Relational Presheaves, Change of Base and Weak Simulation

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Abstract

We show that considering labelled transition systems as relational presheaves captures several recently studied examples in a general setting. This approach takes into account possible algebraic structure on labels. We show that left (2-)adjoints to change-of-base functors between categories of relational presheaves with relational morphisms always exist and, as an application, that weak closure (in the sense of Milner) of a labelled transition system can be understood as a left adjoint to a change-of-base functor.

Keywords: labelled transition systems, simulation, weak simulation, relational presheaves, category theory

1. Introduction

A famous application of coalgebra [3, 4] is as an abstract setting for the study of labelled transition systems (LTS). Indeed, an LTS with label set A is a coalgebra for the functor $\mathcal{P}(A \times -)$. LTSs are thus the objects of the category of coalgebras for this functor. The arrows, in concurrency theoretical terminology, are functional bisimulations. This category of coalgebras (modulo size issues) has a final object that gives a canonical notion of equivalence, although other approaches are available in general[1]. Ordinary bisimulations can be understood as spans of coalgebra morphisms. The coalgebraic approach has been fruitful: amongst many notable works we mention Turi and Plotkin's elegant approach to structural operational semantics congruence formats via bialgebras [50].

In another influential approach, Winskel and Nielsen [52] advocated the use of presheaf categories as a general semantic universe for the study of labelled transition systems. Morphisms turn out to be functional simulations, functional *bi*simulations can be characterised as open maps with respect to a canonical (via the Yoneda embedding) choice of path category [24]. Ordinary bisimulations are then spans of open maps, with some side conditions.

[1] See [48] for an overview of the notions that appear in the literature.

Both the coalgebraic approach and the presheaf approach have generated much subsequent research and have found several applications that we do not account for here. Concentrating on the theory of labelled transition systems, there are some limitations to both approaches. For example both take for granted that the set of labels A is monolithic and has no further structure. In fact, several labelled transition systems have sets of labels that are monoids [9] or even categories [30, 16, 35]. Such examples are more challenging to capture satisfactorily with the aforementioned approaches but some progress has been made—for instance, Bonchi and Montanari [7] captured labelled transition systems on reactive systems (in the sense of Leifer and Milner [30]) as certain coalgebras on presheaves.

There is also a mismatch between notions typically studied by concurrency theorists or researchers in the operational semantics of concurrent languages and the morphisms in categories of coalgebras or in presheaves. From the point of view of process theory, the morphisms in the aforementioned categories are not the notions typically studied: *functional* simulations and *functional* bisimulations[2] instead of ordinary simulations and ordinary bisimulations.

Finally, and perhaps most importantly, the most natural notions of equivalence in applications are often weak (in the sense of Milner) and these tend to be technically challenging to capture in the coalgebra or presheaf settings. One can think of weak bisimulation as ordinary bisimulation on a labelled transition system that has been "saturated" with the silent τ-actions, but this is just another way of saying that τ is made the identity of a monoid of actions. We will develop this idea in Section 6. Because the coalgebraic and presheaf approaches were not designed with a view to accommodate such *algebraic structure* on the set of labels, some work has to be performed in order to talk about weak equivalences, see for example [47, 15].

This paper proposes the universe of relational presheaves[3] as an abstract setting for the study of labelled transition systems. Relational presheaves are lax functors [28] (or, equivalently, morphisms of bicategories [5]) $\mathbf{C}^{\mathrm{op}} \to \mathbf{Rel}$, where **Rel** is the 2-category of relations with objects sets, arrows relations, and 2-cells inclusions of relations (in particular, the hom-categories are partial orders—some authors refer to such 2-categories as locally partially ordered). The applications that we study in this paper are related to the previous use of relational presheaves and enriched category theory in automata theory [41, 25, 26]. Relational presheaves have been called by various other

[2]Although Hughes and Jacobs [21] and Hasuo [18, 19] have also studied simulations coalgebraically—the latter using oplax morphisms on coalgebras in Kleisli categories; these are conceptually closely related to morphisms between relational presheaves. Recently, Levy [31] has considered similarity coalgebraically using the notion of *relators*, which also have some technical similarities with relational presheaves.

[3]Using Rosenthal's terminology [41].

names: specification structures [1], relational variable sets [17], dynamic sets [49], with several applications in Computer Science and related fields. Functors to **Rel** as a way of giving semantics to flowcharts were considered as early as 1972 by Burstall [11].

The typical examples of $\mathbf{C}$ that we shall consider will be monoids (i.e. categories with one object) and categories of contexts, in the sense of Leifer and Milner [30, 45]. Ordinary A-labelled transition systems can be seen as instances of the former by considering the free monoid A^*. A more interesting example where the label set is a non-free monoid is the LTS considered in [9] or the LTSs that arise from expressions in the algebra of Span(**Graph**) [27]. Weak derived LTS for reactive systems in the sense of Jensen [23], are examples of relational presheaves for $\mathbf{C}$ a category of contexts. Tile logics [16] can also be seen as relational presheaves where $\mathbf{C}$ is not merely a monoid: here the vertical composition of tiles gives the algebraic structure on the tile transitions.

There are two different, natural choices for morphisms between relational presheaves. One, suggested by the connection with enriched category theory, is to take functional oplax natural transformations, which Rosenthal refers to as rp-morphisms [41, Definition 3.3.2]. Indeed, the category of relational presheaves and rp-morphisms $\mathscr{R}(\mathbf{C})$ is equivalent to the category of categories enriched over the free quantaloid on $\mathbf{C}$ [41, Theorem 3.3.1]. In our examples rp-morphisms turn out to be functional simulations. Here, change-of-base functors u^* always have left-adjoints and they have right-adjoints whenever u satisfies the weak factorisation lifting property [36]. One consequence is that $\mathscr{R}(\mathbf{C})$ has limits that are computed pointwise.

A second choice for morphisms is to also consider all (ie not necessarily functional) oplax natural transformations, which Rosenthal refers to as generalized rp-morphisms [41, Definition 3.3.3]: in our examples these turn out to be *ordinary* simulations. They have been studied to a lesser extent and are the morphisms on which we shall focus in this paper. Following Rosenthal, we denote the 2-category of relational presheaves and generalized rp-morphisms by $\mathscr{R}^*(\mathbf{C})$. Given relational presheaves $h, h' \in \mathscr{R}^*(\mathbf{C})$, morphisms $h \to h'$ organise themselves in a sup-lattice, with the order inherited from **Rel**, and joins are preserved by composition, in other words $\mathscr{R}^*(\mathbf{C})$ is a quantaloid [41, Proposition 3.3.2]. The mathematical universe of relational presheaves with relational morphisms is rich. Most importantly, some familiar constructions from the concurrency theory literature can be characterised as adjunctions. For example, we shall show that the weak-closure of a labelled transition system is actually a left (2-)adjoint to a change-of-base functor. In this sense, this paper continues the programme of [51] in that category theory is used to clarify and distill constructions commonly used in the study of models of concurrency.

Explicitly, the contributions of this paper are:

(i) the insight that the theory of relational presheaves and (generalized) rp-morphisms is a suitable mathematical universe for the study of labelled transition systems whose sets of labels bear algebraic structure.

(ii) the proof that the construction of left adjoints to change-of-base functors in $\mathscr{R}(\mathbf{C})$ ([36]) works also in the larger 2-category of relational presheaves and *generalized* rp-morphisms $\mathscr{R}^*(\mathbf{C})$ (Theorem 5.1). This fact does not seem to have been noticed before.

(iii) as an application of this result, we show how weak-closure of an LTS can be considered as a left 2-adjoint to a suitable change-of-base functor. This construction can be viewed as an instance of a general mathematical theory of weak simulations.

A previous conference version of this article appeared in [46]. In this extended version we add Theorem 5.1, together with a more thorough exposition of examples and the constructions that appeared in the original conference version.

Related work. Quantaloids have previously been used in the study of process equivalence by Abramsky and Vickers [2]. Simulations and bisimulations have been investigated in the setting of categories enriched over quantaloids by Schmitt and Worytkiewicz [44]. The fact that simulations are oplax natural transformations in a relational setting was previously observed by Jürgen Koslowski [29] and Ichiro Hasuo [18, 19].

2. Examples

In order to motivate the more abstract developments we start here with a sightseeing tour of examples of labelled transition systems that will turn out to be relational presheaves.

Example 2.1 (Ordinary labelled transition systems)**.** A labelled transition system is a triple (X, A, T) where $T \subseteq X \times A \times X$. The set X is called the statespace, A is a monolithic set of atomic actions and T is a set of transitions. We usually write $x \xrightarrow{a} x'$ for $(x, a, x') \in T$.

One way to describe such a structure using the coalgebraic approach is to use the functor $\mathcal{P}(A \times -)$, where $\mathcal{P}$ is the (covariant) powerset endofunctor on **Set**. An LTS with label set A is a coalgebra for the $\mathcal{P}(A \times -)$ **Set**-endofunctor, i.e. a function

$$h : X \to \mathcal{P}(A \times X). \tag{1}$$

There are several examples where the set A has more structure, in particular, when it's a monoid or even the set of arrows of a category. We list some of these examples below.

Example 2.2 (Weak labelled transition systems)**.** Consider a special "silent" action τ that we intend to be unobservable. Define a *weak* LTS to be $(X, A + \{\tau\}, T)$ where the set T of transitions is reflexive-transitive closed wrt the τs, i.e. it is closed under the rules below

$$\frac{}{P \xrightarrow{\tau} P} \qquad \frac{P \xrightarrow{\tau} Q \quad Q \xrightarrow{a} R \quad R \xrightarrow{\tau} S}{P \xrightarrow{a} S} \tag{2}$$

It is easy to show that bisimilarity and weak bisimilarity, in the sense of Milner, coincide on a weak LTS. A weak LTS can be obtained from an LTS with τ-labels via reflexive-transitive τ-closure. In this paper we shall show that this construction arises as a left adjoint to a change-of-base functor (Corollary 6.1).

Example 2.3 (Monoidal structure on labels)**.** An LTS with monoidal structure on labels is (X, M, T) where M is a monoid with multiplication $\star$ and unit ι, and where the set T of transitions is closed under the following two rules:

$$\frac{}{P \xrightarrow{\iota} P} \qquad \frac{P \xrightarrow{a} Q \quad Q \xrightarrow{b} R}{P \xrightarrow{a \star b} R} \tag{3}$$

Notice that such LTSs are actually examples of categories, with objects the states, identities ι, and composition given by $\star$. See [9, 10] for recent examples of such LTS in concurrency theory literature.

Labelled transition systems with a monoidal structure on labels have been called *non-deterministic M-dynamics* [41, Definition 4.1.1] and they have received attention in applications of enriched category theory to automata theory.

Example 2.4 (Contexts as labels)**.** The contexts-as-labels approach was introduced by Leifer and Milner [30], and developed further in [42, 43, 23, 6, 7]. It is a general setting for the study of labelled transition systems in process calculi such as CCS [32] and the π-calculus [14, 34]. In those calculi there is an underlying reduction semantics and the labelled semantics is closely related: the labels can be understood to represent minimal contexts that allow a reductions to occur. See [38, 39] for a detailed analysis of the relationship between reduction and labelled semantics in concrete calculi. The generality of Leifer and Milner's approach allows one to consider intuitions and constructions from process calculi in models that are not obviously syntactic; for example the theory of reactive systems serves as the foundation of the behavioural theory of bigraphs [33].

Here we give only a very brief summary of Leifer and Milner's theory of reactive systems: suppose that $\mathbf{C}$ is a small category with objects interfaces and arrows contexts; see [45] for a number of concrete examples. There is a chosen object 0 that serves as the ground interface, arrows with domain 0 are then the *ground terms.*

Given a set of *reaction*-rules $\mathcal{R}$, a family of pairs of arrows $r_i, r'_i : 0 \to X_i$ indexed by some set I, one generates a *reaction relation* $\longrightarrow\!\!\!\triangleright$ by closing the rules with respect to reactive contexts: the arrows of a subcategory $\mathbf{D}$ of $\mathbf{C}$ that is composition reflecting (if $g \circ f$ is in $\mathbf{D}$ then both f is in $\mathbf{D}$ and g is in $\mathbf{D}$).

$$c \longrightarrow\!\!\!\triangleright c' \quad \stackrel{\text{def}}{=} \quad \exists\, d \text{ in } \mathbf{D},\, i \in I,\, (r_i, r'_i) \in \mathcal{R}.\ c = d \circ r_i \wedge c' = d \circ r'_i$$

The collection $R = (\mathbf{C},\, 0,\, \mathbf{D},\, \mathcal{R})$ is called a *reactive system.*

Given a reactive system R, one can *derive* an LTS that has transitions of the form $t \xrightarrow{c} t'$ where $t : 0 \to X$, $c : X \to Y$, $t' : 0 \to Y$ such that $c \circ t \longrightarrow\!\!\!\triangleright t'$ and c is the smallest context that makes this reduction possible. The notion of smallest is typically captured via a universal property, in $\mathbf{C}$, of relative (local) pushouts. In symbols:

$$t \xrightarrow{c} t' \quad \stackrel{\text{def}}{=} \quad \exists d \text{ in } \mathbf{D},\, i \in I,\, (r_i, r'_i) \in \mathcal{R},\, c \circ t = d \circ r_i,\, t' = d \circ r'_i,$$

$$\begin{tikzcd} & Y & \\ X \arrow[ur, "c"] & & X_i \arrow[ul, "d"'] \\ & 0 \arrow[ul, "t"] \arrow[ur, "r_i"'] & \end{tikzcd} \quad \text{is a pushout diagram in } \mathbf{C}/Y.$$

Jensen [23, Definition 3.18] introduced the notion of weak bisimilarity for reactive systems (see also [8]), a construction that for every reactive system R with local pushouts, gives a reactive system $\mathcal{W}(R)$. The intuition is that reactions of R are "closed under composition" to form reactions in $\mathcal{W}(R)$: for example (r, r') and (s, s') in R are composed to $(p \circ r,\, q \circ s')$ in $\mathcal{W}(R)$ where $p \circ r' = q \circ s$ give a local pushout. This composition is iterated, and "identity" reactions, or nullary compositions, (r, r) are added to $\mathcal{W}(R)$ for all ground terms r. Bisimilarity on the LTS derived from $\mathcal{W}(R)$ acts as a kind of weak bisimilarity in examples. We omit the details here and just mention two properties that are satisfied by this LTS:

1. $u \xrightarrow{\text{id}} u$ for all $u \in \mathbf{C}$,
2. if $u \xrightarrow{a} v$ and $v \xrightarrow{b} w$ then $u \xrightarrow{b \circ a} w$.

Both are a consequence of [23, Lemma 3.2] and rely on the properties of local pushouts.

Example 2.5 (Tile systems)**.** Tile systems [16] are family of models that generalise several kinds of rule-based computational systems, notably including SOS operational semantics [37].

The theory of tile systems is founded on double categories, in which the vertical dimension can be viewed as (double) labelled transitions between the arrows of the horizontal category, considered as the statespace. This vertical dimension forms a category, and hence this labelled transition system is closed under composition and has identities.

More concretely, one can view the double cell below—in which s and t are arrows of the horizontal category (states) while a and b are arrows of the vertical category (labels)—as a transition $s \xrightarrow[a]{b} t$.

$$\begin{tikzcd} k \arrow[r, "s"] \arrow[d, "a"'] \arrow[dr, phantom, "\alpha"] & l \arrow[d, "b"] \\ m \arrow[r, "t"'] & n \end{tikzcd}$$

Of particular interest to us are the following facts: for each $s : k \to l$ in the horizontal category there is a double cell

$$\begin{tikzcd} k \arrow[r, "s"] \arrow[d, "\mathrm{id}_k"'] \arrow[dr, phantom, "\mathrm{id}_s"] & l \arrow[d, "\mathrm{id}_l"] \\ k \arrow[r, "s"'] & l \end{tikzcd}$$

where id_k, id_l are the identity arrows of the vertical category; written in transition form this is the transition $s \xrightarrow[\mathrm{id}_k]{\mathrm{id}_l} s$. Moreover, given two cells

$$\begin{tikzcd} k \arrow[r, "s"] \arrow[d, "a"'] \arrow[dr, phantom, "\alpha"] & l \arrow[d, "b"] \\ m \arrow[r, "t"] \arrow[d, "c"'] \arrow[dr, phantom, "\beta"] & n \arrow[d, "d"] \\ p \arrow[r, "u"'] & q \end{tikzcd}$$

their pasting is also a double cell: from the point of view of labelled transition systems, whenever $s \xrightarrow[a]{b} t$ and $t \xrightarrow[c]{d} u$, then also $s \xrightarrow[c \circ a]{d \circ b} u$.

3. LTSs: from coalgebras to relational presheaves

It is typical to consider an LTS as a coalgebra (1) for the $\mathcal{P}(A \times -)$ endofunctor on set. This approach emphasises the statespace but wraps the set of labels A within the definition of the functor. In our examples the labels have an algebraic structure that arises from composition of transitions, so in order to consider possible extra structure of A we need to bring the labels out as first class citizens so:

$$\frac{\dfrac{h : X \to \mathcal{P}(A \times X)}{h' : X \to \mathcal{P}(X)^A}}{h'' : A \to \mathcal{P}(X)^X} \tag{4}$$

That is, to give a standard $\mathcal{P}(A \times -)$ coalgebra is to give an A-indexed family of $\mathcal{P}(-)$ coalgebras.

For the sequel, it makes sense to use the fact that $\mathcal{P}$ is a monad, and in particular we can use the composition in the Kleisli category $Kl(\mathcal{P})$.[4] We can replace the conclusion of (4) by:

$$h : A \to Kl(\mathcal{P})(X, X) \tag{5}$$

If A is a monoid, then h should preserve the structure in some way. But monoids are exactly one-object categories. So for a general category $\mathbf{C}$, (5) generalises to a mapping

$$h : \mathbf{C} \to Kl(\mathcal{P}) \tag{6}$$

We shall now elucidate what properties the mapping (6) ought to satisfy. Of course, $Kl(\mathcal{P})$ is another name for $\mathbf{Rel}$, the locally partially ordered 2-category with objects sets, arrows relations and 2-cells inclusions.

A natural choice for (6) is that of *relational presheaf*, that is a lax functor from $\mathbf{C}^{\mathrm{op}}$ to $\mathbf{Rel}$:

$$h : \mathbf{C}^{\mathrm{op}} \to \mathbf{Rel} \tag{7}$$

The laxness here means that:

$$h(b); h(a) \subseteq h(a; b) \qquad \mathrm{id}_{h(x)} \subseteq h(\mathrm{id}_x) \tag{8}$$

We take as morphisms of $\varphi : h \to h'$ of relational presheaves $\mathbf{C}^{\mathrm{op}} \to \mathbf{Rel}$ the oplax natural transformations [17, 36], Rosenthal [41, Definition 3.3.3] calls them generalized rp-morphisms. To give such a morphism is to give, for each $C \in \mathbf{C}$, a relation $\varphi_C \colon h(C) \nrightarrow h'(C)$ such that the (lax) naturality condition is satisfied for each $f : D \to C$ in $\mathbf{C}$, ie:

$$\begin{tikzcd} hC \arrow[r, "\varphi_C", "\shortmid" marking] \arrow[d, "hf"', "\shortmid" marking] \arrow[dr, phantom, "\subseteq"] & h'C \arrow[d, "h'f", "\shortmid" marking] \\ hD \arrow[r, "\varphi_D"', "\shortmid" marking] & h'D \end{tikzcd} \tag{9}$$

The 2-category with objects relational presheaves and morphisms as above will be denoted $\mathscr{R}^*(\mathbf{C})$. The subcategory with functional morphisms, that is, those $\varphi \colon h \to h'$ where φ_C in (9) is a function[5] for all $C \in \mathbf{C}$, will be denoted $\mathscr{R}(\mathbf{C})$.

Returning to our motivational examples, consider $\mathbf{C}$ to be a category with one object, ie a monoid $(M, \star, \iota)$. We will write $\mathscr{R}^*(M)$ to emphasise that we are considering the one object case.

[4] Jacobs, Hasuo and Sokolova have used coalgebras in Kleisli categories in order to consider trace semantics coalgebraically [22, 20]. Hasuo studied a general notion of simulation as lax morphism of coalgebras in Kleisli categories [18, 19].

[5] These are Rosenthal's *rp-morphisms*.

Proposition 3.1. *To give an LTS with a monoidal structure on labels (Example 2.3) is to give a relational presheaf, i.e. an object of $\mathscr{R}^*(M)$. Morphisms of $\mathscr{R}^*(M)$ are precisely the simulations between such LTSs.*

Proof. To give a functor h from M to **Rel** is to pick a set X_h, and for each $a \in M$, a relation $ha : X_h \twoheadrightarrow X_h$: now the required LTS has X_h as its statespace, and its set of transitions is $R \subset X \times A \times X$ where $(x, a, y) \in R$ precisely when $(y, x) \in ha$. It is easy to verify that the laxness conditions (8) mean exactly that the LTS is closed under the rules (3). Thus to give such an LTS is also to give a functor from M to **Rel**.

Now, given two relational presheaves $h, h' : M \to \mathbf{Rel}$, to give a morphism $\varphi : h \to h'$ is to give a relation $\varphi : X_h \twoheadrightarrow X_{h'}$ that satisfies the inclusion illustrated below, for each $a \in M$. Here the contravariance of h, h' plays a role.

$$\begin{tikzcd} X_h \arrow[r, "\varphi", "|" marking] \arrow[d, "ha"', "-" marking] \arrow[dr, phantom, "\subseteq"] & X_{h'} \arrow[d, "h'a", "-" marking] \\ X_h \arrow[r, "\varphi"', "|" marking] & X_{h'} \end{tikzcd}$$

Indeed, in the diagram above, for $(y, x') \in h_a; \varphi$ means that $x \xrightarrow{a} y$ and $x \varphi x'$ for some $x \in X$. Using the laxness condition, we must have also that $(y, x') \in \varphi; h'a$, and this means that there exists y' with $x' \xrightarrow{a} y'$ and $y \varphi y'$. This is precisely the condition on φ being a simulation. □

The category $\mathscr{R}(M)$ has been called M–**Dyn**, with the objects called non-deterministic M-dynamics [41, Definition 4.1.1].

Any ordinary LTS (Example 2.1) can be considered a relational presheaf by taking $\mathbf{C} = A^*$ and considering the LTS of traces. Not all relational presheaves in $\mathscr{R}^*(A^*)$ are ordinary labelled transition systems in the sense of Example 2.1. For instance, in a general relational presheaf in $\mathscr{R}^*(A^*)$ there can be transitions $p \xrightarrow{\epsilon} q$ for $p \neq q$ and transitions $p \xrightarrow{ab} p'$ without an intermediate p'' such that $p \xrightarrow{a} p''$ and $p'' \xrightarrow{b} p'$.

In fact, ordinary LTSs with labels in A form the full subcategory of $\mathscr{R}^*(A^*)$ with objects the ordinary (not lax) functors, which we will refer to as $\mathrm{LTS}(A)$.

Proposition 3.2. *To give an ordinary LTS with label set A (Example 2.1) is to give a functor $(A^*)^{\mathrm{op}} \to \mathbf{Rel}$. Let $\mathrm{LTS}(A)$ denote the corresponding full subcategory of $\mathscr{R}^*(A^*)$. The morphisms of $\mathrm{LTS}(A)$ are thus the simulations.* □

Taking a reactive system R with category of contexts $\mathbf{C}$, the derived LTS on $\mathcal{W}(R)$ [23] is a relational presheaf. This is a direct consequence of [23, Lemma 3.22].

Proposition 3.3. *Given a reactive systems R with a category of contexts $\mathbf{C}$ that has all local pushouts, the derived LTS on $\mathcal{W}(R)$ (Example 2.4) is an object of $\mathscr{R}^*(\mathbf{C})$.* $\square$

Proof. Consider the derived LTS h on $\mathcal{W}(R)$. It can be considered as a relational presheaf as follows: each object is $X \in \mathbf{C}$ sent to the set of ground terms $t : 0 \to X$. Arrows $c : X \to Y$ are sent to the relation $\{ (t', t) \mid t \xrightarrow{c} t' \}$. The properties of h given in Example 2.4 ensure that (8) is satisfied. $\square$

The contexts as labels example is somewhat similar to non-deterministic $\mathscr{A}$-dynamics ([41, Definition 4.2.1]) in that categories of contexts often resemble algebraic theories in the sense of Lawvere. Here we do not require either that that products exist in $\mathbf{C}$ or are preserved by the functor to $\mathbf{Rel}$.

Similarly, taking $\mathbf{C}$ to be the product of the vertical category of any tile system with itself allows (the LTS generated by) any tile system to be considered as a relational presheaf.

Proposition 3.4. *Consider the labelled transition system obtained from a tile system (Example 2.5). Then the labelled transition system is an object of $\mathscr{R}^*(\mathcal{V} \times \mathcal{V})$, where $\mathcal{V}$ is the vertical category of the tile system.*

Proof. An object $(k, l) \in \mathcal{V} \times \mathcal{V}$ is sent to the set $\{ s : k \to l \mid s \text{ in } \mathcal{H} \}$ where $\mathcal{H}$ is the horizontal category. An arrow $(a : k \to m, b : l \to n)$ in $\mathcal{V} \times \mathcal{V}$ is sent to the relation $\left\{ (t, s) \;\middle|\; \begin{array}{ccc} k & \xrightarrow{s} & l \\ {\scriptstyle a}\downarrow & \alpha & \downarrow{\scriptstyle b} \\ m & \xrightarrow[t]{} & n \end{array} \right\}$. The properties highlighted in Example 2.5 ensure that (8) is satisfied. $\square$

Propositions 3.3 and 3.4 concern relational presheaves that have indexing categories that are not merely monoids. Roughly speaking, such relational presheaves send each object of the indexing category to a statespace with sort given by the object. Morphisms of relational presheaves then give a natural notion of a sort-respecting simulation.

4. Relational presheaves as labelled transition systems

We have shown how several examples of LTS can be considered as relational presheaves. Going in the other direction, the familiar Grothendieck construction instantiated to relational presheaves yields categories, which we can consider as LTSs. The intuition given at the end of the previous section is that relational presheaves send an object to a statespace with sort given by that object. The Grothendieck construction forgets sorts and allows us to consider the relational presheaf as a transition system on a global statespace.

Definition 4.1. Let $h : \mathbf{C}^{\mathrm{op}} \to \mathbf{Rel}$ be a relational presheaf. Then the labelled transition system $\Gamma(h)$ has:

states: pairs (C, x) where $C \in \mathbf{C}$ and $x \in h(C)$

transitions: $(C, x) \xrightarrow{f} (D, y)$ if $C \xrightarrow{f} D$ is in $\mathbf{C}$ and $x \in h(f)(y)$.

As a consequence of (8), $\Gamma(h)$ is closed under the rules of (3). Indeed, if $\mathbf{C}$ has one object then $\Gamma(h)$ is an LTS with a monoidal structure on labels, in the sense of Example 2.3.

Proposition 4.2. *Morphisms $\varphi : h \to h'$ are in a 1-1 correspondence with simulations from $\Gamma(h)$ to $\Gamma(h')$.* $\square$

Niefield [36] amongst other authors have studied $\mathscr{R}(\mathbf{C})$ and we can use some of the rich theory of relational presheaves to yield insights about the examples that we have identified. An example is the following result.

Proposition 4.3. *Limits exist and are computed pointwise in $\mathscr{R}(\mathbf{C})$.*

Proof. Corollary 3.2 in [36]. $\square$

Categories $\mathscr{R}^*(\mathbf{C})$ of relational presheaves with *relational* morphisms have received less attention in the literature. They are locally partially-ordered 2-categories, with natural transformations ordered pointwise by inclusion, thus inheriting their 2-categorical structure from that of **Rel**—these are the modifications [28] in this simple setting. It is well-known that the hom-posets of $\mathscr{R}^*(\mathbf{C})$ are complete suplattices and this fact generalises the observation that simulations are closed under unions. Sups commute with composition and so $\mathscr{R}^*(\mathbf{C})$ are themselves examples of quantaloids [41]. We will concentrate on $\mathscr{R}^*(\mathbf{C})$ in the remainder of the paper.

5. Change of base

In this section we show that change-of-base functors between categories of relational presheaves and relational morphisms have left adjoints. The construction is the same as in categories of relational presheaves and functional morphisms, given for example in [36]. The novelty here is that we show that the universal property also holds wrt the larger universe of relational morphisms.

Given a functor $p : \mathbf{C} \to \mathbf{D}$, the *change-of-base* 2-functor $p^* : \mathscr{R}^*(\mathbf{D}) \to \mathscr{R}^*(\mathbf{C})$ is defined as follows:

- on objects: a relational presheaf $g \in \mathscr{R}^*(\mathbf{D})$ is mapped to the relational presheaf $gp \in \mathscr{R}^*(\mathbf{C})$.
- on arrows: a (relational) natural transformation $\varphi : g \to g'$ is mapped to $\varphi p : gp \to g'p$ where $(\varphi p)_c \stackrel{\text{def}}{=} \varphi_{pc}$.

- on 2-cells: if $\varphi_d \subseteq \psi_d$ for all $d \in \mathbf{D}$ then clearly $(\varphi p)_c \subseteq (\psi p)_c$ for all $c \in \mathbf{C}$.

Theorem 5.1. *Given a functor $p : \mathbf{C} \to \mathbf{D}$, the change-of-base 2-functor $p^* : \mathscr{R}^*(\mathbf{D}) \to \mathscr{R}^*(\mathbf{C})$ has a left 2-adjoint $\Sigma_p : \mathscr{R}^*(\mathbf{C}) \to \mathscr{R}^*(\mathbf{D})$.*

Proof. The construction of $\Sigma_p : \mathscr{R}^*(\mathbf{C}) \to \mathscr{R}^*(\mathbf{D})$ is as follows:

- On objects, given $h \in \mathscr{R}^*(\mathbf{C})$ and $d \in \mathbf{D}$, let:

$$\Sigma_p(h)(d) \stackrel{\text{def}}{=} \coprod_{c \in \mathbf{C},\, pc = d} hc.$$

  Given $v : d' \to d$ in $\mathbf{D}$, $((c, x), (c', x')) \in \Sigma_p(h)(v)$ exactly when there is an arrow $u : c' \to c$ in $\mathbf{C}$ with $pu = v$ and $(x, x') \in h(u)$.

- On arrows, given $\varphi : h \nrightarrow h'$ in $\mathscr{R}^*(\mathbf{C})$ we define

$$\Sigma_p(\varphi)_d \stackrel{\text{def}}{=} \coprod_{c \in \mathbf{C},\, pc = d} \varphi_c : \Sigma_p(h)(d) \nrightarrow \Sigma_p(h')(d) \tag{10}$$

- On 2-cells, if $\varphi_c \subseteq \psi_c$ for all $c \in \mathbf{C}$ then evidently from (10) also $\Sigma_p(\varphi)_d \subseteq \Sigma_p(\psi)_d$.

Notice that

$$(p^* \Sigma_p h)_c = (\Sigma_p h)_{pc} = \coprod_{c' \in \mathbf{C},\, pc' = pc} hc'$$

Now for each $h \in \mathscr{R}^*(\mathbf{C})$, we can define $\eta_h : h \to p^* \Sigma_p h$ in $\mathscr{R}^*(\mathbf{C})$ to be, at each c, the (graph of the) obvious injection $hc \to \coprod_{c' \in \mathbf{C},\, pc' = pc} hc'$. To check that η is 2-natural it suffices to note that, for any $\varphi : h \to h'$ in $\mathscr{R}^*(\mathbf{C})$ the diagram below obviously commutes:

$$\begin{tikzcd}
hc \arrow[r, "(\eta_h)_c"] \arrow[d, "\varphi_c"'] & \coprod_{c',\, pc' = pc} hc' \arrow[d, "\coprod_{c',\, pc' = pc} \varphi_c"] \\
h'c \arrow[r, "(\eta_{h'})_c"'] & \coprod_{c',\, pc' = pc} h'c'
\end{tikzcd}$$

On 2-cells there is nothing to verify.

We now verify that η satisfies the universal property of units. Suppose that $\alpha : h \nrightarrow p^* g$ for some $g \in \mathscr{R}^*(\mathbf{D})$.

$$\begin{tikzcd}
hc \arrow[r, "(\eta_h)_c"] \arrow[dr, "\alpha_c"'] & \coprod_{c',\, pc' = pc} hc' \arrow[d, dotted, "\beta_{pc}"] \\
& gpc
\end{tikzcd} \tag{11}$$

Define $\beta : \Sigma_p h \to g$ in $\mathscr{R}^*(\mathbf{D})$ at d to be the relation $\beta_d : \coprod_{c:pc=d} hc \twoheadrightarrow gd$ that precomposed with each coproduct injection hc (where $pc = d$) is α_c. Then clearly the diagram above commutes.

If also $\beta' : \Sigma_p h \to g$ satisfies (11) then it follows that at each injection it must agree with α; since relations from a coproduct are defined by their value on their injections this means that $\beta = \beta'$. □

6. Weak labelled transition systems

Consider an LTS with silent τ labels. Closing wrt the rules in (2), considering τ as the identity of a monoid of actions and forming the traces yields a labelled transition systems on which strong and weak equivalences coincide. This kind of weak-closure of an LTS is technically somewhat similar to the approaches advocated in [40] and [15]. Here we shall make this construction precise and characterise it with a universal property. In particular we will show that weak-closure is actually the left adjoint to a change-of-base functor, making this example an application of the construction in the proof of Theorem 5.1.

Let $A_\tau \stackrel{\text{def}}{=} A + \{\tau\}$. There is a function $A_\tau \to A^*$ that sends each $a \in A$ to itself (considered as a word of length one) and τ to ϵ. This extends to a morphism from the free monoid

$$u : A_\tau^* \to A^* \tag{12}$$

that yields a change-of-base functor

$$u^* : \mathscr{R}^*(A^*) \to \mathscr{R}^*(A_\tau^*).$$

In terms of LTSs, it is easy to see that u^* acts on objects by closing a transition system with actions in A^* by the following rule, and on arrows (simulations) as identity.

$$\frac{P \xrightarrow{\epsilon} Q}{P \xrightarrow{\tau} Q} \tag{13}$$

The left adjoint $\Sigma_u : \mathscr{R}^*(A_\tau^*) \to \mathscr{R}^*(A^*)$ can be obtained by following the construction in the proof of Theorem 5.1. It can be understood in this particular case as first closing an LTS wrt to rules (2) and then renaming τ to ϵ.

Corollary 6.1. *There is a 2-adjunction*

$$\begin{tikzcd} \mathscr{R}^*(A_\tau^*) \arrow[r, bend left, "\Sigma_u"] \arrow[r, phantom, "\perp"] & \mathscr{R}^*(A^*) \arrow[l, bend left, "u^*"] \end{tikzcd} \tag{14}$$

where Σ_u can be understood on objects as closing an LTS wrt to the rules (2) *and renaming τ to ϵ. On arrows Σ_u acts as identity .*

The definition of Σ_u gives us a concise way to capture weak simulation—a relation $\varphi\colon h \twoheadrightarrow h'$, where $h, h' \in \mathscr{R}^*(A_\tau^*)$ is a weak simulation precisely when it is defines an arrow $\varphi\colon \Sigma_u h \twoheadrightarrow \Sigma_u h'$ in $\mathscr{R}^*(A^*)$.

Proposition 6.2. *The category of (ordinary) labelled transition systems with label set A_τ and weak simulations is the category with*

- *objects those of $LTS(A_\tau)$ (the full subcategory of $\mathscr{R}^*(A_\tau^*)$ with objects ordinary functors) and*
- *morphisms h to h' given by $\mathscr{R}^*(A^*)(\Sigma_u(h), \Sigma_u(h'))$.*

□

Restricting to the subcategories with functional morphisms, u defined in (12) satisfies the weak factorisation lifting property and hence u^* also has a right adjoint $\Pi_u : \mathscr{R}(A_\tau^*) \to \mathscr{R}(A^*)$ [36, Theorem 4.1]. In terms of LTS terminology, Π_u maps an ordinary LTS $L = (X, A_\tau, T)$ to the following LTS with labels in A:

states: those $x \in X$ for which $x \xrightarrow{\tau} x$ in L

transitions: the non-τ transitions of L.

$\Pi_u(h)$ can thus be understood as the largest τ-reflexive sub-LTS, restricted to the non-τ actions. It is not difficult to verify that the Π_u defined in this way is a right adjoint to u^*.

7. Conclusions and future work

We have observed that several examples of labelled transition systems, especially when there is algebraic structure on labels, can be considered as relational presheaves. The morphisms between such LTSs are simulations and the resulting categories of LTSs have rich structure due to their mathematical status. As an example, we characterised weak closure as a left adjoint to a change-of-base functor.

There are several future directions. The theory of bisimulation needs to be developed: in the examples that we have examined bisimulations are those morphisms that remain morphisms when reversed via the underlying involution in **Rel**, it remains to be seen whether this view is fruitful in the wider, general picture. For example, it is of interest that functional bisimulations in our examples can be characterised as *maps* in $\mathscr{R}^*(\mathbf{C})$, that is, those 1-cells which have a right adjoint in the 2-categorical sense.

There are several other settings in which there is an obvious monoidal structure on labels. One example is reversible transition systems where standard constructions [12, 13] may be characterised as universal when viewed as relational presheaves.

Morphisms in $\mathscr{R}^*(\mathbf{C})$, change-of-base functors and their adjoints give a general approach to defining simulations (amongst other equivalences), morphisms and canonical constructions in models where such notions have not been obvious; for example in the theory of reactive systems. Finally, **Rel** can be generalised to **Span** and other (bi)categories, yielding more general notions of labelled transition systems. Another possibility would be to consider Kleisli categories on monads other than $\mathcal{P}$; for instance the subdistribution monad $\mathcal{D}$ for probabilistic notions of transition systems, following Hasuo [18, 19].

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