PROFUNCTOR OPTICS, A CATEGORICAL UPDATE (EXTENDED ABSTRACT)

BRYCE CLARKE, DEREK ELKINS, JEREMY GIBBONS, FOSCO LOREGIAN, BARTOSZ MILEWSKI, EMILY PILLMORE, MARIO ROMÁN

Abstract. *Optics* are bidirectional data accessors that capture data transformation patterns such as accessing subfields or iterating over containers. *Profunctor optics* are composable representations of optics in terms of polymorphic functions. Our work provides some original families of optics and derivations, including an elementary one for *traversals* that solves an open problem posed by Milewski. We generalize a classic result by Pastro and Street on Tambara theory and use it to describe *mixed* $\mathcal{V}$-enriched profunctor optics and to endow them with $\mathcal{V}$-category structure.

1. Introduction

1.1. **Optics.** *Optics* are an abstract representation of some common patterns in bidirectional data accessing. The most widely known optics are *lenses*, pairs of functions *view* ($S \to A$) and *update* ($S \times A \to S$) that respectively retrieve and modify a subfield of a data structure (Figure 1).

```
data Address = Address
  { street'  :: String
  , city'    :: String
  , country' :: String }

viewStreet :: Address -> String
viewStreet = street'

updateStreet :: Address -> String -> Address
updateStreet a s = a {street' = s}

example :: Address
example = Address
  { street' = "221b Baker Street"
  , city' = "London"
  , country' = "UK" } }

>>> viewStreet example
"221b Baker Street"
```

Figure 1. Lenses are pairs of functions 'view' and 'update' that capture the repeating pattern of accessing subfields.

It is routine to describe how lenses can be composed to access nested subfields. However, it is tedious, and it becomes increasingly difficult as other data accessors enter the stage. We would like optics to behave *modularly*; in the sense that, given two optics, it should be possible to join them into a composite optic that directly accesses the innermost subfield. Perhaps surprisingly, many implementations allow the programmer to wrap optics into a different representation and then use *ordinary function composition* to construct composite optics. How is it possible to compose two constructs that are not functions using ordinary function composition? Implementations provided by popular libraries such as lens [Kme18], mezzolens [O'C15] in Haskell, or profunctor-optics [FMH+19] in Purescript, achieve this effect by using different representations of optics in terms of polymorphic functions and the Yoneda lemma. This text focuses on the encoding known as *profunctor representation*, which is based in the isomorphism between lenses (and optics in general) and functions polymorphic over profunctors with a particular algebraic structure called a *Tambara module*. Optics under this encoding are called *profunctor optics*.

1.2. **Coend Calculus.** *Coend calculus* is a branch of category theory that describes the behaviour of *ends* and *coends*, certain universal objects associated to profunctors $P : \mathbf{C}^{op} \times \mathbf{C} \to \mathcal{V}$. Theorems involving ends and coends can be proved by means of their universal properties; an extensive discussion of this topic is in [Lor19]. Via the calculus induced by these universal properties, it is possible to construct isomorphisms between objects of a category by means of a chain of 'deduction rules'.

As an example, let us consider how two forms of the Yoneda lemma can be written in terms of ends and coends. Ends and coends are usually denoted by subscripted and superscripted integrals, respectively. This notation makes Yoneda lemma resemble a 'Dirac delta' integration rule, whereas commutativity of limits becomes a form of 'Fubini rule'.

$$\begin{array}{ll} \displaystyle\int_{X\in\mathcal{V}} \mathcal{V}(\mathbf{C}(A,X), FX) & \displaystyle\int^{X\in\mathbf{C}} \mathbf{C}(A,X)\otimes FX \\ \cong \quad \{\text{Yoneda reduction}\} & \cong \quad \{\text{Coyoneda reduction}\} \\ FA & FA \end{array}$$

FIGURE 2. Yoneda lemma in terms of ends and coends.

$$\begin{array}{ll} \displaystyle\int_{X_1\in\mathbf{C}}\int_{X_2\in\mathbf{C}} P(X_1,X_2,X_1,X_2) & \displaystyle\int^{X_1\in\mathbf{C}}\int^{X_2\in\mathbf{C}} P(X_1,X_2,X_1,X_2) \\ \cong \quad \{\text{Fubini rule (for ends)}\} & \cong \quad \{\text{Fubini rule (for coends)}\} \\ \displaystyle\int_{X_2\in\mathbf{C}}\int_{X_1\in\mathbf{C}} P(X_1,X_2,X_1,X_2) & \displaystyle\int^{X_2\in\mathbf{C}}\int^{X_1\in\mathbf{C}} P(X_1,X_2,X_1,X_2) \end{array}$$

FIGURE 3. Commutativity of ends and coends.

1.3. **Contributions.** Our first contribution is a generalization of Pastro and Street's *double* construction [PS08] that captures mixed and enriched optics (§4). Their work characterized copresheaves over these *doubles* as *Tambara modules* [PS08, Proposition 6.1]; we follow this idea to characterize copresheaves of mixed optics to be suitably generalized Tambara modules. As a corollary, we extend the result that justifies the profunctor representation of optics used in functional programming to the case of enriched and mixed optics (Theorem 4.3), endowing them with $\mathcal{V}$-category structure.

Our second contribution is the derivation of many optics, both existing in the literature and novel, from a unified definition (Definition 2.1). We present a new family of optics, which we dub *algebraic lenses* (Definition 3.5), that unifies new examples with some optics already present in the literature, such as Boisseau's *achromatic lens* [Boi17, §5.2]. We introduce a new derivation showing that *monadic lenses* [ACG$^+$16] are mixed optics. Similarly, a new derivation showing that Myers' *lenses in a symmetric monoidal category* [Spi19, §2.2] are mixed optics (Proposition 3.4). We give a unified definition of *lens* (Definition 3.1), that can be specialized to all of these previous examples (algebraic lens, monadic lens, lens in a symmetric monoidal category). Finally, we present a new derivation of the optic known as *traversal* (Definition 3.11) in terms of a monoidal structure originally described by Kelly [Kel05, §8] for the study of non-symmetric operads (Proposition 3.13), as well as a generalization.

2. Optics

The structure that is common to all optics is that they divide a bigger data structure of type $S \in \mathbf{C}$ into some *focus* of type $A \in \mathbf{C}$ and some *context* or *residual* $M \in \mathbf{M}$ around it. We cannot access the context but we can still use its shape to update the original data structure, replacing the current focus by a new one. The definition will capture this fact imposing a quotient relation on the possible contexts; this quotient is expressed by the dinaturality condition of a coend. The category of contexts $\mathbf{M}$ will be monoidal, allowing us to compose optics with contexts M and N into an optic with context $M \otimes N$. Finally, as we want to capture type-variant optics, we leave open the possibility of the new focus being of a different type $B \in \mathbf{D}$, possibly in a different category, which yields a new data structure of type $T \in \mathbf{D}$.

Definition 2.1 (after Mil17, BG18, Ril18)**.** Let $(\mathbf{M}, \otimes, I, a, \lambda, \rho)$ be a monoidal $\mathcal{V}$-category [Day70]. Let it act on two arbitrary $\mathcal{V}$-categories $\mathbf{C}$ and $\mathbf{D}$ with strong monoidal $\mathcal{V}$-functors $(\circledL)\colon \mathbf{M} \to [\mathbf{C}, \mathbf{C}]$ and $(\circledR)\colon \mathbf{M} \to [\mathbf{D}, \mathbf{D}]$; let us write

$$\phi_A \colon A \cong I \circledL A, \quad \phi_{M,N,A} \colon M \circledL N \circledL A \cong (M \otimes N) \circledL A,$$
$$\varphi_B \colon B \cong I \circledR B, \quad \varphi_{M,N,B} \colon M \circledR N \circledR B \cong (M \otimes N) \circledR B,$$

for the structure isomorphisms of the strong monoidal actions $\circledL$ and $\circledR$.

Let $S, A \in \mathbf{C}$ and $T, B \in \mathbf{D}$. An **optic** from (S, T) with the focus on (A, B) is an element of the following object described as a coend

$$\mathbf{Optic}_{\circledL,\circledR}((A, B), (S, T)) \coloneqq \int^{M \in \mathbf{M}} \mathbf{C}(S, M \circledL A) \otimes \mathbf{D}(M \circledR B, T).$$

The two strong monoidal actions Ⓛ and Ⓡ represent the two different ways in which the context interacts with the focus: one when the data structure is decomposed and another one, possibly different, when it is reconstructed. Varying these two actions we will cover many examples from the literature and propose some new ones, as the following table summarizes.

Name	Description	Actions	Base
Adapter	$\mathbf{C}(S,A) \otimes \mathbf{D}(B,T)$	$(\mathbf{Optic}_{\mathrm{id},\mathrm{id}})$	$\mathcal{V}, \otimes$
Lens	$\mathbf{C}(S,A) \times \mathbf{D}(S \bullet B,T)$	$(\mathbf{Optic}_{\times,\bullet})$	$\mathcal{W}, \times$
Monoidal lens	$\mathbf{CCom}(S,A) \times \mathbf{C}(\mathcal{U}S \otimes B,T)$	$(\mathbf{Optic}_{\otimes,\mathcal{U}\times})$	$\mathcal{W}, \times$
Algebraic lens	$\mathbf{C}(S,A) \times \mathbf{D}(\Psi S \bullet B,T)$	$(\mathbf{Optic}_{\mathcal{U}\times,\mathcal{U}\bullet})$	$\mathcal{W}, \times$
Monadic lens	$\mathcal{W}(S,A) \times \mathcal{W}(S \times B,\Psi T)$	$(\mathbf{Optic}_{\times,\rtimes})$	$\mathcal{W}, \times$
Linear lens	$\mathbf{C}(S,[B,T] \bullet A)$	$(\mathbf{Optic}_{\otimes,\bullet})$	$\mathcal{V}, \otimes$
Prism	$\mathbf{C}(S,T \bullet A) \times \mathbf{D}(B,T)$	$(\mathbf{Optic}_{\bullet,+})$	$\mathcal{W}, \times$
Coalg. prism	$\mathbf{C}(S,\Theta T \bullet A) \times \mathbf{D}(B,T)$	$(\mathbf{Optic}_{\mathcal{U}\bullet,\mathcal{U}+})$	$\mathcal{W}, \times$
Grate	$\mathbf{D}([S,A] \bullet B,T)$	$(\mathbf{Optic}_{\{\},\bullet})$	$\mathcal{V}, \otimes$
Glass	$\mathbf{C}(S \times [[S,A],B],T)$	$(\mathbf{Optic}_{\times[],\times[]})$	$\mathcal{W}, \times$
Affine traversal	$\mathbf{C}(S,T + A \otimes \{B,T\})$	$(\mathbf{Optic}_{+\otimes,+\otimes})$	$\mathcal{W}, \times$
Traversal	$\mathcal{V}(S,\int^{n} A^n \otimes [B^n,T])$	$(\mathbf{Optic}_{\mathrm{Pw},\mathrm{Pw}})$	$\mathcal{V}, \otimes$
Kaleidoscope	$\int_{n} \mathcal{V}([A^n,B],[S^n,T])$	$(\mathbf{Optic}_{\mathrm{App},\mathrm{App}})$	$\mathcal{V}, \otimes$
Setter	$\mathcal{V}([A,B],[S,T])$	$(\mathbf{Optic}_{\mathrm{ev},\mathrm{ev}})$	$\mathcal{V}, \otimes$
Fold	$\mathcal{V}(S,\mathcal{L}A)$	$(\mathbf{Optic}_{\mathrm{Foldable},*})$	$\mathcal{V}, \otimes$

3. Examples of optics

3.1. **Lenses and prisms.** The basic definition of *lens* [Ole82, Pal07] has been generalized in many different directions. *Monadic lenses* [ACG+16], *lenses in a symmetric monoidal category* [Spi19, §2.2], *linear lenses* [Ril18, §4.8] or *achromatic lenses* [Boi17, §5.2] are some of them. However, these generalizations are not mutually compatible in general. They use the monoidal structure in different ways and introduce monadic effects in different parts of the signature. Many were not presented as *optics*, and a profunctor representation for them was not considered. We present two derivations of lenses as mixed optics that capture all of the variants mentioned before, together with new ones, and endow them with a unified profunctor representation (Theorem 4.3).

Definition 3.1. Let $\mathbf{C}$ be a cartesian $\mathcal{W}$-category with a monoidal $\mathcal{W}$-action $(\bullet)\colon \mathbf{C} \times \mathbf{D} \to \mathbf{D}$ to an arbitrary $\mathcal{W}$-category $\mathbf{D}$. A **lens** is an element of

$$\mathbf{Lens}((A,B),(S,T)) \coloneqq \mathbf{C}(S,A) \times \mathbf{D}(S \bullet B,T).$$

Proposition 3.2. *Lenses are mixed optics (as in Definition 2.1) for the actions of the cartesian product* $(\times)\colon \mathbf{C} \times \mathbf{C} \to \mathbf{C}$ *and* $(\bullet)\colon \mathbf{C} \times \mathbf{D} \to \mathbf{D}$. *That is,* $\mathbf{Lens} \cong \mathbf{Optic}_{(\times,\bullet)}$.

Proof. The universal property of the product can be summarized as it being right adjoint to the diagonal functor $(\Delta)\colon \mathbf{C} \to \mathbf{C}^2$.

$$\int^{C \in \mathbf{C}} \mathbf{C}(S, C \times A) \times \mathbf{D}(C \bullet B, T)$$

$$\cong \quad \{\text{Adjunction } \Delta \dashv (\times)\}$$
$$\int^{C \in \mathbf{C}} \mathbf{C}(S, C) \times \mathbf{C}(S, A) \times \mathbf{D}(C \bullet B, T)$$
$$\cong \quad \{\text{Coyoneda}\}$$
$$\mathbf{C}(S, A) \times \mathbf{D}(S \bullet B, T). \qquad \square$$

Definition 3.3 (Spi19, §2.2)**.** A lens in a symmetric monoidal category $\mathbf{C}$ is a view and update pair where the view is a comonoid homomorphism,

$$\mathbf{mLens}_{\otimes}((A, B), (S, T)) \coloneqq \mathbf{CCom}(S, A) \times \mathbf{C}(\mathcal{U}S \otimes B, T).$$

Proposition 3.4. *Lenses in a symmetric monoidal category are a particular case of Definition 3.1.*

Proof. The category of cocommutative comonoids $\mathbf{CCom}$ over a category $\mathbf{C}$ can be given a cartesian structure in such a way that the forgetful functor $\mathcal{U}\colon \mathbf{CCom} \to \mathbf{C}$ is strict monoidal (see [Fox76], where a stronger result is shown). We can show $\mathbf{mLens}_{\otimes} \cong \mathbf{Optic}_{(\otimes, \bullet)}$ where $(\bullet)$ is given by $S \bullet A \coloneqq \mathcal{U}S \otimes A$. $\square$

Definition 3.5. Let $\Psi\colon \mathbf{C} \to \mathbf{C}$ be a $\mathcal{W}$-monad in a cartesian $\mathcal{W}$-category $\mathbf{C}$. Let $(\bullet)\colon \mathbf{C} \times \mathbf{D} \to \mathbf{D}$ be a monoidal $\mathcal{W}$-action to an arbitrary $\mathcal{W}$-category $\mathbf{D}$. An **algebraic lens** is an element of

$$\mathbf{Lens}_{\Psi}((A, B), (S, T)) \coloneqq \mathbf{C}(S, A) \times \mathbf{D}(\Psi S \bullet B, T).$$

Proposition 3.6. *Algebraic lenses are mixed optics for the actions of the product by the carrier of an algebra* $({}_{\mathcal{U}}\times)\colon \mathrm{EM}_{\Psi} \times \mathbf{C} \to \mathbf{C}$ *and* $({}_{\mathcal{U}}\bullet)\colon \mathrm{EM}_{\Psi} \times \mathbf{D} \to \mathbf{D}$*. That is,* $\mathbf{Lens}_{\Psi} \cong \mathbf{Optic}_{({}_{\mathcal{U}}\times, {}_{\mathcal{U}}\bullet)}$.

Proof. The $\mathcal{W}$-category of algebras is cartesian making the forgetful functor $\mathcal{U}\colon \mathrm{EM}_{\Psi} \to \mathbf{C}$ monoidal; $\mathcal{U}C \times A$ defines a strong monoidal action.

$$\int^{C \in \mathrm{EM}_{\Psi}} \mathbf{C}(S, \mathcal{U}C \times A) \times \mathbf{D}(\mathcal{U}C \bullet B, T)$$
$$\cong \quad \{\text{Adjunction } (\times) \dashv \Delta\}$$
$$\int^{C \in \mathrm{EM}_{\Psi}} \mathbf{C}(S, \mathcal{U}C) \times \mathbf{C}(S, A) \times \mathbf{D}(\mathcal{U}C \bullet B, T)$$
$$\cong \quad \{\text{Adjunction } \Psi \dashv \mathcal{U}\}$$
$$\int^{C \in \mathrm{EM}_{\Psi}} \mathrm{EM}_{\Psi}(\Psi S, C) \times \mathbf{C}(S, A) \times \mathbf{D}(\mathcal{U}C \bullet B, T)$$
$$\cong \quad \{\text{Coyoneda}\}$$
$$\mathbf{C}(S, A) \times \mathbf{D}(\Psi S \bullet B, T). \qquad \square$$

Definition 3.7 (Ril18, §4.8)**.** Let $(\mathbf{D}, \otimes, [])$ be a right closed $\mathcal{V}$-category with a monoidal $\mathcal{V}$-action $(\bullet)\colon \mathbf{D} \otimes \mathbf{C} \to \mathbf{C}$ to an arbitrary $\mathcal{V}$-category $\mathbf{C}$. A **linear lens** is an element of

$$\mathbf{Lens}_{[]}((A, B), (S, T)) \coloneqq \mathbf{C}(S, [B, T] \bullet A).$$

Proposition 3.8. *Linear lenses are mixed optics (as in Definition 2.1) for the actions of the monoidal product* $(\otimes)\colon \mathbf{D} \times \mathbf{D} \to \mathbf{D}$ *and* $(\bullet)\colon \mathbf{D} \times \mathbf{C} \to \mathbf{C}$*. That is,* $\mathbf{Lens}_{[]} \cong \mathbf{Optic}_{\otimes, \bullet}$

Proof. The monoidal product has a right adjoint given by the exponential.

$$\begin{aligned}
&\int^{D\in\mathbf{D}} \mathbf{C}(S, D\bullet A)\otimes \mathbf{D}(D\otimes B, T)\\
\cong\ & \{\text{Adjunction } (-\otimes B)\dashv [B,-]\}\\
&\int^{D\in\mathbf{D}} \mathbf{C}(S, D\bullet A)\otimes \mathbf{D}(D, [B,T])\\
\cong\ & \{\text{Coyoneda}\}\\
&\mathbf{C}(S, [B,T]\bullet A).
\end{aligned}$$

□

Definition 3.9. Let $\mathbf{D}$ be a cocartesian $\mathcal{W}$-category with a monoidal $\mathcal{W}$-action $(\bullet)\colon \mathbf{D}\times\mathbf{C}\to\mathbf{C}$ to an arbitrary category $\mathbf{C}$. A **prism** is an element of

$$\mathbf{Prism}((A,B),(S,T)) \coloneqq \mathbf{C}(S, T\bullet A)\times \mathbf{D}(B,T).$$

In other words, a prism from (S,T) to (A,B) is a lens from (T,S) to (B,A) in the opposite categories $\mathbf{D}^{op}$ and $\mathbf{C}^{op}$. However, they can also be seen as optics from (A,B) to (S,T).

Proposition 3.10. *Prisms are mixed optics (as in Definition 2.1) for the actions of the coproduct* $(+)\colon \mathbf{D}\times\mathbf{D}\to\mathbf{D}$ *and* $(\bullet)\colon \mathbf{C}\times\mathbf{D}\to\mathbf{D}$. *That is,* $\mathbf{Prism}\cong\mathbf{Optic}_{(\bullet,+)}$.

Proof. The coproduct $(+)\colon \mathbf{C}^2\to\mathbf{C}$ is left adjoint to the diagonal functor.

$$\begin{aligned}
&\int^{D\in\mathbf{D}} \mathbf{C}(S, D\bullet A)\times \mathbf{D}(D+B, T)\\
\cong\ & \{\text{Adjunction } (+)\dashv \Delta\}\\
&\int^{D\in\mathbf{D}} \mathbf{C}(S, D\bullet A)\times \mathbf{D}(D, T)\times \mathbf{D}(B,T)\\
\cong\ & \{\text{Coyoneda}\}\\
&\mathbf{C}(S, T\bullet A)\times\mathbf{D}(B,T).
\end{aligned}$$

□

3.2. Traversals.

Definition 3.11. A **traversal** is an element of

$$\mathbf{Traversal}((A,B),(S,T)) \coloneqq \mathcal{V}\left(S, \int^{n\in\mathbb{N}} A^n\otimes [B^n, T]\right).$$

Remark 3.12. Let $(\mathbb{N},+)$ be the free strict monoidal category on one object. Ends and coends indexed by $\mathbb{N}$ coincide with products and coproducts, respectively. Here $A^{(-)}\colon \mathbb{N}\to\mathbf{C}$ is the unique monoidal $\mathcal{V}$-functor sending the generator of $\mathbb{N}$ to $A\in\mathbf{C}$. Each functor $C\colon\mathbb{N}\to\mathcal{V}$ induces a *power series*

$$\mathrm{Pw}_C(A) = \int^{n\in\mathbb{N}} A^n\otimes C_n.$$

This defines an action $\mathrm{Pw}\colon [\mathbb{N},\mathbf{C}]\to[\mathbf{C},\mathbf{C}]$ sending the indexed family to its power series. We propose a derivation of the *traversal* as the optic for power series.

Proposition 3.13. *Traversals are optics (as in Definition 2.1) for power series. That is,* $\mathbf{Traversal}\cong\mathbf{Optic}_{\mathrm{Pw},\mathrm{Pw}}$.

Proof. The derivation generalizes that of linear lenses (Definition 3.7).

$$\begin{aligned}
& \int^{C\in[\mathbb{N},\mathcal{V}]} \mathcal{V}\left(S, \int^{n\in\mathbb{N}} C_n \otimes A^n\right) \otimes \mathcal{V}\left(\int^{n\in\mathbb{N}} C_n \otimes B^n, T\right) \\
\cong\ & \{\text{Continuity}\} \\
& \int^{C\in[\mathbb{N},\mathcal{V}]} \mathcal{V}\left(S, \int^{n\in\mathbb{N}} C_n \otimes A^n\right) \otimes \int_{n\in\mathbb{N}} \mathcal{V}\left(C_n \otimes B^n, T\right) \\
\cong\ & \{\text{Adjunction } (-\otimes B^n) \dashv [B^n, -]\} \\
& \int^{C\in[\mathbb{N},\mathcal{V}]} \mathcal{V}\left(S, \int^{n\in\mathbb{N}} C_n \otimes A^n\right) \otimes \int_{n\in\mathbb{N}} \mathcal{V}\left(C_n, [B^n, T]\right) \\
\cong\ & \{\text{Natural transformation}\} \\
& \int^{C\in[\mathbb{N},\mathcal{V}]} \mathcal{V}\left(S, \int^{n\in\mathbb{N}} C_n \otimes A^n\right) \otimes [\mathbb{N},\mathcal{V}]\left(C_{(-)}, [B^{(-)}, T]\right) \\
\cong\ & \{\text{Coyoneda}\} \\
& \mathcal{V}\left(S, \int^{n\in\mathbb{N}} A^n \otimes [B^n, T]\right).
\end{aligned}$$
□

4. Tambara theory

Profunctor optics are functions polymorphic over profunctors endowed with some extra algebraic structure. This extra structure depends on the family of the optic they represent. For instance, *lenses* are represented by functions polymorphic over *cartesian* profunctors, while *prisms* are represented by functions polymorphic over *cocartesian* profunctors [PGW17, §3]. Milewski notes that the algebraic structures accompanying these profunctors are precisely *Tambara modules* [Mil17], a particular kind of profunctor that has been used to characterize the monoidal centre of convolution monoidal categories [Tam06]. Because of this correspondence, categories of lenses or prisms can be obtained as particular cases of the "Doubles for monoidal categories" defined by Pastro and Street [PS08, §6].

Generalizing the construction of these "Doubles for monoidal categories", we obtain that copresheaves over the standard $\mathcal{V}$-category structure of optics (as in Definition 2.1) are *Tambara modules.*

Definition 4.1. Let $(\mathbf{M}, \otimes, I)$ be a monoidal $\mathcal{V}$-category with two monoidal actions $(Ⓛ)\colon \mathbf{M}\otimes\mathbf{C}\to\mathbf{C}$ and $(Ⓡ)\colon \mathbf{M}\otimes\mathbf{D}\to\mathbf{D}$. A generalized **Tambara module** consists of a $\mathcal{V}$-profunctor $P\colon \mathbf{C}^{op}\otimes\mathbf{D}\to\mathcal{V}$ endowed with a family of morphisms

$$\alpha_{M,A,B}\colon P(A,B)\to P(M Ⓛ A, M Ⓡ B)$$

$\mathcal{V}$-natural in $A\in\mathbf{C}$ and $B\in\mathbf{D}$ and $\mathcal{V}$-dinatural $M\in\mathbf{M}$, which additionally satisfies the two equations

$$\begin{gathered}
\alpha_{I,A,B}\circ P(\phi_A, \varphi_B^{-1}) = \mathrm{id}, \\
\alpha_{M\otimes N,A,B}\circ P(\phi_{M,N,A}, \varphi_{M,N,B}^{-1}) = \alpha_{M,N Ⓛ A, N Ⓡ B}\circ \alpha_{N,A,B},
\end{gathered}$$

for every $M, N\in\mathbf{M}$, every $A\in\mathbf{C}$ and every $B\in\mathbf{D}$.

4.1. **Profunctor representation theorem.** Let us zoom out to the big picture again. It has been observed that optics can be composed using their profunctor representation; that is, profunctor optics can be endowed with a natural categorical structure. On the other hand, we have generalized the double construction by Pastro and Street [PS08] to abstractly obtain the category **Optic**. The last missing piece that makes both coincide is the *profunctor representation theorem*, which will justify the profunctor representation of optics and their composition in profunctor form being the usual function composition.

The profunctor representation theorem for the case $\mathcal{V} = \mathbf{Sets}$ and non-mixed optics has been discussed by Boisseau and Gibbons [BG18, Theorem 4.2]. Although our statement is more general and the proof technique is different, the main idea is the same. In both cases, the key insight is the following lemma, already described by Milewski [Mil17].

Lemma 4.2 ("Double Yoneda" from Milewski [Mil17])**.** *For any $\mathcal{V}$-category $\mathbf{A}$, the hom-object between X and Y is $\mathcal{V}$-naturally isomorphic to the object of $\mathcal{V}$-natural transformations between the functors that evaluate copresheaves in X and Y; that is,*

$$\mathbf{A}(X, Y) \cong [[\mathbf{A}, \mathcal{V}], \mathcal{V}](-(X), -(Y)).$$

The isomorphism is given by the canonical maps $\mathbf{A}(X, Y) \to \mathcal{V}(FX, FY)$ for each $F \in [\mathbf{A}, \mathcal{V}]$. Its inverse is given by computing its value on the identity on the $\mathbf{A}(X, -)$ component.

Proof. In the functor $\mathcal{V}$-category $[\mathbf{A}, \mathcal{V}]$, we can apply the Yoneda embedding to two representable functors $\mathbf{A}(Y, -)$ and $\mathbf{A}(X, -)$ to get

$$\mathrm{Nat}(\mathbf{A}(Y, -), \mathbf{A}(X, -)) \cong \int_F \mathcal{V}\Big([\mathbf{A}(X, -), F], [\mathbf{A}(Y, -), F]\Big).$$

Here reducing by Yoneda lemma on both the left hand side and the two arguments of the right hand side, we get the desired result. □

Theorem 4.3 (Profunctor representation theorem)**.**

$$\int_{P \in \mathbf{Tamb}} \mathcal{V}(P(A, B), P(S, T)) \cong \mathbf{Optic}((A, B), (S, T)).$$

Proof. We apply Double Yoneda (Lemma 4.2) to the $\mathcal{V}$-category **Optic** and then use that copresheaves over it are precisely Tambara modules. □

5. Conclusions

We have extended a result by Pastro and Street to a setting that is useful for optics in functional programming. Using it, we have refined some of the optics already present in the literature to mixed optics, providing derivations for each one of them. We have also described new optics. A Haskell implementation is available at [PR20].

5.1. **Related work.** Pastro and Street [PS08] first described the construction of *doubles* in their study of Tambara theory. Their results can be reused for *optics* thanks to the observations of Milewski [Mil17]. The profunctor representation theorem and its implications for functional programming have been studied by Boisseau and Gibbons [BG18]. We combine their approach with Pastro and Street's to get a proof of a more general version of this theorem.

The case of mixed optics was first mentioned by Riley [Ril18, §6.1], but his work targeted a more restricted case. Specifically, the definitions of *optic* given by Riley [Ril18, Definition 2.0.1] and Boisseau [BG18, §4.5] deal only with the particular case in which $\mathcal{V} = \mathbf{Sets}$, the categories $\mathbf{C}$ and $\mathbf{D}$ coincide, and the two actions are the same. Riley uses the results of Jaskelioff and O'Connor [JO15] to propose a description of the traversal in terms of traversable functors [Ril18, §4.6]; our derivation simplifies this approach, which was in principle not suitable for the enriched case.

A central aspect of Riley's work is the extension of the concept of *lawful lens* to arbitrary *lawful optics* [Ril18, §3]. This extension works exactly the same for the optics we define here, so we do not address it explicitly in this text. A first reasonable notion of lawfulness for the case of mixed optics for two actions $(Ⓛ) \colon \mathbf{M} \otimes \mathbf{C} \to \mathbf{C}$ and $(Ⓡ) \colon \mathbf{N} \otimes \mathbf{D} \to \mathbf{D}$ is to use a cospan $\mathbf{C} \to \mathbf{E} \leftarrow \mathbf{D}$ *of actions* to push the two parts of the optic into the same category and then consider lawfulness in $\mathbf{E}$. This can be applied to monadic lenses, for instance, when the monad is copointed.

Acknowledgements

This work was started in the last author's MSc thesis [Rom19]; development continued at the Applied Category School 2019 at Oxford, and we thank the organizers of the School for that opportunity. We also thank Pawel Sobocinski for discussion.

Bryce Clarke is supported by the Australian Government Research Training Program Scholarship. Fosco Loregian and Mario Román are supported by the European Union through the ESF funded Estonian IT Academy research measure (project 2014-2020.4.05.19-0001).

References

[ACG+16] Faris Abou-Saleh, James Cheney, Jeremy Gibbons, James McKinna, and Perdita Stevens. Reflections on monadic lenses. In *A List of Successes That Can Change the World - Essays Dedicated to Philip Wadler on the Occasion of His 60th Birthday*, pages 1–31, Heidelberg, 2016. Springer. 3, 4

[BG18] Guillaume Boisseau and Jeremy Gibbons. What you needa know about Yoneda: Profunctor optics and the Yoneda Lemma (functional pearl). *PACMPL*, 2(ICFP):84:1–84:27, 2018. 3, 8, 9

[Boi17] Guillaume Boisseau. Understanding Profunctor Optics: a Representation Theorem. Master's thesis, University of Oxford, 2017. 3, 4

[Day70] Brian Day. On closed categories of functors. In *Reports of the Midwest Category Seminar IV*, pages 1–38, Heidelberg, 1970. Springer. 3

[FMH+19] Phil Freeman, Brian Marick, Lukas Heidemann, et al. Purescript Profunctor Lenses. Github `https://github.com/purescript-contrib/purescript-profunctor-lenses`, 2015–2019. 2

[Fox76] Thomas Fox. Coalgebras and cartesian categories. *Communications in Algebra*, 4(7):665–667, 1976. 5

[JO15] Mauro Jaskelioff and Russell O'Connor. A representation theorem for second-order functionals. *J. Funct. Program.*, 25, 2015. 9

[Kel05] G. M. Kelly. Basic concepts of enriched category theory. *Reprints in Theory and Applications of Categories*, 1(10):137, 2005. Reprint of the 1982 original. Cambridge Univ. Press, Cambridge; MR0651714. 3

[Kme18] Edward Kmett. Lens library, version 4.16. Hackage `https://hackage.haskell.org/package/lens-4.16`, 2012–2018. 2

[Lor19] Fosco Loregian. Coend calculus. *arXiv preprint arXiv:1501.02503*, 2019. 2

[Mil17] Bartosz Milewski. Profunctor optics: the categorical view. `https://bartoszmilewski.com/2017/07/07/profunctor-optics-the-categorical-view/`, 2017. 3, 7, 8, 9

[O'C15] Russell O'Connor. Mezzolens: Pure Profunctor Functional Lenses. Hackage `https://hackage.haskell.org/package/mezzolens`, 2015. 2

[Ole82] Frank Joseph Oles. *A Category-theoretic Approach to the Semantics of Programming Languages*. PhD thesis, Syracuse University, Syracuse, NY, USA, 1982. AAI8301650. 4

[Pal07] Luke Palmer. Making Haskell nicer for game programming. `https://lukepalmer.wordpress.com/2007/07/26/making-haskell-nicer-for-game-programming/`, archived at `https://web.archive.org/web/20141219182332/http://lukepalmer.wordpress.com/2007/07/26/making-haskell-nicer-for-game-programming/`, 2007. 4

[PGW17] Matthew Pickering, Jeremy Gibbons, and Nicolas Wu. Profunctor optics: Modular data accessors. *Programming Journal*, 1(2):7, 2017. 7

[PR20] Emily Pillmore and Mario Román. Vitrea library, version 0.1.0.0. Hackage `https://hackage.haskell.org/package/vitrea-0.1.0.0`, Github `https://github.com/mroman42/vitrea`, 2019–2020. 8

[PS08] Craig Pastro and Ross Street. Doubles for monoidal categories. *Theory and applications of categories*, 21(4):61–75, 2008. 3, 7, 8, 9

[Ril18] Mitchell Riley. Categories of optics. *arXiv preprint arXiv:1809.00738*, 2018. 3, 4, 5, 9

[Rom19] Mario Román. Profunctor optics and traversals. Master's thesis, University of Oxford, 2019. 9

[Spi19] David I. Spivak. Generalized lens categories via functors $\mathcal{C}^{\mathrm{op}} \to \mathsf{Cat}$. *arXiv preprint arXiv:1908.02202*, 2019. 3, 4, 5

[Tam06] Daisuke Tambara. Distributors on a tensor category. *Hokkaido mathematical journal*, 35(2):379–425, 2006. 7