The Axiom of Choice in Cartesian bicategories

Filippo Bonchi
University of Pisa, Italy

Jens Seeber
IMT School for Advanced Studies Lucca, Italy

Paweł Sobociński
University of Southampton, UK

Abstract

We argue that cartesian bicategories, often used as a general categorical algebra of relations, are also a natural setting for the study of the axiom of choice (AC). In this setting, AC manifests itself as an inequation asserting that every total relation contains a map. The generality of cartesian bicategories allows us to separate this formulation from other set-theoretically equivalent properties, for instance that epimorphisms split. Moreover, via a classification result, we show that cartesian bicategories satisfying choice tend to be those that arise from bicategories of spans.

2012 ACM Subject Classification Theory of computation → Categorical semantics; Theory of computation → Logic

Keywords and phrases Cartesian bicategories, Axiom of choice, string diagrams

Digital Object Identifier 10.4230/LIPIcs.CALCO.2019.9

Introduction

Cartesian bicategories were introduced by Carboni and Walters [8] as a categorical algebra of relations and an alternative to Freyd and Scedrov's allegories [14][1]. In recent years they have been receiving renewed attention by researchers interested in string-diagrammatic languages. Indeed, thanks to the compact closed structure induced by Frobenius bimonoids, cartesian bicategories have proved to be an appropriate mathematical playground for compositional studies of different kinds of feedback systems. For instance, signal flow graphs [21], which are circuit-like specifications of linear dynamical systems, form a cartesian bicategory [3]. Moreover, the fact that cartesianity only holds laxly makes them able to serve as "resource-sensitive" syntax, as outlined in [4], where free cartesian bicategories were proposed as a resource-sensitive generalisation of Lawvere theories.

Free cartesian bicategories were also used in [5], where we showed that their algebraic presentation can be seen as an equational characterisation of well-known logical preorders, namely those arising from query inclusion of conjunctive queries (aka regular logic). The deep relationship between cartesian bicategories and regular logic—already alluded to in [8]—was also recently touched upon by Fong and Spivak [11].

In cartesian bicategories, it is important to distinguish between arbitrary morphisms—which can be thought of as relations—and a certain class of morphisms called *maps*, which can be thought of as functions. A fundamental result [8, Theorem 3.5] states that, for a cartesian bicategory $\mathcal{B}$ satisfying the property of *functional completeness*, (i) the subcategory of maps (denoted by $\mathrm{Map}\,\mathcal{B}$) is regular and (ii) the category of relations over the category of maps ($\mathbf{Rel}(\mathrm{Map}\,B)$) is biequivalent to $\mathcal{B}$. Unfortunately, this beautiful result is not relevant for

[1] RFC Walters referred to the modular law of allegories as a *formica mentale*, a "complication which prevents thought" (`http://rfcwalters.blogspot.com/2009/10/categorical-algebras-of-relations.html`).

8th Conference on Algebra and Coalgebra in Computer Science (CALCO 2019).
Editors: Markus Roggenbach and Ana Sokolova; Article No. 9; pp. 9:1–9:17
Leibniz International Proceedings in Informatics
LIPICS Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

free cartesian bicategories: for instance the categories obtained by the algebraic presentations in [4] and [5] do not arise from the $\mathbf{Rel}(\cdot)$ construction.

For this reason in [5], we needed to rely on an alternative construction $\mathsf{Span}^{\sim}$ that we believe is of independent interest. First, it requires less structure of the underlying category: while $\mathbf{Rel}(\cdot)$ requires a regular category, $\mathsf{Span}^{\sim}$ requires merely the presence of weak pullbacks. Second, while in the category of sets and functions both constructions yield the usual category of relations, as we shall see, there are important cases in which they differ.

Our main contribution is an analogue of the aforementioned result for $\mathsf{Span}^{\sim}$, namely that $\mathsf{Span}^{\sim}\,\mathrm{Map}\,\mathcal{B}$ is biequivalent to $\mathcal{B}$. In this setting, Carboni and Walters' functional completeness can be relaxed to a weaker condition that we call *having enough maps*, but an additional assumption is necessary: $\mathcal{B}$ has to satisfy the *axiom of choice*. Indeed, our main result (Theorem 30) asserts that a cartesian bicategory $\mathcal{B}$ with enough maps satisfies the axiom of choice *if and only if* $\mathcal{B}$ is biequivalent to $\mathsf{Span}^{\sim}\,\mathrm{Map}\,\mathcal{B}$.

This characterisation motivates a closer look at the axiom of choice, one of the best known—and most controversial—axioms of set theory [16]. It has many ZF-equivalent formulations, some requiring only very basic concepts. One is:

Every total relation contains a map.

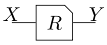
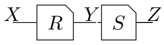
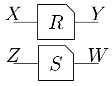
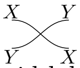
Our starting observation is that this condition is natural to state in the language of cartesian bicategories. Another way of viewing our main result is, therefore, a characterisation of cartesian bicategories with enough maps that satisfy the axiom of choice as precisely those that arise via the $\mathsf{Span}^{\sim}$ construction.

Given the innovations of topos theory [19] in foundations of mathematics, the question of whether or not to accept the axiom of choice is nowadays less absolute (and therefore less heated). Indeed, if a topos is a mathematical "universe", then it holds in some and not in others, thus accepting/rejecting choice turns from a philosophical question into a practical matter. Interpreting choice inside a category does not need the full power of the internal language of a topos – it suffices if the category in question captures basic properties of relations. Cartesian bicategories can therefore be seen as an amusing setting for the study of the axiom of choice. Indeed, the advantage of a weaker language is a finer grained analysis: e.g. we shall see that properties well-known to be equivalent to choice in ZF (e.g. surjective functions split) are different as properties of cartesian bicategories.

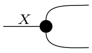
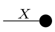
Structure of the paper. We start by giving an overview of a few important concepts of cartesian bicategories in Section 1. In Section 2 we define the axiom of choice in cartesian bicategories, the property of "having enough maps" and discuss ramifications of this, including a useful characterisation. In Section 3 we introduce the $\mathsf{Span}^{\sim}$ construction and prove several useful results that are necessary for showing the classification theorem in Section 4. In Section 5, we compare the constructions $\mathbf{Rel}(\mathcal{C})$ and $\mathsf{Span}^{\sim}\mathcal{C}$ and show that they coincide if regular epis split in $\mathcal{C}$.

We would like to thank Aleks Kissinger and the team behind TikZit, which was used to create the diagrams in this paper.

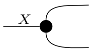
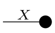
1 Cartesian bicategories

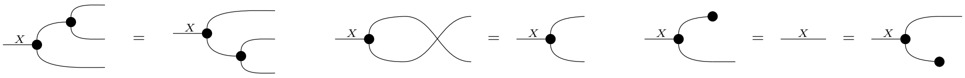
We start by recalling the notion of cartesian bicategory [8]. We will often use *string diagrams* [23] as a graphical notation for morphisms: given a symmetric monoidal category $\mathcal{B}$ with monoidal product $\otimes$ and monoidal unit I, a wire $\overset{X}{\text{---}}$ denotes id_X, the identity for an arbitrary object X of $\mathcal{B}$, while a box  denotes an arbitrary morphism $R\colon X \to Y$ of $\mathcal{B}$; for $R\colon X \to Y$ and $S\colon Y \to Z$, the composition $R;S\colon X \to Z$ is depicted as  ; for $R\colon X \to Y$ and $S\colon Z \to W$, $R \otimes S\colon X \otimes Z \to Y \otimes W$ is depicted as  ; symmetries $\sigma\colon X \otimes Y \to Y \otimes X$ are drawn as  ; the identity for I as an empty diagram . When clear from the context, we will avoid labelling wires.

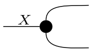
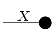
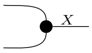
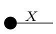
▶ **Definition 1.** A cartesian bicategory is a symmetric monoidal category $\mathcal{B}$ enriched over the category of posets. Every object $X \in \mathcal{B}$ is equipped with morphisms

 $: X \to X \otimes X$ and  $: X \to I$

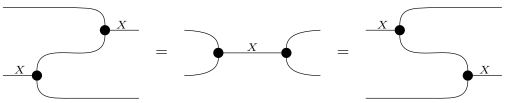
such that

-  and  form a cocommutative comonoid, that is they satisfy



-  and  have right-adjoints  and  respectively, that is

- The Frobenius law holds, that is



- Each morphism $R\colon X \to Y$ is a lax comonoid homomorphism, that is

- The choice of comonoid on every object is coherent with the monoidal structure[2] in the sense that

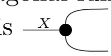
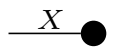
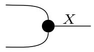
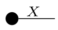
A morphism of cartesian bicategories is a monoidal functor preserving the ordering and the chosen monoids and comonoids.

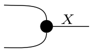
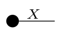
[2] In the original definition of [8] this property is replaced by requiring the uniqueness of the comonoid/-monoid. However, as suggested in [22], coherence seems to be the property of primary interest.

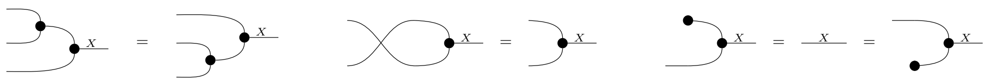
The archetypal example of a cartesian bicategory is the category of sets and relations **Rel**, with cartesian product of sets, hereafter denoted by $\times$, as monoidal product and $1 = \{\bullet\}$ as unit I. To be precise, **Rel** has sets as objects and relations $R \subseteq X \times Y$ as arrows $X \to Y$. Composition and monoidal product are defined as expected:

$$R;S = \{(x,z) \mid \exists y \text{ s.t. } (x,y) \in R \text{ and } (y,z) \in S\},$$

$$R \otimes S = \{\big((x_1,x_2)\,,\,(y_1,y_2)\big) \mid (x_1,y_1) \in R \text{ and } (x_2,y_2) \in S\}.$$

For each set X, the comonoid structure is given by the diagonal function $X \to X \times X$ and the unique function $X \to 1$, considered as relations. That is  $= \{\big(x,(x,x)\big) \mid x \in X\}$ and  $= \{(x,\bullet) \mid x \in X\}$. Their right adjoints are given by their opposite relations:  $= \{\big((x,x),x\big) \mid x \in X\}$ and  $= \{(\bullet,x) \mid x \in X\}$. The reader can easily check that the four inequalities are satisfied and that, moreover, the Frobenius law holds. The right adjoints also enjoy an additional property that holds in any cartesian bicategory.

▶ **Lemma 2.**  *and*  *form a commutative monoid, that is*



To appreciate the property that every morphism is a lax-comonoid homomorphism, it is useful to spell out its meaning in **Rel**: in the first inequality, the left and the right-hand side are, respectively, the relations

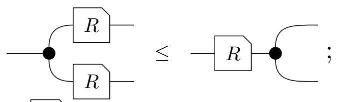
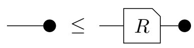
$$\{\big(x,(y,y)\big) \mid (x,y) \in R\} \quad \text{and} \quad \{\big(x,(y,z)\big) \mid (x,y) \in R \text{ and } (x,z) \in R\}, \tag{1}$$

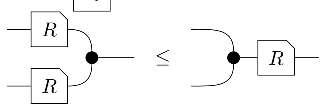
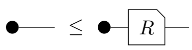
while in the second inequality, they are the relations

$$\{(x,\bullet) \mid \exists y \in Y \text{ s.t. } (x,y) \in R\} \quad \text{and} \quad \{(x,\bullet) \mid x \in X\}. \tag{2}$$

It is immediate to see that the two left-to-right inclusions hold for any relation $R \subseteq X \times Y$, while the right-to-left inclusions hold exactly when R is a function: a relation which is single valued and total[3]. This observation justifies the following definition.

▶ **Definition 3.** Let R be a morphism in a cartesian bicategory. We call R

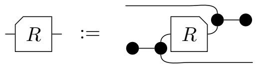
single valued if  ; total if  ;

injective if  ; surjective if  .

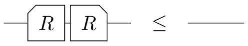
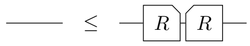
By translating the last two inequalities in **Rel**, similarly to what we have shown in (1) and (2), the reader can immediately check that these correspond to the usual properties of injectivity and surjectivity for relations. Moreover, since the converses of these inequalities hold in cartesian bicategories, the four inequalities are actually equalities.

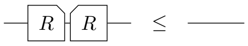
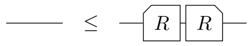
[3] Requiring every morphism to be a comonoid homomorphism would make $\otimes$ the categorical product [13] and thus the whole category would be cartesian. Ensuring just *lax*-comonoid homomorphism makes $\otimes$ a certain kind of bi-limit, called in [8], bi-product. This fact explains the name cartesian bicategory.

We can characterise all the notions of Definition 3 equivalently in terms of *opposite morphisms* R^{op} which are defined for any morphism R as follows:



▶ **Proposition 4.** *Let R be a morphism in a cartesian bicategory.*

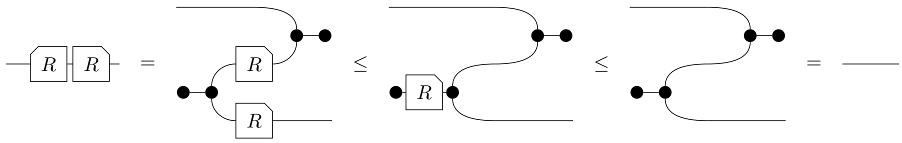
 *iff R is single valued.*  *iff R is total.*

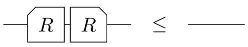
 *iff R is injective.*  *iff R is surjective.*

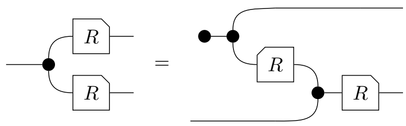
In particular, R is surjective iff R^{op} is total and R is injective iff R^{op} is single valued.

Proof. We show the proofs for single valued and total. The proofs for injectivity and surjectivity are analogous. The last statement follows from the others and the fact that $(R^{\mathrm{op}})^{\mathrm{op}} = R$

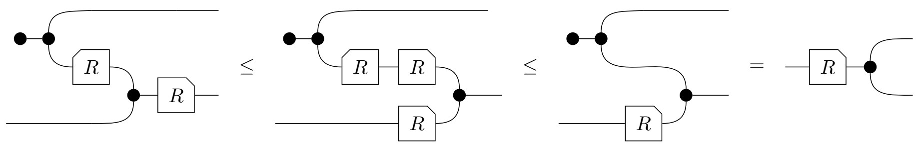
- Let R be single valued. Then



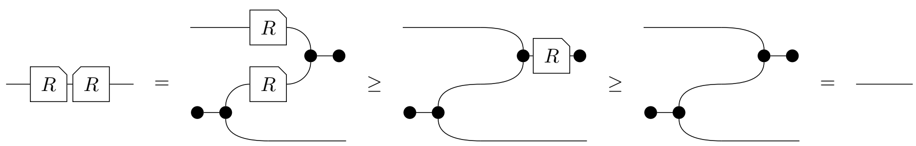
Conversely, if , then by the Frobenius law one gets

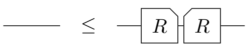


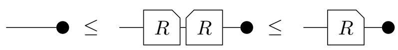
and from there



- Let R be total. Then

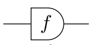
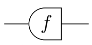


Conversely, if , then



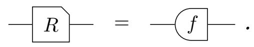
◀

▶ **Definition 5.** A map in a cartesian bicategory is a morphism f that is a comonoid homomorphism, i.e. is single valued and total.

We will write  to denote a map f and  for its opposite. Note that we use lower-case letters for maps and upper-case for arbitrary morphisms. Following the analogy with **Rel**, we will often call arbitrary morphisms of a cartesian bicategory relations.

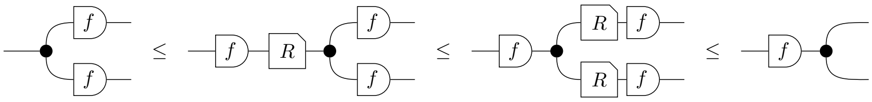
The original treatment of cartesian bicategories in [8] introduces maps as those morphisms that admit a right-adjoint. We show below that this amounts to the same notion.

▶ **Proposition 6.** *A morphism* f *is a map if and only if it has a right adjoint – a morphism* R *such that* $R\,f \leq$ —— *and* —— $\leq f\,R$ *. In that case, necessarily*

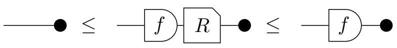


Proof. If f is a map, then f is a right-adjoint by Proposition 4.

On the other hand, if f has a right-adjoint R, then it is a map since



and



Therefore, f is indeed a map and $R = f$ by uniqueness of adjoints. ◀

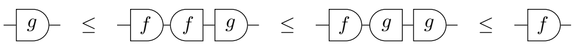
The identity is a map, and maps are easily shown to be closed under composition, so they constitute a category.

▶ **Definition 7.** Given a cartesian bicategory $\mathcal{B}$, we define its category of maps, Map($\mathcal{B}$) to have the same objects of $\mathcal{B}$ and as morphism the maps of $\mathcal{B}$.

By the following proposition, the ordering of $\mathcal{B}$ becomes trivial when restricted to maps.

▶ **Proposition 8.** *Let* f, g *be maps such that* $f \leq g$*. Then* $f = g$*.*

Proof. Since $f \leq g$, also $f \leq g$. Therefore



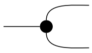
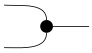
◀

We have seen that **Rel** is source of intuition for cartesian bicategories. There are many other similar examples; for instance **LinRel**, the category of linear relations of vector spaces where the monoidal product is the direct sum of vector spaces. Nevertheless, there are examples of cartesian bicategories that are significantly different, e.g. in which—concretely speaking—the monoidal product does not act as cartesian product on the underlying sets.

▶ **Example 9.** Recall that a prop is a strict symmetric monoidal category where the objects are the natural numbers and monoidal product on objects is addition. The prop **ERel** of *equivalence relations* [6, 9, 10, 12, 25] (also called the prop of *corelations*) has objects natural numbers, where $n \in \mathbb{N}$ is thought of as the finite set $\{0, \ldots, n-1\}$. A morphism $n \to m$ is an equivalence relation on $n + m$. Composition of an equivalence relation on $n + m$ with one on $m + o$ is given by taking the smallest equivalence relation they generate on $n + m + o$ and restricting it to $n + o$. Monoidal product is given by disjoint union.

Another important example is the prop **PERel** of *partial equivalence relations*. These are symmetric and transitive, but not necessarily reflexive, and have been used in the study of the semantics of higher order λ-calculi [17, 24] and quantum computations [15, 18]. In **PERel** a morphism $n \to m$ is a partial equivalence relation on $n + m$; composition similar to that in **ERel**, taking the smallest induced partial equivalence relation. Again $\otimes$ is given by disjoint union. See [25, Definitions 2.52 and 2.63] for additional details.

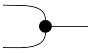
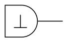
Both **ERel** and **PERel** carry the structure of cartesian bicategories after taking into consideration their posetal enrichment. Here the ordering $\leq$ is the *opposite* of set inclusion: $R \leq S$ iff $R \supseteq S$. Note that for **PERel**, we need some extra care. We consider partial equivalence relations $R, S\colon n \to m$ as equivalence relations $\bar{R}, \bar{S}$ over $(n+m) \cup \{\bot\}$ and then take $R \leq S$ iff $\bar{R} \supseteq \bar{S}$. In particular, notice that the completely undefined partial equivalence relation is represented by the chaotic relation on $(n+m) \cup \{\bot\}$, and is thus—according to this ordering—the least element in its homset.

To define the comonoid structure it is enough to consider 1, since for arbitrary n it is forced by coherence (Definition 1). For both **ERel** and **PERel**  $: 1 \to 2$ is the equivalence relation equating all the elements of the set $1+2$ and $: 1 \to 0$ equates the single element of the set 1. The monoid structure  $: 2 \to 1$ and $: 0 \to 1$ is defined in a similar way.

In order to illustrate what maps are in these categories, it is convenient to write $[i]_R$ for the set $\{j \mid (i,j) \in R\}$. Both in **ERel** and **PERel** a morphism $R\colon n \to m$ is

$$\text{total iff for all } i, j \in n, (i,j) \in R \text{ implies } i = j, \text{ and} \tag{3}$$

$$\text{single valued iff for all } i \in m, \text{ either } [i]_R = \emptyset \text{ or there is } j \in n \text{ such that } (i,j) \in R. \tag{4}$$

Thus  is single valued but not total; is total but not single valued. In **PERel**, the undefined relation $0 \to 1$, hereafter denoted , is both total and single valued.

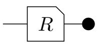
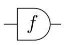
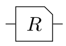
2 Choice in Cartesian bicategories

One of the many equivalent formulations of the axiom of choice in set theory is

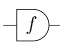
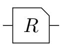
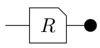
Every total relation contains a map.

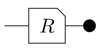
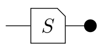
In a total relation every element in the domain is related to at least one element in the codomain. A map is obtained by choosing, for each element in the domain, exactly one related element in the codomain. This can be stated in the language of cartesian bicategories.

▶ **Definition 10** (Choice)**.** Let $\mathcal{B}$ be a cartesian bicategory. We say that $\mathcal{B}$ satisfies the axiom of choice (AC), or that $\mathcal{B}$ has choice, iff the following holds for any morphism $R\colon X \to Y$:

 $\leq$  (R is total) implies $\exists$ map $f\colon X \to Y$ such that  $\leq$  (AC)

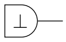
Observe that the converse implication holds in any cartesian bicategory.

▶ **Lemma 11.** *If*  $\leq$  *then* $\leq$ .

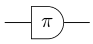
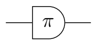
Proof. Obvious, since if S is total and $S \leq R$, then R is total:  $\geq$  $\geq$. ◀

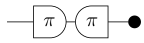
▶ **Example 12.**

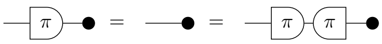
- The usual axiom of choice implies that **Rel** satisfies (AC).
- **ERel** is an example of a cartesian bicategory that does not satisfy (AC). Recall from Example 9 that the ordering is the *reverse* of inclusion. Therefore, for (AC) to hold would mean that every equivalence relation that satisfies (3) could be included in one that satisfies both (3) and (4). Now consider $: 0 \to 1$. As seen in Example 9, it is total, but not single valued. Since equivalence relations have to be reflexive, this is also the only morphism of type $0 \to 1$: clearly AC fails here.

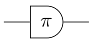
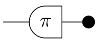
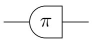
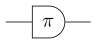
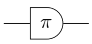
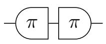
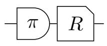
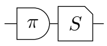
- Interestingly, **PERel** *does* satisfy (AC). For example, $: 0 \to 1$ is included, as an equivalence relation over $(0+1) \cup \{\bot\}$, in  .

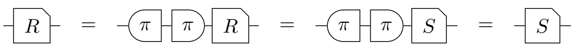
Another common formulation of the axiom of choice in set theory is the assertion that every surjective function $\pi \colon X \to Y$ splits, namely, there exists a function $\rho \colon Y \to X$ such that $\rho \,;\, \pi = id_Y$. A standard categorification of the notion of surjectivity is the notion of epi(morphism): π is epi iff $\pi \,;\, f = \pi \,;\, g$ entails $f = g$. In order to clarify the picture and justify our Definition 10 we will now investigate epimorphisms in cartesian bicategories.

▶ **Lemma 13.** *Let*  *be a map in a cartesian bicategory* $\mathcal{B}$. *Then*  *is an epi in* $\mathcal{B}$ *if and only if it is surjective.*

Proof.
- Let π be an epi in $\mathcal{B}$. Since π is a map, by Proposition 4, $\leq$  and therefore



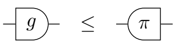
Since  is epi,  $=$ so  is total, hence  is surjective by Proposition 4.
- Assume  is surjective. Then  $=$ by Proposition 4. If now R, S are morphisms such that  $=$  , then



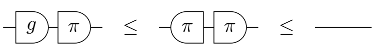
◀

▶ **Lemma 14.** *Surjective maps split in any cartesian bicategory with choice.*

Proof. Let $\pi \colon X \to Y$ be a surjective map. Therefore, $\pi^{\mathrm{op}} \colon Y \to X$ is a total relation, so by (AC) there is a map $g \colon Y \to X$ such that



Now we have

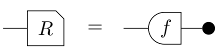


and since both the left hand side and the right hand side of that inequality are maps, we have by Proposition 8 that $g \,;\, \pi = \mathrm{id}_Y$. ◀

2.1 Cartesian bicategories with enough maps

The converse of Lemma 14 does not hold in general. The reason is that a general cartesian bicategory might not have enough maps to "cover" all its morphisms in a suitable sense. In order to prove the converse, we need to assume a saturation property.

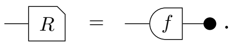
▶ **Definition 15.** We say a cartesian bicategory *has enough maps* if for every morphism $R \colon X \to I$ there is a map $f \colon Z \to X$ such that



The intuition for this notion is the following: a morphism $R\colon X \to I$ can be considered as a predicate on X. Then having enough maps ensures the existence of a function f that picks out the subset of X where R holds.

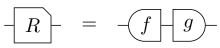
▶ **Example 16.** The description above shows that **Rel** has enough maps. Also **ERel** and **PERel** have both enough maps. We briefly describe the construction for **ERel**, the one for **PERel** is similar. For any morphism $R\colon n \to 0$ in **ERel**, take e to be the number of the equivalence classes of R. Choose a total ordering for these equivalence classes, so that for each $i \in e = \{0, \dots e-1\}$, we denote by R_i the i-th equivalence class of R. Then, define $f\colon e \to n$ as the equivalence on $e+n$

$$R \cup \{(i,j) \mid i \in e \text{ and } j \in R_i\} \cup \{(i,j) \mid j \in e \text{ and } i \in R_j\}.$$

It is immediate to see that f satisfies (3) and (4) and that  .

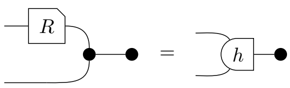
▶ Remark 17. A similar property, *functional completeness*, was already considered in [8]. The important difference is that we don't require f to be mono. Ours is a more general notion: every functionally complete cartesian bicategory also has enough maps.

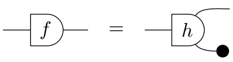
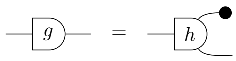
▶ **Lemma 18.** *If a cartesian bicategory has enough maps, then for every morphism $R\colon X \to Y$, there are maps $f\colon Z \to X$ and $g\colon Z \to Y$ such that*

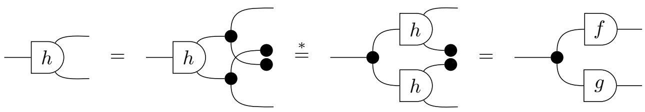


We call this a comap-map factorisation *of R.*

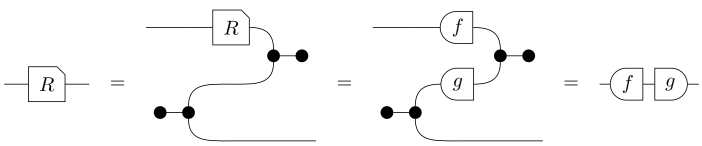
Proof. Since there are enough maps, there is a map $h\colon Z \to X \otimes Y$ such that



Let  and , then

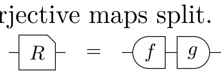


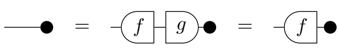
where the step marked $*$ uses coherence of the comonoid and the fact that h is a map. Therefore we have



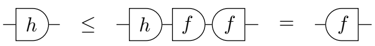
◀

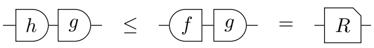
▶ **Proposition 19.** *A cartesian bicategory with enough maps satisfies* (AC) *iff surjective maps split.*

Proof. By Proposition 14, it suffices to prove that (AC) holds if surjective maps split. So let $R\colon X \to Y$ be a total relation and take a comap-map factorisation  with maps f, g. Since R is total,



so f is surjective. Since surjective maps split, there exists a map h that is a pre-inverse of f, so $h \mathbin{;} f = \mathrm{id}$. Then



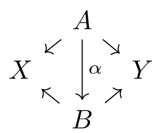
and therefore , so R contains a map. ◀

3 The Span$^\sim$ construction

In [5], we used a construction that, for any category with finite limits, gives rise to a cartesian bicategory. Since it plays a key role in our main result (Theorem 30), we recall the construction and extend it to arbitrary categories with finite products and *weak* pullbacks.

We start by recalling the standard notion of bicategory of spans.

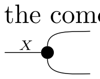
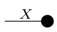
▶ **Definition 20** (Span)**.** Let $\mathcal{C}$ be a finitely complete category. A *span* from X to Y is a pair of arrows $X \leftarrow A \rightarrow Y$ in $\mathcal{C}$. A morphism $\alpha \colon (X \leftarrow A \rightarrow Y) \Rightarrow (X \leftarrow B \rightarrow Y)$ is an arrow $\alpha \colon A \rightarrow B$ in $\mathcal{C}$ s.t. the diagram on the right commutes. Spans $X \leftarrow A \rightarrow Y$ and $X \leftarrow B \rightarrow Y$ are *isomorphic* if α is an isomorphism. For $X \in \mathcal{C}$, the *identity span* is $X \xleftarrow{\mathrm{id}_X} X \xrightarrow{\mathrm{id}_X} X$. The composition of $X \leftarrow A \xrightarrow{f} Y$ and $Y \xleftarrow{g} B \rightarrow Z$ is $X \leftarrow A \times_{f,g} B \rightarrow Z$, obtained by taking the pullback of f and g. This data defines the bicategory [1] $\mathsf{Span}(\mathcal{C})$: the objects are those of $\mathcal{C}$, the arrows are spans and 2-cells are homomorphisms. Finally, $\mathsf{Span}(\mathcal{C})$ has monoidal product given by the product in $\mathcal{C}$, with unit the final object $1 \in \mathcal{C}$.



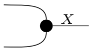
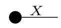
To avoid the complications of non-associative composition, it is common to consider a *category* of spans, where isomorphic spans are equated: let $\mathsf{Span}^{\leq}\mathcal{C}$ be the monoidal category that has isomorphism classes of cospans as arrows. Note that, when going from bicategory to category, after identifying isomorphic arrows it is usual to simply discard the 2-cells. Differently, we consider $\mathsf{Span}^{\leq}\mathcal{C}$ to be locally preordered with $(X \leftarrow A \rightarrow Y) \leq (X \leftarrow B \rightarrow Y)$ if there exists a morphism $\alpha \colon (X \leftarrow A \rightarrow Y) \Rightarrow (X \leftarrow B \rightarrow Y)$. It is an easy exercise to verify that this (pre)ordering is well-defined and compatible with composition and monoidal product. Note that, in general, $\leq$ is a genuine preorder: i.e. it is possible that $(X \rightarrow A \leftarrow Y) \leq (X \rightarrow B \leftarrow Y) \leq (X \rightarrow A \leftarrow Y)$ without the cospans being isomorphic.

Since $\mathsf{Span}^{\leq}\mathcal{C}$ is preorder enriched, rather than poset enriched, it is *not* a cartesian bicategory. However, one can transform a preorder enriched category into a poset enriched one with a simple construction: for $\mathsf{Span}^{\leq}\mathcal{C}$, one first defines $\sim = \leq \cap \geq$, namely $(X \leftarrow A \rightarrow Y) \sim (X \leftarrow B \rightarrow Y)$ iff there exists $\alpha \colon (X \leftarrow A \rightarrow Y) \Rightarrow (X \leftarrow B \rightarrow Y)$ and $\beta \colon (X \leftarrow B \rightarrow Y) \Rightarrow (X \leftarrow A \rightarrow Y)$, and then one takes equivalence classes of morphisms of $\mathsf{Span}^{\leq}\mathcal{C}$ modulo $\sim$. It is worth observing that pullbacks are no longer necessary to compose $\sim$-equivalence classes of spans: weak pullbacks are sufficient, since non-isomorphic weak pullbacks of the same cospan all belong to the same $\sim$-equivalence class.

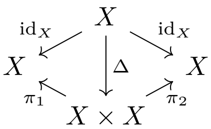
▶ **Definition 21** (Span$^\sim$)**.** Let $\mathcal{C}$ be a category with finite product and weak pullbacks. The posetal category $\mathsf{Span}^{\sim}\mathcal{C}$ has the same objects as $\mathcal{C}$ and as morphisms $\sim$-equivalence classes of spans. The order is defined as in $\mathsf{Span}^{\leq}\mathcal{C}$. Composition is given by weak pullbacks in $\mathcal{C}$. Identities, monoidal product and unit are as in $\mathsf{Span}(\mathcal{C})$.

Like in **Rel**, the comonoid structure is given for any object X by the diagonal and final morphism in $\mathcal{C}$:  is the span $X \leftarrow X \rightarrow X \times X$ and  is $X \leftarrow X \rightarrow 1$.

▶ **Proposition 22.** *Let $\mathcal{C}$ be a category with finite product and weak pullbacks. Then* $\mathsf{Span}^{\sim}\mathcal{C}$ *is a cartesian bicategory.*

Proof. For  we take the span $X \times X \leftarrow X \rightarrow X$ and for  we take $1 \leftarrow X \rightarrow X$.

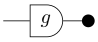
With this information, one has only to check that the inequalities in Definition 1 hold: each of them is witnessed by a commutative diagrams in $\mathcal{C}$. As an example, we illustrate $\leq$ . The left hand side is the span $X \xleftarrow{\mathrm{id}_X} X \xrightarrow{\mathrm{id}_X} X$. The right hand side is the composition of $X \xleftarrow{\mathrm{id}_X} X \xrightarrow{!} 1$ and $1 \xleftarrow{!} X \xrightarrow{\mathrm{id}_X} X$. Since the product $X \xleftarrow{\pi_1} X \times X \xrightarrow{\pi_2} X$ is a pullback of $X \xrightarrow{!} 1 \xleftarrow{!} X$, the composition turns out to be exactly the span $X \xleftarrow{\pi_1} X \times X \xrightarrow{\pi_2} X$. Now the diagonal $\Delta\colon X \to X \times X$ makes the following diagram in $\mathcal{C}$ commute. Therefore Δ witnesses the inequality $\leq$ .



◀

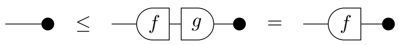
The following is a characterisation of $\mathsf{Span}^{\sim}\mathcal{C}$ maps: a span $X \leftarrow A \rightarrow Y$ is a map iff it is $\sim$-equivalent to $X \xleftarrow{\mathrm{id}_X} X \xrightarrow{f} Y$ for some f in $\mathcal{C}$. Moreover it is surjective iff f is a split epi.

▶ **Proposition 23.** *Let $\mathcal{C}$ be a category with finite products and weak pullbacks. Then* $\mathrm{Map}(\mathsf{Span}^{\sim}\mathcal{C}) \cong \mathcal{C}$ *and surjective maps in* $\mathsf{Span}^{\sim}\mathcal{C}$ *are exactly split epis in* $\mathcal{C}$.

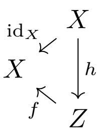
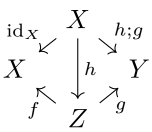
Proof. Since $\mathcal{C}$ has finite products, it is endowed with a cartesian monoidal structure. This means in particular that $=$  for all g in $\mathcal{C}$.

Let $F\colon \mathcal{C} \to \mathsf{Span}^{\sim}\mathcal{C}$ be the identity on objects and mapping a morphism $f\colon X \to Y$ to the span $X \xleftarrow{\mathrm{id}_X} X \xrightarrow{f} Y$. It is easy to check that F is a monoidal functor.
Since every morphism in $\mathcal{C}$ is a comonoid homomorphism, F factors as $\mathcal{C} \xrightarrow{F'} \mathrm{Map}(\mathsf{Span}^{\sim}\mathcal{C}) \to \mathsf{Span}^{\sim}\mathcal{C}$. To conclude that F' is an isomorphism, it is enough to show that every $\mathsf{Span}^{\sim}\mathcal{C}$ map is the $\sim$-equivalence class of some span $X \xleftarrow{\mathrm{id}_X} X \xrightarrow{f} Y$.

Now, if $X \xleftarrow{f} Z \xrightarrow{g} Y$ is a map in $\mathsf{Span}^{\sim}\mathcal{C}$, in particular

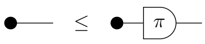
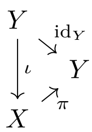


Therefore, by Definition of the ordering in $\mathsf{Span}^{\sim}\mathcal{C}$, there is a morphism $h\colon X \to Z$ such that

 commutes, and therefore .

The two spans are thus equal in $\mathsf{Span}^{\sim}\mathcal{C}$, since they are both maps.

We can now prove the second part of the proposition. If $\pi\colon X \to Y$ is a map in $\mathcal{C}$ such that $F(\pi)$ is surjective in $\mathsf{Span}^{\sim}\mathcal{C}$, then we have

 and therefore there is $\iota\colon Y \to X$ such that 

so π is a split epi. The converse direction is obvious. ◀

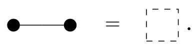
▶ **Proposition 24.** $\mathsf{Span}^{\sim}\mathcal{C}$ *has enough maps.*

Proof. In a cartesian bicategory, for all $R\colon X \to I$ we have $R \leq \overset{X}{-}\bullet$. In the special case when R is a map $g\colon X \to I$, by Proposition 8, it holds that $g = \overset{X}{-}\bullet$. Now take a morphism $R\colon X \to I$ in $\mathsf{Span}^{\sim}\mathcal{C}$. By definition, R is a span $X \xleftarrow{f} A \xrightarrow{g} I$. Observe that by Proposition 23, both f and g are maps in $\mathsf{Span}^{\sim}\mathcal{C}$. Therefore $g = \overset{X}{-}\bullet$ and $\mathsf{Span}^{\sim}\mathcal{C}$ has enough maps. ◀

By Proposition 19, the two propositions above entail the following.

▶ **Corollary 25.** $\mathsf{Span}^{\sim}\mathcal{C}$ *satisfies* (AC).

▶ **Example 26.** Let **FinSet** be the category with natural numbers as objects and as morphisms functions (as in Example 9, natural numbers are regarded as finite sets). The category $\mathsf{Span}^{\sim}\,\mathbf{FinSet}^{\mathrm{op}} = \mathsf{Cospan}^{\sim}\,\mathbf{FinSet}$ satisfies (AC) by Corollary 25. This category is particularly relevant for different reasons. First, it is the cartesian bicategory on one object (see [5, Theorem 31]) or, using the terminology in [4], it is the Carboni-Walters category freely generated by the empty Frobenius theory. Moreover, after forgetting its posetal enrichment, it is the PROP **Frob** of special Frobenius bimonoids which appears to be of fundamental importance in several works (e.g. in [2, 20]). Finally, the cartesian bicategory of equivalence relations, **ERel** from Example 9, can be obtained as quotient of $\mathsf{Cospan}^{\sim}\,\mathbf{FinSet}$: to pass from cospans to equivalence relations, it suffices to equate

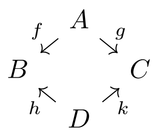


Since **ERel** does not satisfy (AC), by Corollary 25, there is no category $\mathcal{C}$, such that **ERel** is $\mathsf{Span}^{\sim}\mathcal{C}$. Instead, **PERel** can be put in $\mathsf{Span}^{\sim}$ form: it is $\mathsf{Span}^{\sim}\,\mathbf{FinSet}_p^{\mathrm{op}} = \mathsf{Cospan}^{\sim}\,\mathbf{FinSet}_p$ for $\mathbf{FinSet}_p$ being defined as **FinSet** but with partial functions as morphisms. Indeed, as we will see in the next section, any cartesian bicategory with enough maps that satisfies (AC) arises from the $\mathsf{Span}^{\sim}$ construction.

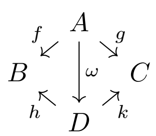
4 Characterising cartesian bicategories with choice

In this section, we prove our characterisation result (Theorem 30). First, we observe that (AC) allows us to construct maps witnessing certain inequalities.

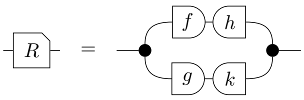
▶ **Lemma 27.** *Let $\mathcal{B}$ be a cartesian bicategory with choice and*

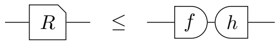
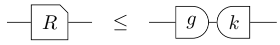


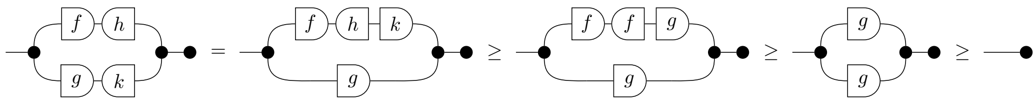
a diagram of maps such that $-\boxed{f}\!-\!\boxed{g}- \;\leq\; -\boxed{h}\!-\!\boxed{k}-$. *Then there is a map $\omega\colon A \to D$ (called witness) such that the following diagram commutes.*



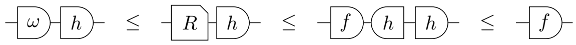
Proof. Consider $R\colon A \to D$ given by



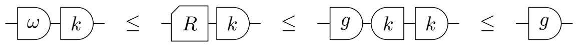
One readily checks that  and  .
R is total, since



so by the axiom of choice, there is a map $\omega \leq R$. This satisfies

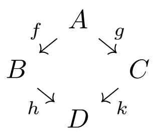


and

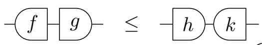
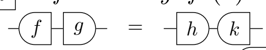
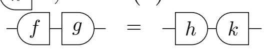


and since both side are maps we have equality by Proposition 8. ◀

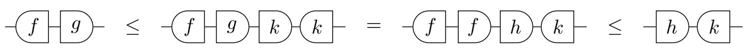
▶ **Lemma 28.** *Let $\mathcal{B}$ be a cartesian bicategory and consider the following diagram in* $\mathrm{Map}\,\mathcal{B}$.

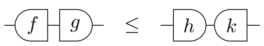


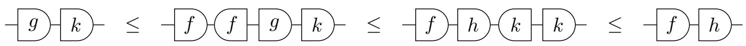
(5)

1.  *if and only if* (5) *commutes.*
2. *If $\mathcal{B}$ has choice and*  *, then* (5) *is a weak pullback.*
3. *If $\mathcal{B}$ has choice and enough maps then*  *iff* (5) *is a weak pullback.*

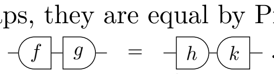
Proof. **1.** If the diagram commutes, then

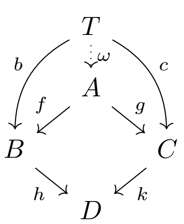


Conversely, if  , then

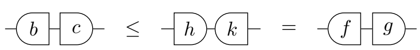


and since both sides are maps, they are equal by Proposition 8.

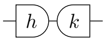
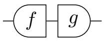
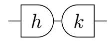
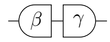
2. Let now $\mathcal{B}$ satisfy (AC) and  . We want to show that (5) is a weak pullback. Given a commutative diagram of solid arrows below,

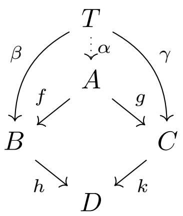


we need to construct the dotted arrow. By Lemma 28.1, we get

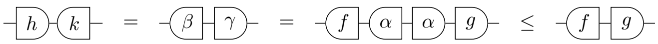


and therefore by Lemma 27 we get $\omega\colon T \to A$ as desired.

3. Let now $\mathcal{B}$ have also enough maps and let (5) be a weak pullback. By Lemma 28.1, it suffices to prove that  $\leq$  . By Lemma 18, take  $=$  to be a factorisation with $\beta\colon T \to B$ and $\gamma\colon T \to C$. By Lemma 28.1, the external square of the following diagram commutes.



Since (5) is a weak pullback, there is $\alpha\colon T \to A$ making the above commute. With this we get



◀

By the third point of the above lemma and Lemma 18 we immediately get the following.

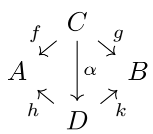
▶ **Corollary 29.** *Let $\mathcal{B}$ be a cartesian bicategory with enough maps and choice. Then* $\mathrm{Map}(\mathcal{B})$ *has weak pullbacks given by the comap-map factorisation.*

We can now state our main result.

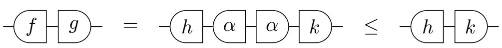
▶ **Theorem 30.** *Let $\mathcal{B}$ be a cartesian bicategory with enough maps. $\mathcal{B}$ satisfies* (AC) *if and only if there is a category $\mathcal{C}$ with products and weak pullbacks such that $\mathcal{B} \cong \mathsf{Span}^{\sim}\mathcal{C}$. More precisely,* (AC) *holds if and only if there exists a functor $F\colon \mathsf{Span}^{\sim}\,\mathrm{Map}(\mathcal{B}) \to \mathcal{B}$ that is an isomorphism.*

Proof. By Corollary 25, $\mathsf{Span}^{\sim}\,\mathrm{Map}(\mathcal{B})$ satisfies (AC), so if there is an isomorphism F also $\mathcal{B}$ satisfies (AC).

Now assume that $\mathcal{B}$ has enough maps and satisfy (AC). Since every morphism in $\mathrm{Map}(\mathcal{B})$ is a comonoid homomorphism, $\mathrm{Map}(\mathcal{B})$ is a cartesian monoidal category and hence has finite products, see [8, Theorem 1.6]. It furthermore has weak pullbacks by Corollary 29. We define $F\colon \mathsf{Span}^{\sim}\,\mathrm{Map}(\mathcal{B}) \to \mathcal{B}$ to be the identity on object and mapping a span $X \xleftarrow{f} Z \xrightarrow{g} Y$ into the composite $f^{\mathrm{op}} \mathbin{;} g$ in $\mathcal{B}$. To prove that F preserves the ordering, observe that if



is a commutative diagram in $\mathrm{Map}(\mathcal{B})$, then



That F indeed preserves composition follows from the weak pullback being given by comap-map factorisation (Corollary 29). The functor F is identity-on-objects and full by Lemma 18. By Lemma 27, the functor reflects the ordering. Therefore it is faithful, hence an equivalence.

◀

5 Related work

Another common example of cartesian bicategories, considered in [8], is the category of relations of a regular category. The following definitions can be found in [7].

▶ **Definition 31.** Let $\mathcal{C}$ be a category. A kernel pair of a morphism $f\colon X \to Y$ is a pair of $p_1, p_2\colon P \to X$ such that the diagram on the right is a pullback. An epimorphism is regular if it is the coequaliser of some pair of morphisms. $\mathcal{C}$ is regular if it has finite limits, coequalisers of kernel pairs and regular epis are stable under pullback.

$$\begin{array}{ccc} P & \xrightarrow{p_1} & X \\ {\scriptstyle p_2}\downarrow & & \downarrow{\scriptstyle f} \\ X & \xrightarrow{f} & Y \end{array}$$

Regular categories admit a well-behaved factorisation system, where every morphism factors as a regular epi followed by a mono. The factorisation is used to define the cartesian bicategory of relations of a regular category.

▶ **Definition 32.** Given a regular category $\mathcal{C}$, let $\mathbf{Rel}(\mathcal{C})$ be the category with the same objects as $\mathcal{C}$ and morphisms $X \to Y$ jointly mono spans, i.e. spans $X \xleftarrow{f} A \xrightarrow{g} Y$ such that the induced map $A \xrightarrow{\langle f,g\rangle} X \times Y$ is mono. For an arbitrary span, $X \xleftarrow{f} A \xrightarrow{g} Y$, its *image* is the jointly mono span given by taking the regular epi-mono factorisation of $A \xrightarrow{\langle f,g\rangle} X \times Y$. The composition of two jointly mono spans is given by first composing them as spans via pullback and then taking the image of the resulting span. The identity $X \to X$ is given by the jointly mono span $X \xleftarrow{\mathrm{id}_X} X \xrightarrow{\mathrm{id}_X} X$. Similar to $\mathsf{Span}^\sim\mathcal{C}$, the categorical product of $\mathcal{C}$ induces a monoidal product on $\mathbf{Rel}(\mathcal{C})$. Furthermore, the ordering is defined as for $\mathsf{Span}^\sim\mathcal{C}$: $(X \leftarrow A \to Y) \leq (X \leftarrow B \to Y)$ if there exists a morphism of spans $\alpha\colon (X \leftarrow A \to Y) \Rightarrow (X \leftarrow B \to Y)$.

Since $\mathbf{Rel}(\mathcal{C})$ is a cartesian bicategory [8, Example 1.4], it is important to compare the $\mathbf{Rel}(\mathcal{C})$ and $\mathsf{Span}^\sim\mathcal{C}$ constructions. In general the two do not coincide. To see this, it is enough to take $\mathbf{FinSet}^{\mathrm{op}}$: $\mathbf{Rel}(\mathbf{FinSet}^{\mathrm{op}})$ is $\mathbf{ERel}$ which, as discussed in Example 26, is a proper quotient of $\mathsf{Span}^\sim\,\mathbf{FinSet}^{\mathrm{op}}$. This is an instance of a more general fact:

▶ **Proposition 33.** *There is a full monoidal functor* $F\colon \mathsf{Span}^\sim\mathcal{C} \to \mathbf{Rel}(\mathcal{C})$ *given by mapping a span to its image.*

The proof of the above and the following statements can be found in the Appendix.

▶ **Lemma 34.** $\mathbf{Rel}(\mathcal{C})$ *has enough maps.*

▶ **Proposition 35.** $\mathsf{Span}^\sim\mathcal{C} \cong \mathbf{Rel}(\mathcal{C})$ *if and only if* (AC) *holds in* $\mathbf{Rel}(\mathcal{C})$.

It is known that surjective maps in $\mathbf{Rel}(\mathcal{C})$ are precisely regular epis in $\mathcal{C}$, see [8, Theorem 3.5]. Using Proposition 19, we have the following.

▶ **Corollary 36.** $\mathsf{Span}^\sim\mathcal{C} \cong \mathbf{Rel}(\mathcal{C})$ *iff regular epis split in* $\mathcal{C}$.

References

1 Jean Bénabou. Introduction to bicategories. In *Reports of the Midwest Category Seminar*, pages 1–77. Springer, 1967.

2 Filippo Bonchi, Fabio Gadducci, Aleks Kissinger, Paweł Sobociński, and Fabio Zanasi. Rewriting modulo symmetric monoidal structure. In *Proceedings of the 31st Annual ACM/IEEE Symposium on Logic in Computer Science*, pages 710–719. ACM, 2016.

3 Filippo Bonchi, Joshua Holland, Dusko Pavlovic, and Pawel Sobocinski. Refinement for signal flow graphs. In *28th International Conference on Concurrency Theory (CONCUR 2017)*. Schloss Dagstuhl-Leibniz-Zentrum fuer Informatik, 2017.

4 Filippo Bonchi, Dusko Pavlovic, and Pawel Sobocinski. Functorial semantics for relational theories. *CoRR*, abs/1711.08699, 2017. URL: `http://arxiv.org/abs/1711.08699`, `arXiv:1711.08699`.

5 Filippo Bonchi, Jens Seeber, and Pawel Sobocinski. Graphical Conjunctive Queries. In Dan Ghica and Achim Jung, editors, *27th EACSL Annual Conference on Computer Science Logic (CSL 2018)*, volume 119 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 13:1–13:23, Dagstuhl, Germany, 2018. Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik. URL: `http://drops.dagstuhl.de/opus/volltexte/2018/9680`, `doi:10.4230/LIPIcs.CSL.2018.13`.

6 Roberto Bruni and Fabio Gadducci. Some algebraic laws for spans (and their connections with multirelations). In *Proc. RelMiS 2001, ENTCS*, volume 44, 2001.

7 Carsten Butz. Regular categories and regular logic. *BRICS Lecture Series LS-98-2*, 1998.

8 Aurelio Carboni and Robert FC Walters. Cartesian bicategories i. *Journal of pure and applied algebra*, 49(1-2):11–32, 1987.

9 Brandon Coya and Brendan Fong. Corelations are the prop for extraspecial commutative frobenius monoids. *Theory and Applications of Categories*, 32(11):380–395, 2017.

10 Brendan Fong. *The Algebra of Open and Interconnected Systems*. PhD thesis, University of Oxford, 2016.

11 Brendan Fong and David I Spivak. Graphical regular logic. *arXiv preprint arXiv:1812.05765*, 2018.

12 Brendan Fong and Fabio Zanasi. Universal Constructions for (Co)Relations: categories, monoidal categories, and props. *Logical Methods in Computer Science*, Volume 14, Issue 3, September 2018. URL: `https://lmcs.episciences.org/4796`, `doi:10.23638/LMCS-14(3:14)2018`.

13 Thomas Fox. Coalgebras and cartesian categories. 4(7):665–667. URL: `https://doi.org/10.1080/00927877608822127`, `doi:10.1080/00927877608822127`.

14 Peter J Freyd and Andre Scedrov. *Categories, allegories*, volume 39. Elsevier, 1990.

15 Ichiro Hasuo and Naohiko Hoshino. Semantics of higher-order quantum computation via geometry of interaction. *Annals of Pure and Applied Logic*, 168(2):404–469, 2017.

16 Horst Herrlich. *Axiom of choice*. Springer, 2006.

17 Bart Jacobs. *Categorical logic and type theory*, volume 141. Elsevier, 1999.

18 Bart Jacobs and Jorik Mandemaker. Coreflections in algebraic quantum logic. *Foundations of physics*, 42(7):932–958, 2012.

19 Peter T Johnstone. *Sketches of an elephant: A topos theory compendium*, volume 2. Oxford University Press, 2002.

20 Stephen Lack. Composing props. *Theory and Applications of Categories*, 13(9):147–163, 2004.

21 Samuel J Mason. Feedback theory-some properties of signal flow graphs. *Proceedings of the IRE*, 41(9):1144–1156, 1953.

22 E. Patterson. Knowledge Representation in Bicategories of Relations. *ArXiv e-prints*, June 2017. `arXiv:1706.00526`.

23 Peter Selinger. A survey of graphical languages for monoidal categories. In *New structures for physics*, pages 289–355. Springer, 2010.

24 Thomas Streicher. *Semantics of type theory: correctness, completeness and independence results*. Springer Science & Business Media, 2012.

25 Fabio Zanasi. *Interacting Hopf Algebras: the theory of linear systems*. PhD thesis, Ecole Normale Supérieure de Lyon, 2015.

A Proof of Section 5

Proof of Proposition 33. F preserves the ordering, because of the universal property of the image. Therefore F is well-defined, since equivalent spans in $\mathsf{Span}^{\sim}\mathcal{C}$ are mapped to isomorphic spans in $\mathbf{Rel}(\mathcal{C})$. It is easily verified that F preserves identities and composition. Since monos and regular epis are closed under product, F preserves the monoidal structure. Finally, F is full because any jointly monic span is its own image. ◀

Proof of Lemma 34. By [8, Theorem 3.5], $\mathbf{Rel}(\mathcal{C})$ is functionally complete (see Remark 17). ◀

Proof of Proposition 35. If $\mathsf{Span}^{\sim}\mathcal{C} \cong \mathbf{Rel}(\mathcal{C})$, then $\mathbf{Rel}(\mathcal{C})$ satisfies (AC) by Corollary 25. If on the other hand, $\mathbf{Rel}(\mathcal{C})$ satisfies (AC), then, since it has enough maps, Proposition 19 guarantees that surjective maps split. Now surjective maps in $\mathbf{Rel}(\mathcal{C})$ are regular epis in $\mathcal{C}$ ([8, Theorem 3.5]), hence the latter split in $\mathcal{C}$. We prove that in that case F—defined and shown to be full in Proposition 33—is furthermore faithful, for which it suffices to show that it reflects the ordering. So let $A \xleftarrow{f} B \xrightarrow{g} C$ and $A \xleftarrow{f'} B' \xrightarrow{g'} C$ be spans such that the image of the first is included in the image of the second under F. Assume without loss of generality that A is terminal, which we can achieve by bending the input around to the right. Let $B \xrightarrow{\pi} J \xrightarrow{\iota} C$ be a regular epi-mono factorisation of g and likewise for g'. Then there is a morphism $\alpha\colon J \to J'$

$$\begin{array}{ccccc} B & \overset{\pi}{\twoheadrightarrow} & J & \overset{\iota}{\rightarrowtail} & C \\ & & \downarrow\alpha & \nearrow_{\iota'} & \\ B' & \overset{\pi'}{\twoheadrightarrow} & J' & & \end{array}$$

Since π' is regular epi, it splits by the preceding observation, and thus there is $\beta\colon B \to B'$ such that

$$\begin{array}{ccc} B & \xrightarrow{g} & C \\ \downarrow\beta & \nearrow_{g'} & \\ B' & & \end{array} \quad .$$

It follows that the inclusion between the spans holds in $\mathsf{Span}^{\sim}\mathcal{C}$. ◀