, 1), (\mathbf{1}, 1))$. Natural transformations $\alpha \colon P \to Q$ can be lifted to $\alpha_* \colon (P, p) \to (Q, \alpha(p))$ in a unique way, determining a discrete opfibration $\mathbf{Prof}_*(\mathbf{A}, \mathbf{B}) \to \mathbf{Prof}(\mathbf{A}, \mathbf{B})$ for every pair of categories $\mathbf{A}$ and $\mathbf{B}$. □

Proposition 4.2. *Diagrams on the base category can be lifted to open diagrams.*

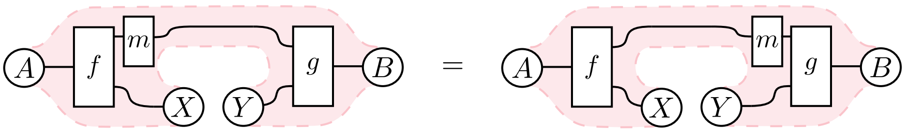
Proof. Let $\mathbf{C}$ be a small category. There exists a pseudofunctor $\mathbf{C} \to \mathbf{Prof}_*$ sending every object $A \in \mathbf{C}$ to the 0-cell pair $(\mathbf{C}, A)$ and every morphism $f \in \mathbf{C}(A, B)$ to the 1-cell pair $(\hom_{\mathbf{C}}, f)$. Moreover, when $(\mathbf{C}, \otimes, I)$ is monoidal, the pseudofunctor is lax and oplax monoidal (weak pseudofunctor in [MV18]), with oplaxators being left adjoint to laxators (see §8.3). This can be called an *op-ajax monoidal pseudofunctor*, following the notion of *ajax monoidal functor* from [FS18]. □

The graphical calculus for open diagrams can then be interpreted as the graphical calculus of pointed profunctors enhanced with a pseudofunctorial box, in the same vein as the functor boxes of [Mel06]. Similar *"internal diagrams"* have been described before by [BDSPV15] (and summarized in [Hu19]) as a "graphical mnemonic notation".

4.2. **Example: Categories of Optics.** The lens $\langle g, f\rangle \colon (A, B) \to (X, Y)$ is depicted as the following open diagram.



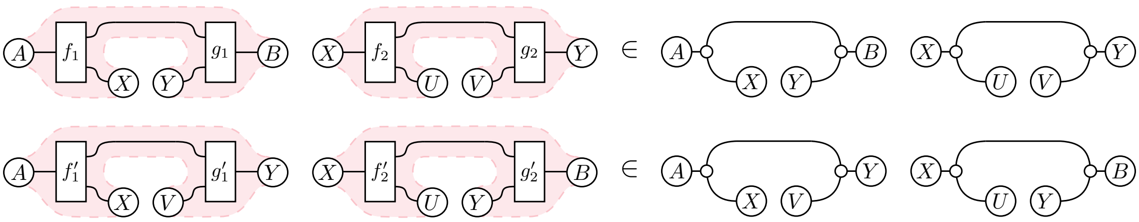
The quotienting that makes $\langle g, (m \otimes \mathrm{id}_X) \circ f\rangle = \langle g \circ (m \otimes \mathrm{id}_Y), f\rangle$ is explicit in this graphical calculus. The following two diagrams are equal in the category $\mathbf{Prof}_*$: they represent the same set and the same element within it.



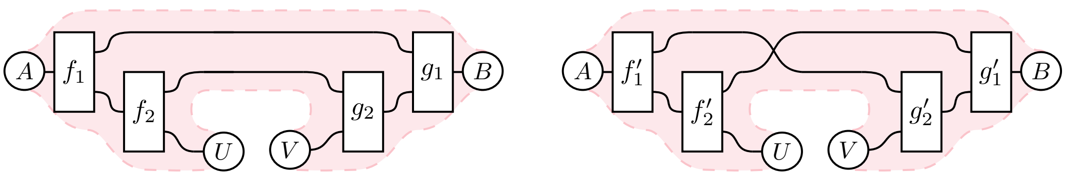
Note that we cannot speak of equality between open diagrams with different shapes, for they belong to different sets. We could however speak of equality between two open diagrams such that the shape of the first can be deformed into the shape of the second. The deformation determines an isomorphism between the sets defined by the shapes. Equality of elements on isomorphic sets is understood to be equality after applying the isomorphism.

For instance, the following two elements are equal under the deformation given by counitality of the pseudocomonoid structure.

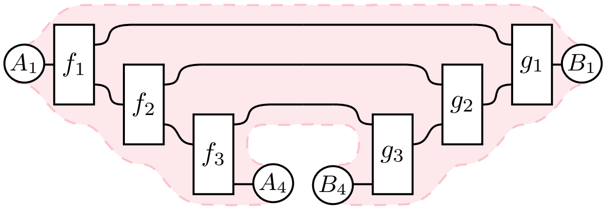
We will use open diagrams to justify that both compositions from Example 3.1 determine a category. Consider two pairs of lenses of suitable types.

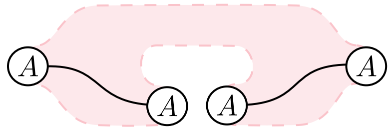


We can use Proposition 4.1 to lift the two compositions in Example 3.1 to two deformations of open diagrams that send the two pairs of lenses to the following two open diagrams, respectively.

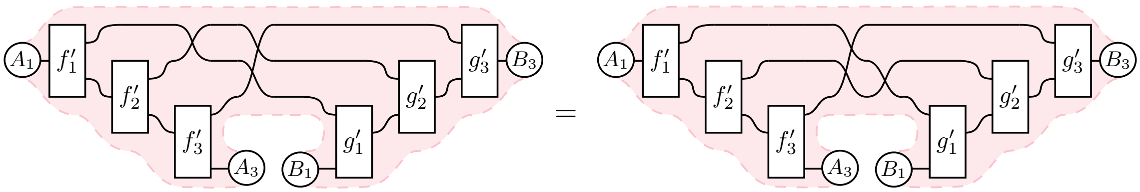


Let us show that a category can be defined from the first composition. Consider three lenses o_i for $i = 1, 2, 3$. We have two ways of composing them, as $o_1 \circ (o_2 \circ o_3)$ or $(o_1 \circ o_2) \circ o_3$, but they both give rise to the same final diagram, thanks to associativity of the base monoidal category. The identity is the diagram on the right.





For the second composition, checking associativity amounts to the following equality. The identity is the same as in the previous case.



The graphical calculus is hiding at the same time the details of two structures. The first is the quotient relation given by the coend in the monoidal bicategory of profunctors; the second is the coherence of the base monoidal category inside the pseudofunctorial box.

5. Related and Further Work

The graphical calculus for profunctors can be seen as a direction in which the graphical calculus for the cartesian bicategory of relations [BPS17, FS18] can be categorified. A notion of *cartesian bicategory* generalizing relations is discussed in [CKWW08]. For a slightly different future direction, we could try to relate this work to many of the interesting applications of *compact closed bicategories* (see [Sta13]); such as *resistor networks*, *double-entry bookeeping* [KSW08] or *higher linear algebra* [KV94].

Certain shapes open diagrams have been described in the literature. Specifically, finite *combs* were used as notation by [CDP08, KU17, Ril18]; the relation with lenses is described in [Rom20]. Previous graphical calculi for lenses and optics [Hed17, Boi20] have elegantly captured some aspects of optics by working on the Kleisli or Eilenberg-Moore categories of the Pastro-Street monoidal monad [PS08]. The present approach diverges from previous formalisms by using the monoidal bicategory structure of profunctors. It is more general than considering combs, as it can express arbitrary shapes in non-symmetric monoidal categories. In any case, it enables us to reason about categories of optics themselves; the results on optics of [CEG$^+$20] can be greatly simplified in this calculus. We believe that it is closer to, and it provides a formal explanation to the *diagrams with holes* of [Ril18, Definition 2.0.1], which were missing from previous approaches.

Most of our first part can be repeated for arbitrary monoidal bicategories such as enriched profunctors or spans. Multiple approaches to open systems (decorated cospans [Fon15], structured cospans [BC19]) could be related in this way to open diagrams, but we have not explored this possibility yet. Another potential direction is to repeat this reasoning for the case of double categories and obtain a "tile" version of these diagrams (see [Mye16, HS19]).

6. Conclusions

We have presented a way to study and compose *processes* in monoidal categories that do not necessarily have the usual shape of a square box without losing the benefits of the usual language of monoidal categories. Direct applications seem to be circuit design, see

[CDP08], or the theory of optics [CEG+20]. This technique is justified by the formalism of coend calculus [Lor19] and string diagrams for monoidal bicategories [Bar14]. We also argue that the graphical representation of coend calculus is helpful to its understanding: contrasting with usual presentations of coends that are usually centered around the Yoneda reductions, the graphical approach seems to put more weight in the non-reversible transformations while making most applications of Yoneda lemma transparent. Regarding open diagrams, we can think of many other applications that have not been described in this text: we could speak of multiple categories at the same time and combine open diagrams of any of them using functors and adjunctions. This work has opened many paths that we aim to further explore.

We have been working in the symmetric monoidal bicategory of profunctors for simplicity, but the same results extend to the symmetric monoidal bicategory of $\mathcal{V}$-profunctors for $\mathcal{V}$ a Bénabou cosmos [Lor19, §5]. We can even consider arbitrary monoidal bicategories and drop the requirements for symmetry, copying or discarding. Finally, there is an important shortcoming to this approach that we leave as further work: the present graphical calculus is an extremely good tool for *coend calculus*, but it remains to see if it is so for *(co)end calculus*. In other words, *ends* "enter the picture" only as natural transformations (see [Wil10]), and this can feel limiting even if, after applying Yoneda embeddings, it usually suffices for most applications. As it happens with diagrammatic presentations of regular logic [BPS17, FS18], the existential quantifier plays a more prominent role. Diagrammatic approaches to obtaining the universal quantifier in a situation like this go back to Peirce and are described by [HS20].

7. Acknowledgements

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8. Appendix

8.1. The Monoidal Bicategory of Profunctors.

Definition 8.1. There exists a symmetric monoidal bicategory **Prof** having as 0-cells the (small) categories $\mathbf{A}, \mathbf{B}, \mathbf{C}, \dots$; as 1-cells from $\mathbf{A}$ to $\mathbf{B}$, the profunctors $\mathbf{A}^{op} \times \mathbf{B} \to \mathbf{Set}$; as 2-cells, the natural transformations; and as tensor product, the cartesian product of categories [Lor19]. Two profunctors $P \colon \mathbf{A}^{op} \times \mathbf{B} \to \mathbf{Set}$ and $Q \colon \mathbf{B}^{op} \times \mathbf{C} \to \mathbf{Set}$ compose into the profunctor $(P \diamond Q) \colon \mathbf{A}^{op} \times \mathbf{C} \to \mathbf{Set}$ defined by

$$(P \diamond Q)(A, C) \coloneqq \int^{B \in \mathbf{B}} P(A, B) \times Q(B, C).$$

The unit of composition in the category $\mathbf{A}$ is the hom-profunctor $\hom_{\mathbf{A}} \colon \mathbf{A}^{op} \times \mathbf{A} \to \mathbf{Set}$. Strong unitality, $(\text{——}\boxed{P}\text{—}) \cong (\text{—}\boxed{P}\text{—}) \cong (\text{—}\boxed{P}\text{——})$, is given by the Yoneda isomorphisms.

$$(\hom_{\mathbf{A}} \diamond P)(A, B) \coloneqq \int^{A' \in \mathbf{A}} \hom_{\mathbf{A}}(A, A') \times P(A', B) \cong P(A, B)$$
$$(P \diamond \hom_{\mathbf{B}})(A, B) \coloneqq \int^{B' \in \mathbf{B}} P(A, B') \times \hom_{\mathbf{B}}(B', B) \cong P(A, B).$$

Strong associativity $(\text{—}\boxed{P_1}\text{—}\boxed{P_2}\text{—}) \diamond (\text{—}\boxed{P_3}\text{—}) \cong (\text{—}\boxed{P_1}\text{—}) \diamond (\text{—}\boxed{P_2}\text{—}\boxed{P_3}\text{—})$ follows from continuity and associativity of the cartesian product of sets. The invertible 2-cells that realise unitality and associativity satisfy the pentagon and triangular equations.

The monoidal product of two profunctors $P_1 \colon \mathbf{A}_1^{op} \times \mathbf{B}_1 \to \mathbf{Set}$ and $P_2 \colon \mathbf{A}_2^{op} \times \mathbf{B}_2 \to \mathbf{Set}$ is the profunctor $(P_1 \otimes P_2) \colon (\mathbf{A}_1 \times \mathbf{A}_2)^{op} \times (\mathbf{B}_1 \times \mathbf{B}_2) \to \mathbf{Set}$ defined by

$$(P_1 \otimes P_2)(A_1, A_2, B_1, B_2) \coloneqq P_1(A_1, B_1) \times P_2(A_2, B_2).$$

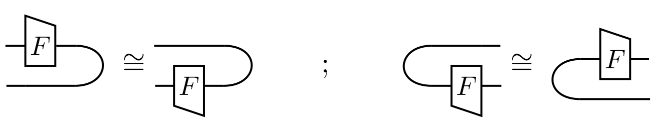
The unit of the monoidal structure is the terminal category. Unitality and associativity follow from those on sets. In the case of profunctors, unitors and associator are not only equivalences but isomorphisms of categories, with strictly commuting pentagons and triangles.

An alternative approach is to construct this symmetric monoidal bicategory from the double symmetric bicategory of profunctors, see [HS19].

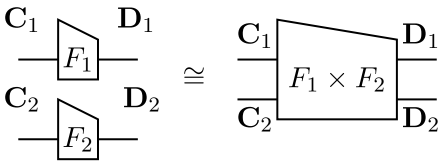
Every category has a dual, its opposite category. There are profunctors $(\mathbf{A}^{op} \times \mathbf{A}) \times \mathbf{1} \to \mathbf{Set}$ and $\mathbf{1}^{op} \times (\mathbf{A}^{op} \times \mathbf{A})^{op} \to \mathbf{Set}$ given by variations of the hom-profunctors; these are represented by caps and cups. Profunctors circulate through the caps and cups as expected thanks to the Yoneda lemma. See [Sta13] for the description as a compact closed bicategory.

Definition 8.2 (Yoneda Embedding of Functors)**.** Let $F \colon \mathbf{C} \to \mathbf{D}$ be a functor. It can be embedded as a profunctor $(\text{—}\boxed{F}\text{—}) \colon \mathbf{C}^{op} \times \mathbf{D} \to \mathbf{Set}$ or as a profunctor $(\text{—}\boxed{F}\text{—}) \colon \mathbf{D}^{op} \times \mathbf{C} \to \mathbf{Set}$. Moreover, every functor has an opposite, so it can also be embedded as a profunctor $(\text{—}\boxed{F}\text{—}) \colon (\mathbf{D}^{op})^{op} \times \mathbf{C}^{op} \to \mathbf{Set}$ or as a profunctor $(\text{—}\boxed{F}\text{—}) \colon (\mathbf{C}^{op})^{op} \times \mathbf{D}^{op} \to \mathbf{Set}$. In particular, $F \dashv G$ precisely when $(\text{—}\boxed{F}\text{—}) \cong (\text{—}\boxed{G}\text{—})$.

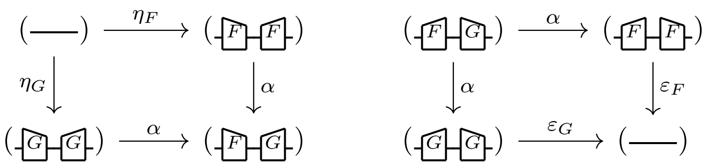
The suggestive shape of the boxes (from [CK17]) is matched by their semantics. Every category has a dual (namely, its opposite category) and functors circulate as expected through the cups and the caps that represent dualities.



Proposition 8.3. *Both Yoneda embeddings are strong monoidal pseudofunctors* $\mathbf{Cat} \to \mathbf{Prof}$*, fully faithful on the 2-cells. Pseudofunctoriality gives* (-[F]-[G]-) $\cong$ (-[G ∘ F]-) *and its counterpart. Monoidality gives the following isomorphism and its mirrored counterpart.*



Proposition 8.4 (Functors are Left Adjoints)**.** *In the category of profunctors, functors are left adjoints, in the sense that there exist morphisms* $\eta_F \colon$ (——) $\to$ (-[F]-[F]-) *and* $\varepsilon_F \colon$ (-[F]-[F]-) $\to$ (——) *and they verify the zig-zag identities. Moreover, every natural transformation commutes with these dualities in the sense that the following are two commutative squares.*[2]



A partial converse holds: a left adjoint profunctor is representable when its codomain is Cauchy complete; see [Bor94].

Proposition 8.5. *Every object* $\mathbf{A}$ *of the category of profunctors has already a canonical pseudocomonoid structure lifted from* $\mathbf{Cat}$ *and given by* (-●<) $:= \mathbf{A}(-^0, -^1) \times \mathbf{A}(-^0, -^2)$ *and* (-●) $:= 1$*; but also a pseudomonoid structure given by* (>●-) $:= \mathbf{A}(-^1, -^0) \times \mathbf{A}(-^2, -^0)$*, and* (●-) $:= 1$*. These structures* copy *and* discard *representable and corepresentable functors, respectively; but they also laxly copy and discard arbitrary profunctors.*

Proof. This is a consequence of the fact the diagonal and discard functors $(\Delta) \colon \mathbf{A} \to \mathbf{A} \times \mathbf{A}$ and $(!) \colon \mathbf{A} \to \mathbf{1}$ copy and discard functors in $\mathbf{Cat}$. Pseudofunctoriality of both Yoneda embeddings sends them to the profunctors we are describing in $\mathbf{Prof}$.

On the other hand, arbitrary profunctors are laxly copied and discarded. For instance, the following coend derivation shows that a profunctor $P \colon \mathbf{A}^{op} \times \mathbf{B} \to \mathbf{Set}$ is laxly copied. In the case of representable profunctors, this is an isomorphism.

$$\int^{X} P(A, X) \times \hom_{\mathbf{A}}(X, Y_1) \times \hom_{\mathbf{B}}(X, Y_2)$$
$$\to$$
$$P(A, Y_1) \times P(A, Y_2)$$
$$\cong$$
$$\int^{X_1, X_2} \hom_{\mathbf{A}}(A, X_1) \times \hom_{\mathbf{A}}(A, X_2) \times P(X_1, Y_1) \times P(X_2, Y_2). \qquad \square$$

8.2. The Monoidal Bicategory of Pointed Profunctors.

Definition 8.6. A *pointed category* $(\mathbf{A}, X)$ is a category $\mathbf{A}$ equipped with a chosen object X, which can be regarded as a functor from the terminal category. There exists a symmetric monoidal bicategory $\mathbf{Prof}_*$ having as 0-cells pairs $(\mathbf{A}, X)$ where $\mathbf{A}$ is a (small) category and $X \in \mathbf{A}$ is an object of that category; 1-cells from $(\mathbf{A}, X) \to (\mathbf{B}, Y)$ pairs (P, p) given by a profunctor $P \colon \mathbf{A}^{op} \times \mathbf{B} \to \mathbf{Set}$ and a point $p \in P(X, Y)$; 2-cells from $(P, p) \to (Q, q)$

[2] The graphical calculus of the bicategory makes these equations much clearer. We are emphasizing the monoidal bicategory structure here only for the sake of coherence.

are natural transformations $\eta \colon P \to Q$ such that $\eta_{X,Y}(p) = q$. Composition of 1-cells $(P, p) \colon (\mathbf{A}, X) \to (\mathbf{B}, Y)$ and $(Q, q) \colon (\mathbf{B}, Y) \to (\mathbf{C}, Z)$ is given by $(Q \diamond P, \langle q, p \rangle)$, where $\langle q, p \rangle \in (Q \diamond P)(X, Z)$ is the equivalence class under the coend of the pair (q, p). The identity 1-cell in $(\mathbf{A}, X)$ is $(\hom_{\mathbf{A}}, \mathrm{id}_X) \colon (\mathbf{1}, 1) \to (\mathbf{A}, X)$.

Strong unitality $\ell_{(P,p)} \colon (\hom_{\mathbf{A}} \diamond P, \langle \mathrm{id}_X, p \rangle) \to (P, p)$ and $\varrho_{(P,p)} \colon (P \diamond \hom_{\mathbf{A}}, \langle p, \mathrm{id}_X \rangle) \cong (P, p)$ is given by the Yoneda isomorphisms.

$$\ell_P \colon \int^{Z \in \mathbf{A}} \hom_{\mathbf{A}}(X, Z) \times P(Z, Y) \cong P(X, Y)$$

$$\varrho_P \colon \int^{Z \in \mathbf{A}} P(X, Z) \times \hom_{\mathbf{A}}(Z, Y) \cong P(X, Y)$$

The Yoneda isomorphisms are such that $\ell_P \langle \mathrm{id}_X, p \rangle = \mathrm{id}_X \circ p = p$ and $\varrho_P \langle p, \mathrm{id}_Y \rangle = p \circ \mathrm{id}_Y = p$. This confirms they are valid 2-cells of $\mathbf{Prof}_*$.

Strong associativity $a_{(P,p,Q,q,R,r)} \colon ((P \diamond Q) \diamond R, \langle \langle p, q \rangle, r \rangle) \to (P \diamond (Q \diamond R), \langle p, \langle q, r \rangle \rangle)$ is given by the isomorphism described by continuity and associativity of the cartesian product.

$$\int^V \left(\int^U P(X, U) \times Q(U, V) \right) \times R(V, Y) \cong \int^U P(X, U) \times \left(\int^V Q(U, V) \times R(V, Y) \right).$$

It is defined by $a(\langle \langle p, q \rangle, r \rangle) = \langle p, \langle q, r \rangle \rangle$, proving that it is a valid 2-cell of $\mathbf{Prof}_*$.

We will show now it is also symmetric monoidal. There is a distinguished object $(\mathbf{1}, 1)$, given by the terminal category and its only object. There is a pseudofunctor $(\otimes) \colon \mathbf{Prof}_* \times \mathbf{Prof}_* \to \mathbf{Prof}_*$ defined

- on 0-cells by $(\mathbf{A}, X) \otimes (\mathbf{B}, Y) \coloneqq (\mathbf{A} \times \mathbf{B}, (X, Y))$;
- on 1-cells by $(P, p) \otimes (Q, q) \coloneqq (P \otimes Q, (p, q))$;
- on 2-cells by $(\gamma \otimes \delta) \colon P \diamond Q \to P' \diamond Q'$ the transformation that applies $\gamma \colon P \to P'$ and $\delta \colon Q \to Q'$ to both factors of the coend,

$$\left(\int^Z P(X, Z) \times Q(Z, Y) \right) \to \left(\int^Z P'(X, Z) \times Q'(Z, Y) \right);$$

- natural isomorphisms $((P \diamond Q) \otimes (P' \diamond Q'), (\langle p, q \rangle, \langle p', q' \rangle)) \cong ((P \otimes P') \diamond (Q \otimes Q'), \langle (p, p'), (q, q') \rangle)$ and $(\hom_{\mathbf{A} \times \mathbf{B}}, \mathrm{id}_{A \times B}) \cong (\hom_{\mathbf{A}} \times \hom_{\mathbf{B}}, (\mathrm{id}_A, \mathrm{id}_B))$.

The symmetric monoidal structure requires the following natural transformations.

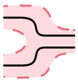
- the left unitor $\lambda^{\otimes}_P \colon (\mathrm{id}_{\mathbf{1}} \otimes P, (\mathrm{id}_*, p)) \to (P, p)$ and right unitor $\rho^{\otimes}_P \colon (P \otimes \mathrm{id}_{\mathbf{1}}, (p, \mathrm{id}_*)) \to (P, p)$;
- the associator $\alpha^{\otimes}_{P,Q,R} \colon ((P \otimes Q) \otimes R, ((p, q), r)) \to (P \otimes (Q \otimes R), (p, (q, r)))$, given by the associator of the cartesian category of sets;
- the braiding $\beta \colon (P \otimes Q, (p, q)) \to (Q \otimes P, (q, p))$;
- such that the two relevant squares and the relevant hexagon commute, making it a syllepsis and a symmetry.

8.3. **Pseudofunctor box.**

Proposition 8.7. *Let* $\mathbf{A}$ *be a small category. There exists a pseudofunctor* $\mathbf{A} \to \mathbf{Prof}_*$ *sending every object* $A \in \mathbf{A}$ *to the 0-cell pair* $(\mathbf{A}, A)$ *and every morphism* $f \in \hom_{\mathbf{A}}(A, B)$ *to the 1-cell pair* $(\hom_{\mathbf{A}}, f)$*. Moreover, when* $(\mathbf{A}, \otimes, I)$ *is monoidal, the pseudofunctor is lax and oplax monoidal (weak pseudofunctor in* [MV18]*, with oplaxators being left adjoint to laxators). This would be an op-ajax monoidal pseudofunctor, following the notion of* ajax monoidal functor *from* [FS18].

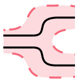
We only sketch the construction. The invertible 2-cells witnessing pseudofunctoriality use the fact that the Yoneda isomorphisms (unitors and associators) send pairs of points to their composition, coinciding with the composition on the base category $\mathbf{A}$.

The following natural transformations make the functor lax monoidal.

$$\Big(\quad \Big)$$  $:= (\mathrm{hom}(-, - \otimes -), \mathrm{id}_{A \otimes B})\colon (\mathbf{A} \times \mathbf{A}, (A, B)) \to (\mathbf{A}, A \otimes B)$

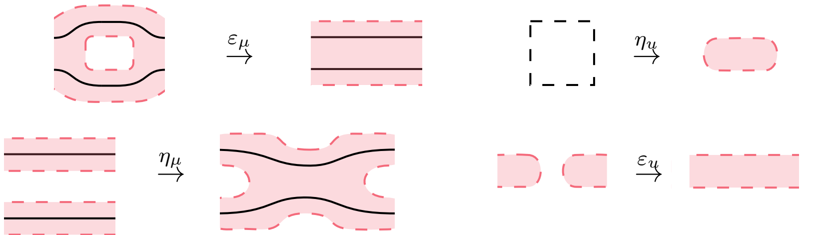
$\Big(\quad \Big)$  $:= (\mathrm{hom}(I, -), \mathrm{id}_I)\colon (\mathbf{1}, *) \to (\mathbf{A}, I)$

The following natural transformations make the functor oplax monoidal.

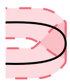
$\Big(\quad \Big)$  $:= (\mathrm{hom}(- \otimes -, -), \mathrm{id}_{A \otimes B})\colon (\mathbf{A}, A \otimes B) \to (\mathbf{A} \times \mathbf{A}, (A, B))$

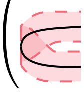
$\Big(\quad \Big)$  $:= (\mathrm{hom}(-, I), \mathrm{id}_I)\colon (\mathbf{A}, I) \to (\mathbf{1}, *)$

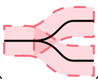
Composition and identities give the counits and units of the adjunctions. The fact that identity is the unit for composition makes the following transformations be 2-cells of $\mathbf{Prof}_*$.

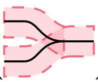


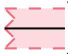
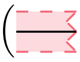
The following morphisms follow the cups, caps, splitting and merging structure from $\mathbf{Prof}$ in $\mathbf{Prof}_*$. Morphisms circulate through them as expected: turning to morphisms in the opposite category, being copied and discarded.

$\Big(\quad \Big)$  $:= (\mathrm{hom}(-, -), \mathrm{id}_A)\colon (\mathbf{A} \times \mathbf{A}^{op}, (A, A)) \to (\mathbf{1}, 1),$

$\Big(\quad \Big)$  $:= (\mathrm{hom}(-, -), \mathrm{id}_A)\colon (\mathbf{1}, 1) \to (\mathbf{A} \times \mathbf{A}^{op}, (A, A)),$

$\Big(\quad \Big)$  $:= (\mathrm{hom}(-^0, -^1) \times \mathrm{hom}(-^0, -^2), (\mathrm{id}_A, \mathrm{id}_A))\colon (\mathbf{A}, A) \to (\mathbf{A} \times \mathbf{A}, (A, A)),$

$\Big(\quad \Big)$  $:= (\mathrm{hom}(-^1, -^0) \times \mathrm{hom}(-^2, -^0), (\mathrm{id}_A, \mathrm{id}_A))\colon (\mathbf{A} \times \mathbf{A}, (A, A)) \to (\mathbf{A}, A),$

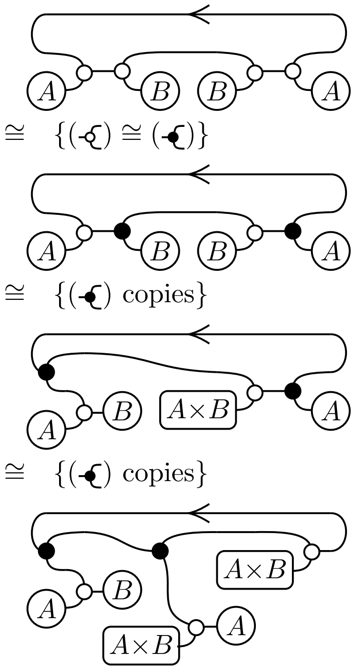
$\Big(\quad \Big)$  $:= (1, *)\colon (\mathbf{1}, 1) \to (\mathbf{A}, A);$ $\quad$ $\Big(\quad \Big)$  $:= (1, *)\colon (\mathbf{A}, A) \to (\mathbf{1}, 1).$

Proposition 8.8. *Let $\mathbf{A}$ be a category. For every $A \in \mathbf{A}$, there exist 1-cells*

$$(\mathit{hom}_{\mathbf{A}}(A, -), \mathrm{id}_A)\colon (\mathbf{1}, 1) \to (\mathbf{A}, A) \quad \text{and} \quad (\mathrm{hom}_{\mathbf{A}}(-, A), \mathrm{id}_A)\colon (\mathbf{A}, A) \to (\mathbf{1}, 1)$$

given by the Yoneda embeddings of A and the identity morphism. Composition and identities define an adjunction.

8.4. **Example: Learners.** A *learner* [FST19, Definition 4.1] in a cartesian category is given by a *parameters* object $P \in \mathbf{C}$, an *implementation* morphism $i\colon P \times A \to B$, and *update* morphism $u\colon P \times A \times B \to P$, and a *request* morphism $r\colon P \times A \times B \to A$. A monoidal generalization, dinatural on the parameters object, has been proposed in [Ril18, Definition 6.4.1]; the following derivation shows how it particularizes into the cartesian case.



$$\int^{P,Q} \mathbf{C}(P \times A, Q \times B) \times \mathbf{C}(Q \times B, P \times A)$$

$\cong$ {Universal property of the product}

$$\int^{P,Q} \mathbf{C}(P \times A, Q) \times \mathbf{C}(P \times A, B) \times \mathbf{C}(Q \times B, P \times A)$$

$\cong$ {Yoneda lemma}

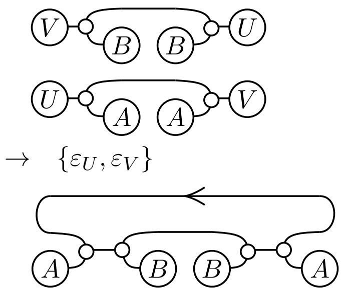
$$\int^{P} \mathbf{C}(P \times A, B) \times \mathbf{C}(P \times A \times B, P \times A)$$

$\cong$ {Universal property of the product}

$$\int^{P} \mathbf{C}(P \times A, B) \times \mathbf{C}(P \times A \times B, A) \times \mathbf{C}(P \times A \times B, P)$$

FIGURE 13. From monoidal to cartesian learners.

Proposition 8.9. *A pair of lenses $(U, V) \to (A, A)$ and $(V, U) \to (B, B)$ define a learner.*



$$\int^{M,N} \mathbf{C}(U, M{\otimes}A) \times \mathbf{C}(M{\otimes}A, V) \times \mathbf{C}(V, N{\otimes}B) \times \mathbf{C}(N{\otimes}B, U)$$

$\to$ {Composition along U and V}

$$\int^{P} \mathbf{C}(P \times A, B) \times \mathbf{C}(P \times A \times B, A) \times \mathbf{C}(P \times A \times B, P)$$

FIGURE 14. From lenses to learners.