Quasitoposes, Quasiadhesive Categories and Artin Glueing

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Abstract. Adhesive categories are a class of categories in which pushouts along monos are well-behaved with respect to pullbacks. Recently it has been shown that any topos is adhesive. Many examples of interest to computer scientists are not adhesive, a fact which motivated the introduction of quasiadhesive categories. We show that several of these examples arise via a glueing construction which yields quasitoposes. We show that, surprisingly, not all such quasitoposes are quasiadhesive and characterise precisely those which are by giving a succinct necessary and sufficient condition on the lattice of subobjects.

1 Introduction

Adhesive categories, introduced in [8], are a class of categories where pushouts along monos exist and are well-behaved with respect to pullbacks. They capture several examples of interest to computer scientists, in particular presheaf toposes. Amongst other applications, they have allowed the generalisation of several aspects of the theory of graph transformations. Several results which before were proved concretely at the level of the category of graphs and graph homomorphisms **Graph** have been generalised and shown to hold in any adhesive category. Because adhesive categories also enjoy useful closure properties, such theory is widely applicable.

Recently it has been shown by the second and the third authors that toposes are adhesive categories [10]. This result, while perhaps not surprising, is useful because topos theory is a well-established branch of mathematics with wide relevance to diverse fields such as logic, geometry and topology. Adhesive categories have less structure than toposes, meaning for example that they enjoy more closure properties (for instance, adhesive categories are closed under coslice). In particular, there are adhesive categories which are not toposes.

Early in the development of the theory of adhesive categories it became clear that the class of adhesive categories was too restrictive for several important examples, notably many arising from the theory of algebraic specifications. In such categories the class of regular monos (the monos which arise as equalisers) differs from the class of all monos. Since it is easy to show that all monos in an adhesive category are regular, it is immediate that the examples are not instances of

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adhesive categories. However, it was also clear that pushouts along *regular* monos enjoyed many of the properties which pushouts along monos enjoy in adhesive categories – for instance such pushouts are also pullbacks and regular monos are stable under pushout. The examples motivated the theory of quasiadhesive categories in which pushouts along regular monos are well-behaved with respect to pullbacks. An example of interest to computer scientists is the category of termgraphs [3]. There have been several other attempts at generalising the original definition of adhesivity, notably adhesive HLR categories [5] and weak adhesive HLR categories [6].

Here we return to some of the examples which first motivated the introduction of quasiadhesive categories and show that they all arise as instances of a glueing construction. As a consequence of this, we show that the categories are quasitoposes. Roughly, quasitoposes are to toposes as quasiadhesive categories are to adhesive categories, in the sense that much of the structure is assumed only of regular monos (in a topos, as in an adhesive category, all monos are regular). In fact, the nomenclature of topos theory motivated the name 'quasiadhesive'.

The central result of the paper can be considered surprising: quasitoposes are a generalisation of toposes roughly as quasiadhesive categories are a generalisation of adhesive categories and since as mentioned previously, toposes are adhesive, one could expect also that quasitoposes are quasiadhesive. As we shall show, this is not the case. In fact, we shall characterise which quasitoposes are quasiadhesive: precisely those where unions of regular subobjects are regular in the subobject lattice. The proof of the main result is interesting in part because it constructs a direct counterexample in any quasitopos in which the condition fails. An example of a quasitopos which is not quasiadhesive is the category of binary relations **BRel**, also known as the category of simple graphs.

Returning to our examples, we take advantage of the fact that they arise uniformly and exhibit a sufficient and necessary condition on the functor along which gluing occurs for the resulting category to be quasiadhesive. This characterisation allows us immediately to derive which of the categories are quasiadhesive. For instance, the category of injective functions, **Inj** is a quasiadhesive quasitopos while the quasitopos **Spec** of algebraic specifications is not quasiadhesive.

Structure of the paper. In §2 we recall the definitions of adhesive and quasiadhesive categories. In §3 we introduce our main motivating examples. We show that these examples arise uniformly as a certain full subcategory of a category obtained by Artin glueing in §4 and prove that such categories are quasitoposes. We prove that a quasitopos is quasiadhesive if and only if unions of regular subobjects are regular in §5. Using this result, we show a necessary and sufficient condition on the glueing functor which allows us to immediately show which of the examples are quasiadhesive. We conclude in §6.

2 Preliminaries

Here we shall briefly recall the notions of adhesive and quasiadhesive categories together with a few of their properties. Adhesive and quasiadhesive categories rely on the notion of a van Kampen (VK) square. Van Kampen squares are pushouts which satisfy a certain axiomatic condition.

Definition 1. A *van Kampen square* is a pushout which satisfies the following: for any commutative cube in which it is the bottom face and which has the left and rear faces pullbacks, the front and right faces are pullbacks if and only if the top face is a pushout. Another way of stating the "only if" part of the above condition is that such a pushout is required to be stable under pullback.

$$\begin{tikzcd}
C' \arrow[rr] \arrow[dr] \arrow[dd] & & B' \arrow[dr] \arrow[dd] & \\
& A' \arrow[rr] \arrow[dd] & & D' \arrow[dd] \\
C \arrow[rr] \arrow[dr] & & B \arrow[dr] & \\
& A \arrow[rr] & & D
\end{tikzcd}$$

A category $\mathbf{C}$ is adhesive when it has pullbacks, pushouts along monos and such pushouts are VK squares. All monos are regular in an adhesive category.

Lemma 2. *Monos and regular monos coincide in any adhesive category.*

Proof. See [9, Lemma 4.9]. □

A category $\mathbf{C}$ is quasiadhesive when it has pullbacks, pushouts along regular monos and such pushouts are VK squares. An adhesive category is a quasiadhesive category where all the monos are regular, in this sense adhesive categories can be regarded as "degenerate" quasiadhesive categories.

3 Motivating Examples

In this section we introduce several examples which fail to be adhesive. As we shall see in Section 4, all of them can be seen as instances of a particular construction and as a consequence all are quasitoposes.

Example 3 (Injective Functions). Let $\mathbf{Inj}$ be the category with objects the injective functions $m\colon X \to Y$ in $\mathbf{Set}$ and arrows commutative diagrams as illustrated to the right.

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "m"'] & X' \arrow[d, "m"] \\
Y \arrow[r, "g"'] & Y'
\end{tikzcd}$$

An object of $\mathbf{Inj}$ can be thought of as a set together with a chosen subset, equivalently a set equipped with a unary predicate. The morphisms are those functions which preserve the subset/predicate in the obvious way. The monos are precisely those where g is an injective function. The regular monos are those monos which reflect the predicate; this condition is easily seen to be equivalent to requiring that the resulting diagram of monos in $\mathbf{Set}$ is a pullback.

Example 4 (Binary relations). The category of binary relations, $\mathbf{BRel}$ has as objects triples $\langle V, E, m\rangle$ with $m\colon E \to V \times V$ an injective function. Arrows are the obvious commutative diagrams:

$$\begin{tikzcd}
E \arrow[r, "f_E"] \arrow[d, "m"'] & E' \arrow[d, "m'"] \\
V \times V \arrow[r, "f_V \times f_V"'] & V' \times V'.
\end{tikzcd}$$

It follows easily that **BRel** is equivalent to the category of graphs with at most one edge from one vertex to another (ie $\{ \langle V, E \rangle \mid E \subseteq V \times V \}$), and arrows ordinary graph morphisms. Some authors refer to such objects as simple graphs. As with **Inj**, the monos in **BRel** are easily seen to be the diagrams arising from f_V being injective. Again, the regular monos are those which reflect edges, or equivalently pullback diagrams with all maps being monos.

The following example appeared in [4]. Fix an arbitrary nonempty set $\mathcal{P}$ of *predicates* with an arity function $ar\colon \mathcal{P} \to \mathbb{N}$. Given a set of *atoms* X, a predicate $P \in \mathcal{P}$, and $x_1, \ldots, x_{arP} \in X$, the formal expression $P(x_1, \ldots, x_{arP})$ is called an *atomic formula over* X. Let $AF_{\mathcal{P}}(X)$ denote the set of atomic formulas.

Example 5 (Structures). Let $\mathbf{Str}_{\mathcal{P}}$ be the category where the objects are *structures* – pairs $\langle X, Y \rangle$ where X is a set and $Y \subseteq AF_{\mathcal{P}}(X)$. A structure morphism $\langle f, g \rangle$ where $f\colon X \to X'$ and $g\colon Y \to Y'$ such that $g(P(x_1, \ldots, x_{arP})) \equiv P(fx_1, \ldots, fx_{arP})$ – that is, atomic formulas are preserved.

The monos are clearly those homomorphisms where the map on the underlying sets is injective. The regular monos are those where also predicates are reflected, in the obvious way.

Example 6 (Algebraic specifications). A *signature* is a quadruple $\Sigma = \langle S, P, s, t \rangle$ where S is a set of *sorts*, P is a set of *operators*, $s\colon P \to S^*$ (where S^* denotes the free monoid over S) is a function giving the sorts of the domain of an operator and and $t\colon P \to S$ is the function giving the sort of the codomain of an operator. We write $p\colon s_1 \times \ldots \times s_k \to s$ if $s(p) = s_1 \ldots s_k$ and $t(p) = s$. A *signature morphism* $f\colon \Sigma \to \Sigma'$ is a pair $f = \langle f_S, f_P \rangle$ where $f_S\colon S \to S'$, $f_P\colon P \to P'$ such that if $p\colon s_1 \times \ldots \times s_k \to s$ then $f_P(p)\colon f_S(s_1) \times \ldots \times f_S(s_k) \to f_S(s)$. The category **Sig** of signatures and signature morphisms is a presheaf topos – indeed, it is isomorphic to the category of hypergraphs[1] with edges restricted to having a single target node (but an arbitrary finite number of source nodes).

A *term* $\sigma(\mathtt{x_1} : s_1, \ldots, \mathtt{x_n} : s_n) : s$ over a signature Σ is the obvious formal construction built up from the basic operators of Σ and composition which may contain instances of variables $\mathtt{x_i} : s_i$. To avoid extra complexity, we shall assume that variables appear at most once within a term. Each term has a unique sort s determined by the codomain sort of the root operator in the syntax tree of the term. An *equation* over a signature Σ is a formal expression of the form $\sigma_1(\mathtt{x_1} : s_1, \ldots, \mathtt{x_n} : s_n) : s \equiv \sigma_2(\mathtt{x_1} : s_1, \ldots, \mathtt{x_n} : s_n) : s$.

An *algebraic specification* is a pair $\mathcal{S} = \langle \Sigma, E \rangle$ where Σ is a signature and E is a set of equations over Σ. An algebraic specification morphism $f\colon \mathcal{S} \to \mathcal{S}'$ is a pair $f = \langle \langle f_S, f_P \rangle, f_E \rangle$ where $\langle f_S, f_P \rangle : \Sigma \to \Sigma'$ is a signature morphism and $f_E\colon E \to E'$ is a function satisfying

$$\begin{aligned} f_E\,(\ \sigma_1(\mathtt{x_1} : s_1, \ldots, \mathtt{x_n} : s_n) : s \equiv \sigma_2(\mathtt{x_1} : s_1, \ldots, \mathtt{x_n} : s_n) : s\ ) = \\ f_P(\sigma_1)(\mathtt{x'_1} : f_S(s_1), \ldots, \mathtt{x'_n} : f_S(s_n)) : f_S(s) \equiv \\ f_P(\sigma_2)(\mathtt{x'_1} : f_S(s_1), \ldots, \mathtt{x'_n} : f_S(s_n)) : f_S(s) \end{aligned}$$

[1] Some authors use the term 'multigraphs'.

for some injective renaming of variables $\mathtt{x_i} \mapsto \mathtt{x'_i}$. The category of algebraic specifications and algebraic specification morphisms is denoted **Spec**.

A mono is an algebraic specification morphism with an underlying signature morphism that is injective (on both sorts and operators). A regular mono also reflects equations, in the obvious way.

Example 7 (Safely marked Petri nets). A Petri net is a tuple $N = \langle P, T, s, t\rangle$ where P is a set of places, T is a set of transitions and $s, t \colon T \to P^*$ are, respectively, the sources and targets of a transition. In other words, we think of a net as a multi-graph. A net morphism $f \colon N \to N'$ is a pair $f = \langle f_P, f_T\rangle$ where $f_P \colon P \to P'$, $f_T \colon T \to T'$ such that $s' f_T = f_P^* s$ and $t' f_T = f_P^* t$ (sources and targets of transitions are preserved). Such a choice of morphism can be useful when deriving compositional labelled equivalences for nets, see for instance [13]. The category of Petri nets and morphisms is denoted **PNet**. It is easily seen to be a presheaf topos. A *marked place-transition net* $\mathcal{N}$ is a pair $\langle N, K, k\rangle$ where $N \in \mathbf{PNet}$, K is a set of *tokens*, and $k \colon K \to P$ is a mapping of tokens to places. A place-transition net morphism $f \colon \mathcal{N} \to \mathcal{N}'$ is a pair $f = \langle f_N, f_K\rangle$ where $f_N \colon N \to N'$ is a map of Petri nets and $f_K \colon K \to K'$ is a function between the sets of tokens satisfying $k' f_K = f_P k$ (places of tokens are preserved). Let **PTNet** be the category of marked place-transition nets and morphisms. A *safely marked place-transition net* is a place-transition net where there is at most one token on each place, that is, the function $k \colon K \to P$ is injective. Let **SPTNet** be the full subcategory of **PTNet** consisting of safely marked nets.

A mono in **SPTNet** is a morphism of marked-nets which is injective on the underlying net. A regular mono has the additional property that the marking is reflected.

Lemma 8. *The categories* **Inj** *(Example 3),* **BRel** *(Example 4),* **Str** *(Example 5),* **Spec** *(Example 6), and* **SPTNet** *(Example 7) are not adhesive.*

Proof. Immediate since the classes of monos and regular monos do not coincide in any of these categories (cf Lemma 2). ◻

4 Glueing

In this section we shall demonstrate that the examples discussed in §3 are formed using a particular variant of a general construction known as Artin glueing [2].

More explicitly, we shall see that the examples given in §3 are actually certain full subcategories of categories obtained by glueing. Using a well-known result, categories obtained by Artin glueing are quasitoposes. Using the fact that also the aforementioned full subcategories of quasitoposes are themselves quasitoposes (Theorem 16) we know that the examples are quasitoposes.

We begin by recalling the definition of a quasitopos.

Definition 9. A category **C** is said to be a quasitopos when it satisfies all of the following conditions:

(i) it has finite limits and colimits;
(ii) it is locally cartesian closed;
(iii) it has a regular-subobject-classifier.

Quasitoposes and quasiadhesive categories share several basic properties, as we outline below.

Proposition 10. *The following hold in any quasitopos $\mathcal{E}$ and in any quasiadhesive category* $\mathbf{C}$*:*

(i) pushouts along regular monos are also pullbacks;
(ii) regular monos are stable under pushout;
(iii) unions of regular subobjects are effective[2]*;*

Proof. Quasitoposes: (i) and (ii) see [7, A2.6.2] and (iii) see [7, A1.4.3]. Quasiadhesive categories: (i) and (ii) see [9, Lemma 6.1]. Part (iii) is a straightforward generalisation of [9, Theorem 5.1]. ☐

In addition, quasitoposes admit a number of factorisation systems.

Proposition 11. *In any quasitopos, every arrow can be factorised into an epi followed by a regular mono or by a regular epi followed by a mono.*

Proof. For the regular epi - mono factorisation see [7, Scholium 1.3.5]. For the mono - regular epi factorisation one can use the dual of [7, Scholium 1.3.5] since regular monos are stable under pushout. ☐

Given categories $\mathbf{C}$, $\mathbf{D}$ and a functor $T\colon \mathbf{D} \to \mathbf{C}$, we write $\mathbf{C}/T$ for the category with objects arrows $f\colon C \to TD$ for $C \in \mathbf{C}$, $D \in \mathbf{D}$. An arrow in $\mathbf{C}/T$ is a pair $\langle g, h\rangle : f \to f'$ consisting of an arrow $g\colon C \to C'$ in $\mathbf{C}$ and $h\colon D \to D'$ in $\mathbf{D}$ such that $(Th)f = f'g$.

$$\begin{tikzcd} C \arrow[r, "g"] \arrow[d, "f"'] & C' \arrow[d, "f'"] \\ TD \arrow[r, "Th"'] & TD' \end{tikzcd}$$

Definition 12 (Artin glueing). A category $\mathbf{Z}$ is said to be obtained by Artin glueing if it is the slice category $\mathbf{C}/T$ for some functor $T\colon \mathbf{D} \to \mathbf{C}$ where $\mathbf{C}$ and $\mathbf{D}$ are quasitoposes.

We shall usually perform Artin glueing along a pullback-preserving functor $T\colon \mathbf{C} \to \mathbf{D}$. It is a well-known "folk" theorem[3] that when $\mathbf{C}$ has finite limits (and this will always be the case for us, since $\mathbf{C}$ will be a quasitopos) then T also preserves all finite limits. The category obtained by glueing along such a functor will in fact be a quasitopos; the following theorem is actually a special case of a more general result of Carboni and Johnstone [2, Theorem 3.3]. We will rely on the 'if' direction which was first shown by Rosebrugh and Wood [11].

[2] Unions of subobjects are said to be effective when the union of two subobjects is obtained by pushing out along their intersection.
[3] See [2, Lemma 1.1] for a proof.

Theorem 13. *If* $\mathbf{C}$, $\mathbf{D}$ *are quasitoposes then* $\mathbf{C}/T$ *is a quasitopos iff* T *preserves pullbacks.* □

We shall denote by $\mathbf{C}/\!\!/T$ the full subcategory of $\mathbf{C}/T$ with objects the monos $m\colon C \to TD$ in $\mathbf{C}$.[4] The following lemma lists some properties of regular monos in $\mathbf{C}/T$ and $\mathbf{C}/\!\!/T$.

Lemma 14. *Suppose that* $\mathbf{C}$, $\mathbf{D}$ *are quasitoposes and* $T\colon \mathbf{D} \to \mathbf{C}$ *preserves pullbacks. Then:*

(i) A map $\langle g, h\rangle$ *in* $\mathbf{C}/T$ *is a regular mono iff* g *is a regular mono in* $\mathbf{C}$ *and* h *is a regular mono in* $\mathbf{D}$.
(ii) A map $\langle g, h\rangle$ *in* $\mathbf{C}/\!\!/T$ *is a regular mono iff* g *is a regular mono in* $\mathbf{C}$, h *is a regular mono in* $\mathbf{D}$ *and the resulting square in* $\mathbf{C}$ *is a pullback diagram.*

Proof. ($\Rightarrow$) Since T preserves pullbacks and $\mathbf{C}$ is finitely complete, T preserves equalisers. In particular, this means that equalisers are constructed componentwise in $\mathbf{C}/T$. It is easy to check that the full subcategory $\mathbf{C}/\!\!/T$ is closed with respect to equalisers and the resulting square is a pullback.
($\Leftarrow$) Suppose that g is a regular mono in $\mathbf{C}$ and that h is a regular mono in $\mathbf{D}$. Let α, β be the cokernel pair of g (obtained by pushing out g along itself in $\mathbf{C}$) and φ, ψ be a pair in $\mathbf{D}$ of which h is the equaliser. Since pushouts along regular monos are also pullbacks in $\mathbf{C}$ (cf Proposition 10, (i)), it follows that g is the equaliser of α and β. Using the fact that α and β are the cokernel pair, let f_2 be the unique map such that $f_2\alpha = T\varphi.f_1$ and $f_2\beta = T\psi.f_1$ (see the first diagram below). It follows that $\langle g, h\rangle$ is the equaliser of $\langle \alpha, \varphi\rangle$ and $\langle \beta, \psi\rangle$ in $\mathbf{C}/T$.

$$\begin{tikzcd} C \arrow[r, "g"] \arrow[d, "f"'] & C_1 \arrow[r, shift left, "\alpha"] \arrow[r, shift right, "\beta"'] \arrow[d, "f_1"] & C_2 \arrow[d, dotted, "f_2"] \\ TD \arrow[r, "Th"'] & TD_1 \arrow[r, shift left, "T\varphi"] \arrow[r, shift right, "T\psi"'] & TD_2 \end{tikzcd} \qquad \begin{tikzcd} C \arrow[r, "g"] \arrow[d, "f"'] & C_1 \arrow[r, shift left, "T\varphi.f_1"] \arrow[r, shift right, "T\psi.f_1"'] \arrow[d, "f_1"] & TD_2 \arrow[d] \\ TD \arrow[r, "Th"'] & TD_1 \arrow[r, shift left, "T\varphi"] \arrow[r, shift right, "T\psi"'] & TD_2 \end{tikzcd}$$

For $\mathbf{C}/\!\!/T$, let φ and ψ be a pair in $\mathbf{D}$ of which h is the equaliser. We shall show that g is the equaliser of $T\varphi.f_1$ and $T\psi.f_1$. Indeed, suppose there is a map $x\colon X \to C_1$ such that $T\varphi.f_1x = T\psi.f_1x$. Since T preserves equalisers, there is a unique map $y\colon X \to TD$ such that $Th.y = f_1x$. Using the fact that the square is a pullback, there is a unique map $z\colon X \to C$ such that $fz = y$ and $gz = x$. It follows that $\langle g, h\rangle$ is the equaliser of $\langle T\varphi.f_1, \varphi\rangle$ and $\langle T\psi.f_1, \psi\rangle$ in $\mathbf{C}/\!\!/T$, as illustrated in the second diagram above. □

In Theorem 16 we shall show that if T is a pullback-preserving functor between quasitoposes then also $\mathbf{C}/\!\!/T$ is a quasitopos. Note that when $\mathbf{C}$ and $\mathbf{D}$ are toposes

[4] One could also define $\mathbf{C}/\!\!/T$ to be the full subcategory of $\mathbf{C}/T$ with objects the *regular* monos. In that case, the conclusion of Theorem 16 would still hold and can be proved using a modified version of Lemma 15. For the purposes of this paper the precise definition used is a moot point since in all of our examples both $\mathbf{C}$ and $\mathbf{D}$ are actually toposes.

then $\mathbf{C}/T$ is also a topos; on the other hand, as will be demonstrated by the examples, $\mathbf{C}/\!\!/T$ will usually be only a quasitopos. We first prove a technical lemma.

Lemma 15. *In the following we assume that* $T\colon \mathbf{D} \to \mathbf{C}$ *preserves finite limits.*

(i) *If* $\mathbf{C}$ *and* $\mathbf{D}$ *are cartesian closed then also* $\mathbf{C}/\!\!/T$ *is cartesian closed;*
(ii) *If* $\mathbf{C}$ *and* $\mathbf{D}$ *are locally cartesian closed then also* $\mathbf{C}/\!\!/T$ *is locally cartesian closed;*
(iii) *If* $\mathbf{C}$ *and* $\mathbf{D}$ *have finite colimits and* $\mathbf{C}$ *has regular epi - mono factorisations then* $\mathbf{C}/\!\!/T$ *has finite colimits;*
(iv) *If* $\mathbf{D}$ *is a quasitopos then* $\mathbf{C}/\!\!/T$ *has a regular-subobject-classifier.*

Proof. *(i)* It is well-known (see [2]) that with these assumptions $\mathbf{C}/T$ is cartesian closed. The internal hom of $C_1 \to TD_1$ and $C_2 \to TD_2$ is the pullback $P \to T[D_1, D_2]$ of $[C_1, C_2] \to [C_1, TD_2]$ along $T[D_1, D_2] \to [TD_1, TD_2] \to [C_1, TD_2]$, as illustrated below.

$$\begin{tikzcd}
P \arrow[rr] \arrow[d] & & {[C_1, C_2]} \arrow[d] \\
T[D_1, D_2] \arrow[r] & {[TD_1, TD_2]} \arrow[r] & {[C_1, TD_2]}
\end{tikzcd}$$

Clearly if $C_2 \to TD_2$ is mono then so is $[C_1, C_2] \to [C_1, TD_2]$ and therefore also $P \to T[D_1, D_2]$. Since $\mathbf{C}/\!\!/T$ is a full subcategory of the cartesian-closed category $\mathbf{C}/T$ and is closed under exponentiation, it itself is cartesian-closed.
(ii) Suppose that $C \to TD$ is an object of $\mathbf{C}/\!\!/T$. Then $(\mathbf{C}/\!\!/T)/(C \to TD) = (\mathbf{C}/C)/\!\!/T'$, where $T'\colon \mathbf{D}/D \to \mathbf{C}/C$ is given by first applying T to get $\mathbf{D}/D \to \mathbf{C}/TD$, and then pulling back along $C \to TD$ as in $\mathbf{C}/TD \to \mathbf{C}/C$. Now $\mathbf{C}/C$ and $\mathbf{D}/D$ are cartesian closed by assumption, and T' clearly preserves finite limits, so $(\mathbf{C}/C)/\!\!/T'$ is cartesian closed by *(i)*, and so $\mathbf{C}/\!\!/T$ is locally cartesian closed.
(iii) It is well-known that in this case also $\mathbf{C}/T$ has finite colimits. The presence of the regular epi - mono factorisation system in $\mathbf{C}$ ensures that $\mathbf{C}/\!\!/T$ is a reflective subcategory of $\mathbf{C}/T$ and so it too has finite colimits.
(iv) A regular subobject in $\mathbf{C}/\!\!/T$ is a pullback square, as illustrated, in which the horizontal maps are regular mono. If $\mathbf{D}$ has a regular-subobject-classifier W, then it is straightforward to verify that $TW \to TW$ is a regular-subobject-classifier in $\mathbf{C}/\!\!/T$.

$$\begin{tikzcd}
C' \arrow[r] \arrow[d] & C \arrow[d] \\
TD' \arrow[r] & TD
\end{tikzcd}$$

□

Theorem 16. *If* $\mathbf{C}$, $\mathbf{D}$ *are quasitoposes and* $T\colon \mathbf{D} \to \mathbf{C}$ *preserves pullbacks then* $\mathbf{C}/\!\!/T$ *is a quasitopos.*

Proof. The proof follows using the results of Lemma 15 and applying the usual reduction in the style of [2]. Indeed, if $\mathbf{C}$, $\mathbf{D}$ are quasitoposes and $T\colon \mathbf{D} \to \mathbf{C}$ preserves pullbacks then $T_1\colon \mathbf{D} \to \mathbf{C}/T1$ preserves finite limits (since it preserves pullbacks and the terminal object) and so $(\mathbf{C}/T1)/\!\!/T_1$ is a quasitopos. But this is just $\mathbf{C}/\!\!/T$. □

We have given a direct proof of Theorem 16. A shorter proof would use the fact that the objects of $\mathbf{C}/\!\!/T$ are the separated objects for the closure operator corresponding to the fact that $\mathbf{D}$ is an open sub-quasitopos of $\mathbf{C}/T$.

We shall now demonstrate that the examples of §3 are of the form $\mathbf{D}/\!\!/T$ for specific choices of $\mathbf{C}, \mathbf{D}$ and pullback-preserving $T\colon \mathbf{D} \to \mathbf{C}$. Theorem 16 ensures that such categories are quasitoposes, thus eliminating the need for tedious direct proofs.

Proposition 17. *The categories* **Inj***,* **BRel***,* **Str***,* **Spec** *and* **SPTNet** *are of the form* $\mathbf{C}/\!\!/T$ *for* $T\colon \mathbf{D} \to \mathbf{C}$ *a pullback-preserving functor between toposes.*

Proof. **Inj** (cf Example 3): Let $\mathbf{C}, \mathbf{D} = \mathbf{Set}$ and let T be the identity functor. It is immediate that $\mathbf{Inj} \cong \mathbf{Set}/\!\!/T$.

BRel (cf Example 4): Let $\mathbf{C}, \mathbf{D} = \mathbf{Set}$ and let $TX = X \times X$. It is immediate that $\mathbf{BRel} \cong \mathbf{Set}/\!\!/T$; also, T preserves limits since it is representable – indeed, $TX \cong \mathbf{Set}(2, X)$.

Str (cf Example 5): Let $T\colon \mathbf{Set} \to \mathbf{Set}$ be defined $TX = \sum_{P\in\mathcal{P}} X^{ar(P)} \cong \sum_{P\in\mathcal{P}} \mathbf{Set}(ar(P), X)$. Pullback preservation is immediate. Notice that there is a bijection $TX \cong AF_{\mathcal{P}}(X)$, so to give a subset of atomic formulas is essentially to give a mono with codomain TX; hence it is easy to show that $\mathbf{Str}_{\mathcal{P}} \simeq \mathbf{Set}/\!\!/T$.

Spec (cf Example 6): Let $T\colon \mathbf{Sig} \to \mathbf{Set}$ be the free term functor: a signature $\Sigma = \langle S, P, s, t\rangle$ is taken to the set $T\Sigma$ of all terms. The action on signature morphisms is canonical.

$$\begin{tikzcd} U\Sigma \arrow[r, "q_2"] \arrow[d, "q_1"'] & T\Sigma \arrow[d, "p"] \\ T\Sigma \arrow[r, "p"'] & S^* \times S \end{tikzcd}$$

Let $S^* \times S$ denote the set of nonempty words over S and $p\colon T\Sigma \to S^* \times S$ be the evident map which takes a term to its “type”. Define $U\Sigma$ to be the illustrated pullback diagram in **Set**. It follows that $U\colon \mathbf{Sig} \to \mathbf{Set}$ is a functor. Intuitively, $U\Sigma$ is the set of “well-typed” equations (consisting of two terms of the same result sort and taking an equal number of variables, with each corresponding pair agreeing on the sorts) between terms over Σ. It follows that $\mathbf{Spec} \simeq \mathbf{Set}/\!\!/U$, we omit the proof of the fact that U preserves pullbacks;

SPTNet (cf Example 7): Let $U\colon \mathbf{PNet} \to \mathbf{Set}$ be the forgetful functor which takes a Petri net to its set of places. It follows that $\mathbf{SPTNet} \simeq \mathbf{Set}/\!\!/U$. Using the fact that limits are computed pointwise in **PNet**, U preserves them. □

Corollary 18. *The categories* **Inj***,* **BRel***,* **Str***,* **Spec** *and* **SPTNet** *are quasitoposes.*

Proof. Immediate by Proposition 17 and Theorem 16. □

5 Quasitoposes and Quasiadhesive Categories

In this section we shall characterise precisely which quasitoposes are quasiadhesive. We begin by proving an important property of quasiadhesive categories – regular monos are closed under union. This result forms one direction of a characterisation of quasiadhesive quasitoposes, which appears as Theorem 21. The proof itself is a step-by-step construction of a counterexample at an abstract level and is followed by a concrete example in Corollary 20.

Theorem 19. *In quasiadhesive categories, binary unions of regular subobjects are regular*

Proof. Suppose that $Z \in \mathbf{C}$ and that U and V are two regular subobjects of Z such that $U \cup V$ is not a regular subobject. We shall show that $\mathbf{C}$ cannot be quasiadhesive by constructing an explicit counterexample cube. Let W and X denote respectively $U \cap V$ and $U \cup V$. Let $\overline{X}$ denote the smallest regular subobject of Z which contains X, ie the join of U and V in the lattice of regular subobjects of Z. This object can be obtained by factorising the map $X \to Z$ into an epi followed by a regular mono. Some work is required to show that this factorisation can be obtained in any quasiadhesive category, we omit the details and only give a sketch: one first shows that $X \to Z$ admits a cokernel pair; secondly, one can construct the equaliser $\overline{X} \to Z$ of the cokernel pair by pulling back, giving the regular mono part of the factorisation. Finally one uses a standard argument to show that the map $\overline{x}\colon X \to \overline{X}$ given by the universal property of equalisers is epi.

We obtain objects A and $\overline{A}$ by constructing the pushouts in the left diagram below, where $\overline{v} = \overline{x}v$. Clearly $\overline{v}$ is regular mono by the usual cancellation properties.

$$\begin{tikzcd}
W \arrow[r, dotted, "\pi_2"] \arrow[d, dotted, "\pi_1"'] \arrow[dr, phantom, "(\dagger)"] & V \arrow[r, dotted, "\overline{v}"] \arrow[d, dotted, "v"] & \overline{X} \arrow[d, dotted, "q"] \arrow[dd, bend left=60, "j_2"] \\
U \arrow[r, dotted, "u"'] & X \arrow[r, dotted, "p"'] \arrow[d, "\overline{x}"'] & A \arrow[d, "\overline{a}"] \\
& \overline{X} \arrow[r, dotted, "j_1"'] & \overline{A}
\end{tikzcd}
\qquad
\begin{tikzcd}
W \arrow[r, dotted, "\pi_1"] \arrow[d, dotted, "\pi_1"'] \arrow[dr, phantom, "(\ddagger)"] & U \arrow[r, dotted, "\overline{u}"] \arrow[d, dotted, "i_2"] & \overline{X} \arrow[d, dotted, "q"] \arrow[dd, bend left=60, "\mathrm{id}"] \\
U \arrow[r, dotted, "i_1"'] & B \arrow[r, dotted, "b"'] \arrow[d, "h"'] & A \arrow[d, "r"] \\
& U \arrow[r, dotted, "\overline{u}"'] & \overline{X}
\end{tikzcd}$$

Note that $(\dagger)$ is a pushout by Proposition 10, (iii). All three pushouts are also pullbacks by part (i) of the proposition, since all the horizontal morphisms in the diagram are regular mono. The fact that the lower square is a pullback together with the fact that $\overline{x}$ is not an isomorphism implies that $\overline{a}$ is not an isomorphism.

Now consider the second diagram above in which $(\ddagger)$ is a pushout and $\overline{u} = \overline{x}u$. Using the fact that $\overline{u}\pi_1 = \overline{v}\pi_2$ the pushout of $\overline{X}$ and U along W is A and we obtain a map $b\colon B \to A$ such that the upper right square commutes and $bi_1 = pu$. By the pasting properties of pushouts, this square is also a pushout. Let $h\colon B \to U$ be the codiagonal (the unique map such that $hi_1 = hi_2 = \mathrm{id}_U$) and let r be the unique map which satisfies $rq = \mathrm{id}_{\overline{X}}$ and $rb = \overline{u}h$. The outer rectangle is clearly a pushout and thus the lower square is also a pushout, by cancellation.

Consider the first diagram below, it shows that the pullback of $\overline{A} \to Z$ and $U \to Z$ is the pushout B of U together with itself along W. To see this, let B' denote this pullback and note that the pullback of $U \to Z$ along $\overline{X} \to Z$ is just U and similarly the pullback of U and V is W. Hence all the vertical faces of the diagram are pullbacks and since pushouts along regular monos are stable under pullback, the upper face of the cube must be a pushout diagram; hence $B \cong B'$.

In particular, we can erase all the primes from the diagram.

$$\begin{tikzcd}
 & & U \arrow[d, "\overline{u}"] \arrow[drr, "i'_1"] & & & \\
W \arrow[urr, "\pi_1"] \arrow[d, "\pi_2"'] \arrow[dr, "\pi_1"] & & \overline{X} \arrow[drr, "j_1"] & & B' \arrow[r, "h'"] \arrow[d, "\overline{b}'"] & U \arrow[d, "u'"] \\
V \arrow[urr, "\overline{v}"] \arrow[dr, "\overline{v}"'] & U \arrow[urrr, "i'_2"'] \arrow[d, "\overline{u}"] & & & \overline{A} \arrow[r] & Z \\
 & \overline{X} \arrow[urrr, "j_2"] & & & &
\end{tikzcd}
\qquad
\begin{tikzcd}
B \arrow[r, dotted, "\overline{b}"] \arrow[d, "h"'] \arrow[dr, phantom, "(\star)"] & \overline{A} \arrow[d, "s"] \\
U \arrow[r, dotted, "\overline{u}"'] \arrow[d] & \overline{X} \arrow[d] \\
U \arrow[r, dotted, "u'"'] & Z
\end{tikzcd}$$

In the second diagram above, let s be the codiagonal. As we have established, the outer region is a pullback and it is immediate that the lower square is a pullback also – thus the upper square $(\star)$ must be a pullback by cancellation. We claim that also that $(\star)$ is a pushout.

To see this, assume that there are maps $f\colon \overline{A} \to C$ and $g\colon U \to C$ such that $f\overline{b} = gh$. Since h and s are epi (being codiagonals), it is enough to show that there exists a map $\alpha\colon \overline{X} \to C$ such that $\alpha s = f$. Note that by the construction of $\overline{A}$ a morphism $f\colon \overline{A} \to C$ corresponds to a pair of morphisms $f_1 = fj_1 = faq\colon \overline{X} \to C$ and $f_2 = fj_2\colon \overline{X} \to C$. We shall show that we can take α to be f_1; to show $f_1 s = f$, it is enough to show that the two maps agree when precomposed with j_1 and j_2; thus we need to show that $f_1 = f_2$. Clearly f_1 and f_2 agree when restricted to $V \rightarrowtail \overline{X}$. Also $f_1\overline{u} = fj_1\overline{u} = f\overline{b}i_2 = f\overline{a}bi_2 = f\overline{a}q\overline{u} = fj_2\overline{u} = f_2\overline{u}$. Thus they must agree on the union $\overline{x}\colon X \rightarrowtail \overline{X}$ of these subobjects. And x is epi by construction, so $f_1 = f_2$.

Now consider the illustrated cube; the top and bottom faces are both pushouts, the diagonal edges are regular mono and three of the four vertical faces are pullbacks. But the front face is not a pullback, because as we have observed previously, $\overline{a}\colon A \to \overline{A}$ is not an isomorphism.

$$\begin{tikzcd}
B \arrow[rr, "h"] \arrow[dd, "b"'] \arrow[dr] & & U \arrow[dr, "\overline{u}"] \arrow[dd] & \\
 & A \arrow[rr, "r"] \arrow[dd, "\overline{a}"] & & \overline{X} \arrow[dd] \\
B \arrow[rr, "h"] \arrow[dr, "\overline{b}"'] & & U \arrow[dr] & \\
 & \overline{A} \arrow[rr, "s"'] & & \overline{X}
\end{tikzcd}$$

□

Note that, although we presented the proof of Theorem 19 above as a proof by contradiction, the argument is in fact constructive: it shows that if $\mathbf{C}$ is quasiadhesive then the mono $\overline{a}\colon A \to \overline{A}$ must be an isomorphism, and hence so is $\overline{x}\colon X \to \overline{X}$.

Corollary 20. The construction in the proof of Theorem 19 allows us to conclude that the category of binary relations **BRel** is *not* quasiadhesive. Indeed, considering the objects of **BRel** as simple graphs, it is immediate that the two vertices of $a \to b$ are regular subobjects but their union is not regular. The counterexample constructed starting from these two subobjects is shown in Fig 1.

The following theorem is the main result of this section. It gives a sufficient and necessary condition for a quasitopos to be quasiadhesive. The condition is easy to state and usually straightforward to check – a quasitopos is quasiadhesive if and only if regular subobjects are closed under binary unions.

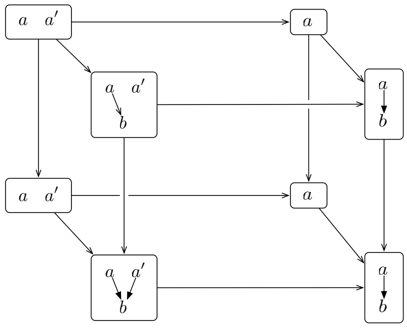

Fig. 1. A counterexample demonstrating that **BRel** is not quasiadhesive

Theorem 21. *Let* **C** *be a quasitopos. Then the following are equivalent:*

(i) **C** *is quasiadhesive.*
(ii) the class of regular subobjects of any object is closed under binary union.

Proof. (i) ⇒ (ii): Is immediate by the conclusion of Theorem 19 and Proposition 11.

(ii) ⇒ (i): We give a brief sketch of a "direct" argument. A shorter but slightly more technical proof relies on connections between quasiadhesivity and Artin glueing in quasitoposes.

Consider the illustrated cube: since we can factorise maps into a regular epi followed by a mono and van Kampen squares are closed under pasting, it suffices to consider the situation where f is a mono or f is a regular epi. The case where f is a mono is straightforward; the reasoning for the case where f is a regular epi is similar to the proof of [10, Theorem 25].

$$\begin{tikzcd}
C' \arrow[rr, "f'"] \arrow[dr] \arrow[dd, "m'"'] & & B' \arrow[dr, "n'"] \arrow[dd] & \\
& A' \arrow[rr, "g'"] \arrow[dd] & & D' \arrow[dd] \\
C \arrow[rr, "f"] \arrow[dr, "m"'] & & B \arrow[dr, "n"'] & \\
& A \arrow[rr, "g"'] & & D
\end{tikzcd}$$

□

When are the hypotheses of Theorem 21 satisfied? Clearly, they hold if **C** is a topos; in particular, the fact that toposes are adhesive is a consequence, since all monos are regular and by the conclusion of Theorem 21 they are quasiadhesive. In this sense, the above result is a generalisation of the main result of [10]. They also hold if **C** is a Heyting algebra, where the only regular monos are isomorphisms.

Proposition 22. *Heyting algebras are quasiadhesive.* □

Clearly, in other quasitoposes, it suffices to check the condition that unions of regular subobjects are regular. We have seen that **BRel** does not satisfy the condition and thus is not quasiadhesive. As our examples are of the form $\mathbf{C}/\!\!/T$, it is useful to understand how unions are computed in such categories.

Lemma 23. *Unions of subobjects in in* $\mathbf{C}/T$ *and* $\mathbf{C}/\!\!/T$ *are computed pointwise. That is, the union of subobjects* $\langle C_1 \rightarrowtail C, D_1 \rightarrowtail D\rangle$ *and* $\langle C_2 \rightarrowtail C, D_2 \rightarrowtail D\rangle$ *of*

$f\colon C \rightarrowtail TD$ is $\langle C_1 \cup C_2 \rightarrowtail C, D_1 \cup D_2 \rightarrowtail D\rangle$, where the unions are formed in **C** *and* **D***, respectively.*

Proof. $\mathbf{C}/T$ is a quasitopos by Theorem 13; $\mathbf{C}/\!\!/T$ is a quasitopos by Theorem 16 and thus unions are effective in both the categories (Proposition 10, (iii)). But all colimits in $\mathbf{C}/T$ are computed pointwise and pullbacks are computed pointwise because T preserves them. Because $\mathbf{C}/\!\!/T$ is reflective in $\mathbf{C}/T$, it suffices to check that the map $C_1 \cup C_2 \to T(D_1 \cup D_2)$ is a mono, but this follows easily since in the commutative diagram below, $C_1 \cup C_2 \to C$ and $C \to TD$ are mono.

$$\begin{tikzcd} C_1 \cup C_2 \arrow[r] \arrow[d] & C \arrow[d] \\ T(D_1 \cup D_2) \arrow[r] & TD \end{tikzcd}$$

□

Recall that in Theorem 16 we showed that for a pullback-preserving functors $T\colon \mathbf{D} \to \mathbf{C}$ between quasitoposes, $\mathbf{C}/\!\!/T$ is a quasitopos. The following result characterises precisely those T for which $\mathbf{C}/\!\!/T$ is also quasiadhesive.

Theorem 24. *Suppose* **C**, **D** *are quasiadhesive quasitoposes and $T\colon \mathbf{D} \to \mathbf{C}$ is pullback-preserving. Then $\mathbf{C}/\!\!/T$ is quasiadhesive iff T preserves unions of regular subobjects*[5].

Proof. ($\Leftarrow$). By Theorem 16, $\mathbf{C}/\!\!/T$ is a quasitopos. Suppose that T preserves unions of regular subobjects. For $i \in \{1, 2\}$ suppose that we have regular subobjects of $C \to TD$ in $\mathbf{C}/\!\!/T$, as illustrated in the first diagram below. Their union is calculated pointwise (Lemma 23), is illustrated in the second diagram. Because **C** and **D** are quasiadhesive, the horizontal maps in the second diagram are regular. Using part (ii) of Lemma 14, it suffices to show that the square is a pullback. We show this directly, suppose there are $\alpha\colon X \to C$ and $\beta\colon X \to T(D_1 \cup D_2)$ such that $m\alpha = Td.\beta$. Using the assumption we can take $T(D_1 \cup D_2) = TD_1 \cup TD_2$ and pull back β to obtain a pushout diagram which decomposes X, as illustrated.

$$\begin{tikzcd} C_i \arrow[r, "c_i"] \arrow[d, "m_i"'] & C \arrow[d, "m"] \\ TD_i \arrow[r, "Td_i"'] & TD \end{tikzcd} \qquad \begin{tikzcd} C_1 \cup C_2 \arrow[r, "c"] \arrow[d, "m_3"'] & C \arrow[d, "m"] \\ T(D_1 \cup D_2) \arrow[r, "Td"'] & TD \end{tikzcd} \qquad \begin{tikzcd} X_3 \arrow[rr] \arrow[dr] \arrow[dd] & & X_2 \arrow[dr, "x_2"] \arrow[dd, "\beta_2"' near start] & \\ & X_1 \arrow[rr, "x_1" near start] \arrow[dd, "\beta_1" near end] & & X \arrow[dd, "\beta"] \\ TD_1 \cup TD_2 \arrow[rr] \arrow[dr] & & TD_2 \arrow[dr] & \\ & TD_1 \arrow[rr] & & TD_1 \cup TD_2 \end{tikzcd}$$

Using the fact that the subobject diagrams are pullbacks, we obtain $h_i : X_i \to C_i$ such that $m_i h_i = \beta_i$ and $c_i h_i = \alpha x_i$. Using the decomposition of X, we obtain a unique map $h\colon X \to C$ such that $hx_i = j_i h_i$ where $j_i : C_i \to C_1 \cup C_2$. A routine calculation confirms that $ch = \alpha$ and $m_3 h = \beta$.

[5] That is, for $D_1 \to D$, $D_2 \to D$ regular subobjects, $T(D_1 \cup D_1) = TD_1 \cup TD_2$ as subobjects of TD.

($\Rightarrow$) Assume that the quasitopos $\mathbf{C}/\!\!/T$ is quasiadhesive, then unions of regular subobjects are regular by Theorem 21. Let $D_1 \to D$ and $D_2 \to D$ be regular monos. They lead to corresponding regular monos in $\mathbf{C}/\!\!/T$ as illustrated in the diagram for $i = 1, 2$.

$$\begin{tikzcd} TD_i \arrow[r] \arrow[d] \arrow[dr, phantom, "(\dagger)"] & TD \arrow[d] \\ TD_i \arrow[r] & TD \end{tikzcd}$$

The union results in the second diagram above, since unions are computed pointwise (cf Lemma 23). Since the union is a regular subobject, the square is a pullback and thus $T(D_1 \cup D_2) = TD_1 \cup TD_2$.

$$\begin{tikzcd} TD_1 \cup TD_2 \arrow[r] \arrow[d] & TD \arrow[d] \\ T(D_1 \cup D_2) \arrow[r] & TD \end{tikzcd}$$

$\square$

Lemma 25. *The category* **Inj** *of injective functions (cf Example 3) and the category* **SPTNet** *of safely-marked nets(cf Example 7) are quasiadhesive. The categories* $\mathbf{Str}_{\mathcal{P}}$ *where* $\mathcal{P}$ *has predicates of arity* > 1 *(cf Example 5) and* **Spec** *(cf Example 6) are not quasiadhesive.*

Proof. Using Theorem 24 and the fact that all of the examples are of the form $\mathbf{C}/\!\!/T$ (cf Proposition 17), it suffices to check whether T in each case preserves unions of regular subobjects. For **Inj**, $T = \mathrm{id}$ and the condition is obviously satisfied. Similarly, the condition clearly holds for the forgetful functor $U\colon \mathbf{PNet} \to \mathbf{Set}$ since unions in the presheaf topos **PNet** are calculated "pointwise". Any polynomial functor $T\colon \mathbf{Set} \to \mathbf{Set}$ which contains powers ≥ 2 does not preserve the union of the two injections of $1 \to 2$, thus $\mathbf{Str}_{\mathcal{P}}$ is not quasiadhesive when $\mathcal{P}$ has predicates of arity ≥ 2. Similarly, the functor $U : \mathbf{Sig} \to \mathbf{Set}$ does not preserve unions, consider the signature $\mathbf{2}$ with a single sort and two unary predicates and the signature $\mathbf{1}$ with a single sort and a single unary predicate. There are two monos $\mathbf{1} \to \mathbf{2}$ but their union is not preserved by U. $\square$

In Theorem 24 we exhibited a necessary and sufficient condition on T for $\mathbf{C}/\!\!/T$ to be quasiadhesive. The following result shows that the category $\mathbf{C}/T$ obtained by glueing is quasiadhesive if additionally T is cartesian and both $\mathbf{C}$ and $\mathbf{D}$ are quasiadhesive.

Lemma 26. *Let* $\mathbf{C}$ *and* $\mathbf{D}$ *be quasitoposes, and* $T\colon \mathbf{D} \to \mathbf{C}$ *a cartesian functor. Then the quasitopos* $\mathbf{C}/T$ *is quasiadhesive iff both* $\mathbf{C}$ *and* $\mathbf{D}$ *are.*

Proof. As observed in Lemma 14, a mono $\langle m, n\rangle : (A', B', f') \to (A, B, f)$ in $\mathbf{C}/T$ is regular iff both m and n are regular monos. Since unions of subobjects are also constructed 'component-wise', it is easy to see that $\mathbf{C}/T$ inherits the condition on unions of regular subobjects if both $\mathbf{C}$ and $\mathbf{D}$ satisfy it. Conversely, if we have a counterexample to the condition in either $\mathbf{C}$ or $\mathbf{D}$, we can obtain one in $\mathbf{C}/T$ by applying the appropriate direct image functor to it, since both these functors preserve regular monos. $\square$

6 Conclusions and Future Work

We have shown that several examples of interest to computer scientists are quasitoposes obtained by using a variant of the Artin glueing construction. We characterised the quasitoposes which are quasiadhesive in terms of a condition on the lattice of subobjects. We have refined this condition to the categories which arise from of the aforementioned construction.

There are two clear directions for future work. Firstly, the fact that not all our examples are quasiadhesive raises the question whether one can find a natural class of categories with less structure and more liberal closure conditions than quasitoposes while at the same time covering the basic properties satisfied by adhesive and quasiadhesive categories; useful for applications of rewriting and other related fields, for examples of such properties see for instance [9, 1, 12]. Secondly, as all of the examples in the present paper are quasitoposes, one could directly evaluate the suitability of rewriting directly on objects in an arbitrary quasitopos and study the resulting theories of parallelism and concurrency.

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