Van Kampen colimits as bicolimits in Span*

Tobias Heindel[1] and Paweł Sobociński[2]

[1] Abt. für Informatik und angewandte KW, Universität Duisburg-Essen, Germany
[2] ECS, University of Southampton, United Kingdom

Abstract. The exactness properties of coproducts in extensive categories and pushouts along monos in adhesive categories have found various applications in theoretical computer science, e.g. in program semantics, data type theory and rewriting. We show that these properties can be understood as a single universal property in the associated bicategory of spans. To this end, we first provide a general notion of Van Kampen cocone that specialises to the above colimits. The main result states that Van Kampen cocones can be characterised as exactly those diagrams in $\mathbb{C}$ that induce bicolimit diagrams in the bicategory of spans $\mathcal{S}pan_{\mathbb{C}}$, provided that $\mathbb{C}$ has pullbacks and enough colimits.

Introduction

The interplay between limits and colimits is a research topic with several applications in theoretical computer science, including the solution of recursive domain equations, using the coincidence of limits and colimits. Research on this general topic has identified several classes of categories in which limits and colimits relate to each other in useful ways; extensive categories [5] and adhesive categories [21] are two examples of such classes.

Extensive categories [5] have coproducts that are "well-behaved" with respect to pullbacks; more concretely, they are disjoint and universal. Extensivity has been used by mathematicians [4] and computer scientists [25] alike. In the presence of products, extensive categories are distributive [5] and thus can be used, for instance, to model circuits [28] or to give models of specifications [11]. Sets and topological spaces inhabit extensive categories while quasitoposes are not, in general, extensive [15].

Adhesive categories [20, 21] have pushouts along monos that are similarly "well-behaved" with respect to pullbacks – they are instances of Van Kampen squares. Adhesivity has been used as a categorical foundation for double-pushout graph transformation [20, 7] and has found several related applications [8, 27]. Toposes are adhesive [22] but quasitoposes, in general, are not [14].

Independently of our work, Cockett and Guo proposed *Van Kampen* (VK) *colimits* [6] as a generalisation of Van Kampen squares. The main examples of VK-colimits include coproducts in extensive categories and pushouts along monos in adhesive categories. Another example is a strict initial object; moreover, in a

* Research partially supported by EPSRC grant EP/D066565/1.

Barr-exact category, any regular epimorphism is a VK-coequaliser of its kernel pair [12, Theorem 3.7(d)].

The definition of VK-colimits relies only on elementary notions of category theory. This feature, while attractive, obscures their relationship with other categorical concepts. More abstract characterisations exist for extensive and adhesive categories. For instance, a category $\mathbb{C}$ is extensive if and only if the functor $+ : \mathbb{C}\downarrow A \times \mathbb{C}\downarrow B \to \mathbb{C}\downarrow A + B$ is an equivalence for any $A, B \in \mathbb{C}$ [23, 5]; adhesive categories can be characterised in a similar manner [21]. Our definition of VK-cocone will be of the latter kind, i.e. in terms of an equivalence of categories. We also provide an elementary characterisation in the spirit of Cockett and Guo.

This paper contains one central result: VK-cocones are those diagrams that are bicolimit diagrams when embedded in the associated bicategory of spans. This characterises "being Van Kampen" as a universal property. We believe that this insight captures and explains the essence of the various aforementioned well-behaved colimits studied in the literature.

Spans are known to theoretical computer scientists through the work of Katis, Sabadini and Walters [16] who used them to model systems with boundary, see also [10]. The bicategory of spans over $\mathbb{C}$ contains $\mathbb{C}$ via a canonical embedding $\Gamma\colon \mathbb{C} \to \mathcal{S}pan_{\mathbb{C}}$, that is the identity on objects and takes each arrow $C \xrightarrow{f} D$ of $\mathbb{C}$ to its *graph* $C \xleftarrow{\mathrm{id}} C \xrightarrow{f} D$. Spans generalise partial maps [26]: those spans that have a monomorphism as their left leg, as well as relations[3]. Bicolimits are the canonical notion of colimit in a bicategory.

There is some interesting related recent work. Milius [25] showed that coproducts are preserved (as a lax-adjoint-cooplimit) in the 2-category of relations over an extensive category $\mathbb{C}$. Cockett and Guo [6] have investigated the general conditions under which partial map categories are join-restriction categories: roughly, certain colimits in the underlying category are required to be VK-cocones.

Structure of the paper. In §1 we isolate the relevant class of bicategories and recall the related notions. We also describe the bicategory of spans $\mathcal{S}pan_{\mathbb{C}}$. In §2 we give a definition of VK-cocones together with an elementary characterisation and several examples. In §3 we recall the definition of bicolimits and prove several technical lemmas that allow us to pass between related concepts in $\mathbb{C}$ and $\mathcal{S}pan_{\mathbb{C}}$. Our main characterisation theorem is proved in §4.

1 Preliminaries

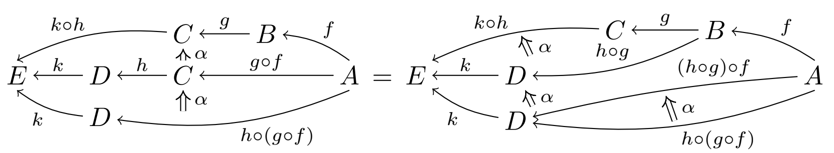
Here we introduce background on bicategories [3] and some notational conventions. For the basic notions of category, functor and natural transformation, the reader is referred to [24]. Our focus is the bicategory of spans over a category $\mathbb{C}$ with a choice of pullbacks (cf. Example 3). In order to avoid unnecessary book-keeping, we only consider bicategories[4] that strictly satisfy the identity axioms.

[3] In the presence of a factorisation system in $\mathbb{C}$.

[4] We have found the bicategory of spans easier to work with than the biequivalent 2-category [19].

Definition 1 (Strictly unitary bicategories). A *strictly unitary* (SU) *bicategory* $\mathscr{B}$ consists of:

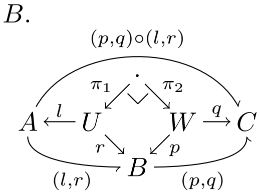
- a collection $\mathrm{ob}\,\mathscr{B}$ of *objects*;
- for $A, B \in \mathrm{ob}\,\mathscr{B}$ a category $\mathscr{B}(A, B)$, the objects and arrows of which are called, respectively, the *arrows* and the *2-cells* of $\mathscr{B}$. Composition is denoted by $\odot$ and referred to as *vertical composition*. Given $(f : A \to B) \in \mathscr{B}(A, B)$, its identity 2-cell will be denoted $\iota_f : f \to f$. Each $\mathscr{B}(A, A)$ contains a special object $\mathrm{id}_A : A \to A$, called the *identity arrow*;
- for $A, B, C \in \mathrm{ob}\,\mathscr{B}$, a functor $c_{A,B,C} : \mathscr{B}(A, B) \times \mathscr{B}(B, C) \to \mathscr{B}(A, C)$ called *horizontal composition*. On objects, $c_{A,B,C}\langle f, g\rangle$ is written $g \circ f$, while on arrows $c_{A,B,C}\langle \gamma, \delta\rangle$ it is $\delta * \gamma$. For any $f : A \to B$ we have $\mathrm{id}_B \circ f = f = f \circ \mathrm{id}_A$;
- for $A, B, C, D \in \mathrm{ob}\,\mathscr{B}$, arrows $f : A \to B$, $g : B \to C$ and $h : C \to D$ an *associativity natural isomorphism* $\alpha_{A,B,C,D}(f, g, h) : h \circ (g \circ f) \to (h \circ g) \circ f$. It satisfies the coherence axioms: for any composable f, g, h, k, we have $\alpha_{f,\mathrm{id},g} = \iota_{g\circ f}$ and also that the following 2-cells are equal:



Example 2. Any (ordinary) category $\mathbb{C}$ is an (SU-)bicategory with trivial 2-cells.

Example 3 (Span bicategory [3]). Assume that $\mathbb{C}$ has a choice of pullbacks that preserves identities: for any cospan $X \xrightarrow{f} Z \xleftarrow{g} Y$ there exists an object $X \times_Z Y$ and span $X \xleftarrow{g'} X \times_Z Y \xrightarrow{f'} Y$ that together with f and g form a pullback square. Moreover if f is id_X then $X \times_Z Y = Y$ and $f' = \mathrm{id}_Y$.[5] $\mathcal{S}pan_{\mathbb{C}}$ has:

- as objects, the objects of $\mathbb{C}$, i.e. $\mathrm{ob}\,\mathcal{S}pan_{\mathbb{C}} = \mathrm{ob}\,\mathbb{C}$;
- as arrows from A to B, the $\mathbb{C}$-spans $A \xleftarrow{l} U \xrightarrow{r} B$. We shall usually write $(l, r) : A \rightharpoonup B$ to denote such a span and refer to U as its *carrier*. When $l = \mathrm{id}$ and $r : A \to B$ we will write simply $(, r) : A \rightharpoonup B$.
  The composition with another span $B \xleftarrow{p} W \xrightarrow{q} C$ is obtained via the chosen pullback as illustrated to the right; however this composition is only associative up to canonical isomorphism. The identity on an object A is the span $A \xleftarrow{\mathrm{id}} A \xrightarrow{\mathrm{id}} A$.



- its 2-cells $\xi : (l, r) \to (l', r')$ are $\mathbb{C}$-arrows $\xi : U \to U'$ between the respective carriers such that $l' \circ \xi = l$ and $r' \circ \xi = r$.

For our purposes it suffices to consider *strict homomorphisms* between SU-bicategories.

[5] Note that this is a harmless assumption since the choice of the particular pullback diagram for any span is insignificant.

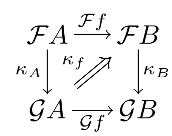
Definition 4 (Strict homomorphisms [3]). Let $\mathscr{A}$ and $\mathscr{B}$ be SU-bicategories. A *strict homomorphism* $\mathcal{F}\colon \mathscr{A} \to \mathscr{B}$ consists of a function $\mathcal{F}\colon \operatorname{ob}\mathscr{A} \to \operatorname{ob}\mathscr{B}$ and a family of functors $\mathcal{F}(A,B)\colon \mathscr{A}(A,B) \to \mathscr{B}(\mathcal{F}A,\mathcal{F}B)$ such that:

(i) for all $A \in \mathscr{A}$, $\mathcal{F}(\mathrm{id}_A) = \mathrm{id}_{\mathcal{F}A}$;
(ii) for all $f\colon A \to B$, $g\colon B \to C$ in $\mathscr{A}$, $\mathcal{F}(g \circ f) = \mathcal{F}(g) \circ \mathcal{F}(f)$;
(iii) $\mathcal{F}\alpha_{A,B,C,D} = \alpha_{\mathcal{F}A,\mathcal{F}B,\mathcal{F}C,\mathcal{F}D}$.

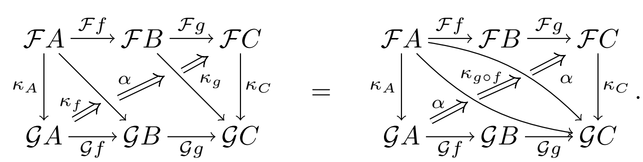
Example 5. The following strict homomorphisms will be of interest to us:

- the covariant embedding $\Gamma\colon \mathbb{C} \to \mathcal{S}pan_{\mathbb{C}}$ which acts as the identity on objects and takes an arrow $f\colon C \to D$ to its graph $(,f)\colon C \rightharpoonup D$;
- $\Gamma\mathcal{F}\colon \mathbb{J} \to \mathcal{S}pan_{\mathbb{C}}$ where $\mathcal{F}\colon \mathbb{J} \to \mathbb{C}$ is a functor;
- given an SU-bicategory $\mathscr{B}$ and $B \in \operatorname{ob}\mathscr{B}$, we shall abuse notation and denote the strict homomorphism from $\mathbb{J}$ to $\mathscr{B}$ which is constant at B by $\Delta_{\mathbb{J}}B$, often omitting the subscript $\mathbb{J}$. Note that in the case of $\mathscr{B} = \mathcal{S}pan_{\mathbb{C}}$, "$\Delta = \Gamma\Delta$".

Definition 6 (Lax transformations). Given strict homomorphisms $\mathcal{F},\mathcal{G}\colon \mathscr{A} \to \mathscr{B}$ between SU-bicategories, a *(lax) transformation* consists of arrows $\kappa_A\colon \mathcal{F}A \to \mathcal{G}A$ for $A \in \mathscr{A}$ and 2-cells $\kappa_f\colon \mathcal{G}f \circ \kappa_A \Rightarrow \kappa_B \circ \mathcal{F}f$ for $f\colon A \to B$ in $\mathscr{A}$ such that:



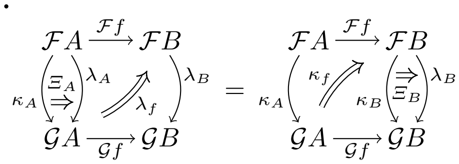
(i) $\kappa_{\mathrm{id}_A} = \iota_{\kappa_A}$ for each $A \in \mathscr{A}$;
(ii) for any $f\colon A \to B$, $g\colon B \to C$ in $\mathscr{A}$, the following 2-cells are equal:



A transformation is said to be *strong* when all the κ_f are invertible 2-cells. Given $B \in \mathscr{B}$ and a homomorphism $\mathcal{M}\colon \mathbb{J} \to \mathscr{B}$, a *pseudo-cocone* $\lambda\colon \mathcal{M} \to \Delta B$ is a synonym for a strong transformation $\lambda\colon \mathcal{M} \to \Delta B$.

Because bicategories have 2-cells, there are morphisms between transformations. They are called *modifications* and are defined as follows.

Definition 7 (Modifications [3, 18]). Given natural transformations κ,λ from $\mathcal{F}$ to $\mathcal{G}$, a modification $\Xi\colon \kappa \to \lambda$ consists of 2-cells $\Xi_A\colon \kappa_A \to \lambda_A$ for $A \in \mathscr{A}$ such that, for all $f\colon A \to B$ in $\mathscr{A}$, $\lambda_f \odot (\iota_{\mathcal{G}f} * \Xi_A) = (\Xi_B * \iota_{\mathcal{F}f}) \odot \kappa_f$.



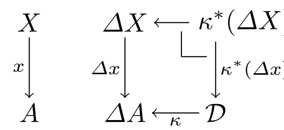
Composition is componentwise, the identity modification on κ is $I_\kappa = \{\iota_{\kappa_A}\}_{A\in\mathscr{A}}$.

Given SU-bicategories $\mathscr{A}$ and $\mathscr{B}$, let $\mathrm{Hom}_l\,[\mathscr{A},\mathscr{B}]$ denote the SU-bicategory of homomorphisms, lax transformations and modifications. Let $\mathrm{Hom}\,[\mathscr{A},\mathscr{B}]$ denote the corresponding SU-bicategory with arrows the strong transformations.

2 Van Kampen cocones

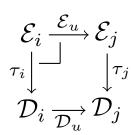
Here we give the definition of Van Kampen cocones together with an elementary characterisation. Let us consider coproducts as a motivating example. A coproduct diagram $A \xrightarrow{i_1} A+B \xleftarrow{i_2} B$ in a category $\mathbb{C}$ is a cocone of the two-object diagram $\langle A, B\rangle$. If $\mathbb{C}$ has pullbacks along coproduct injections then each $x\colon X \to A+B$ gives rise to ${i_1}^*x\colon {i_1}^*X \to A$ and ${i_2}^*x\colon {i_2}^*X \to B$ by pulling back along i_1 and i_2, respectively. Then $x \mapsto \langle {i_1}^*x, {i_2}^*x\rangle$ defines the functor $\langle {i_1}^*\_, {i_2}^*\_\rangle\colon (\mathbb{C}\downarrow A{+}B) \to (\mathbb{C}\downarrow A \times \mathbb{C}\downarrow B)$ on objects. The coproduct satisfies the properties expected in an extensive category when this functor is an equivalence (see [5]).

The situation readily generalises as follows: replace $\langle i_1, i_2\rangle$ by any cocone $\kappa\colon \mathcal{D} \to \Delta A$ from a functor $\mathcal{D}\colon \mathbb{J} \to \mathbb{C}$ to an object A in a category $\mathbb{C}$ with (enough) pullbacks. Any arrow $x\colon X \to A$ induces a natural transformation $\Delta x\colon \Delta X \to \Delta A$ and since also $\kappa\colon \mathcal{D} \to \Delta A$ is a natural transformation, the former can be pulled back along the latter in the functor category $[\mathbb{J}, \mathbb{C}]$ yielding a natural transformation $\kappa^*(\Delta x)\colon \kappa^*(\Delta X) \to \mathcal{D}$.



The described operation extends to a functor $\kappa^*(\Delta\_)$ from $\mathbb{C}\downarrow A$ to (a full subcategory of) $[\mathbb{J}, \mathbb{C}]\downarrow\mathcal{D}$ using the universal property of pullbacks; it maps morphisms with codomain A to *cartesian transformations* with codomain $\mathcal{D}$.

Definition 8 (Cartesian transformations). Let $\mathcal{E}, \mathcal{D} \in [\mathbb{J}, \mathbb{C}]$ be functors and let $\tau\colon \mathcal{E} \to \mathcal{D}$ be a natural transformation. Then τ is a *cartesian (natural) transformation* if all naturality squares are pullback squares, i.e. if the pair $\mathcal{E}_i \xleftarrow{\tau_i} \mathcal{D}_i \xrightarrow{\mathcal{D}_u} \mathcal{D}_j$ is a pullback of $\mathcal{E}_i \xrightarrow{\mathcal{E}_u} \mathcal{E}_j \xleftarrow{\tau_j} \mathcal{D}_j$ for all $u\colon i \to j$ in $\mathbb{J}$.



Let $[\mathbb{J}, \mathbb{C}]\downarrow\mathcal{D}$ be the slice category over $\mathcal{D}$, which has natural transformations with codomain $\mathcal{D}$ as objects. Let $[\mathbb{J}, \mathbb{C}]\Downarrow\mathcal{D}$ denote the full subcategory of $[\mathbb{J}, \mathbb{C}]\downarrow\mathcal{D}$ with the cartesian transformations as objects.

Definition 9 (Van Kampen cocones). Let $\mathbb{C}$ be any category, let $\mathcal{D}\colon \mathbb{J} \to \mathbb{C}$ be a functor, and let $\kappa\colon \mathcal{D} \to \Delta_{\mathbb{J}}A$ be a cocone such that pullbacks along each coprojection κ_i exist ($i \in \mathbb{J}$). Then κ is *Van Kampen* (VK) if the functor $\kappa^*(\Delta_{\mathbb{J}}\_)\colon \mathbb{C}\downarrow A \to [\mathbb{J}, \mathbb{C}]\Downarrow\mathcal{D}$ is an equivalence of categories.

Extensive and adhesive categories have elementary characterisations that are special cases of the following.

Proposition 10 (Elementary VK characterisation). *Suppose that $\mathbb{C}$ has pullbacks and $\mathbb{J}$-colimits, $\mathcal{D}\colon \mathbb{J} \to \mathbb{C}$ is a functor and $\kappa\colon \mathcal{D} \to \Delta_{\mathbb{J}}A$ a cocone such that $\mathbb{C}$ has pullbacks along κ_i ($i \in \mathbb{J}$). Then $\kappa\colon \mathcal{D} \to \Delta_{\mathbb{J}}A$ is Van Kampen iff for every cartesian transformation $\tau\colon \mathcal{E} \to \mathcal{D}$, arrow $x\colon X \to A$ and cocone $\beta\colon \mathcal{E} \to \Delta_{\mathbb{J}}X$ such that $\kappa \circ \tau = \Delta x \circ \beta$, the following are equivalent:*

(i) $\beta\colon \mathcal{E} \to \Delta_{\mathbb{J}}X$ is a $\mathbb{C}$-colimit;
(ii) $\mathcal{D}_i \xleftarrow{\tau_i} \mathcal{E}_i \xrightarrow{\beta_i} X$ is a pullback of $\mathcal{D}_i \xrightarrow{\kappa_i} A \xleftarrow{x} X$ for all $i \in \mathbb{J}$. □

Cockett and Guo's [6] definition of Van Kampen colimits is the equivalence of (i) and (ii) in our Proposition 10. The two definitions are thus very close and coincide in the presence of the relevant pullbacks and colimits.

Remark 11. Under the assumptions of Proposition 10, any Van Kampen cocone $\kappa\colon \mathcal{D} \to \Delta A$ is a colimit diagram of $\mathcal{D}$ in $\mathbb{C}$ (take $\tau = \mathrm{id}_{\mathcal{D}}$ and $x = \mathrm{id}_A$).

Example 12. The following well-known concepts are examples of VK-cocones:

(i) a strict initial object is a VK-cocone for the functor from the empty category;
(ii) a coproduct diagram in an extensive category [5] is a VK-cocone for a functor from the discrete two object category;

A VK-cocone from a span is what has been called a *Van Kampen square* [21].

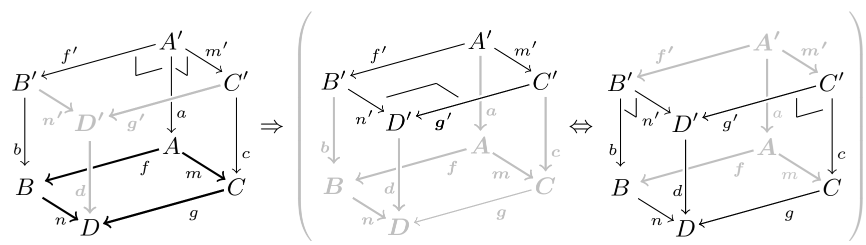


Fig. 1: Van Kampen square

Example 13 (Van Kampen square). A commutative square $B \leftarrow f - A - m \rightarrow C$, $B - n \rightarrow D \leftarrow g - C$ is *Van Kampen* when for each commutative cube as illustrated in Figure 1 on the left that has pullback squares as rear faces, its *top face is a pushout square* if and only if its *front faces are pullback squares* (cf. Figure 1).

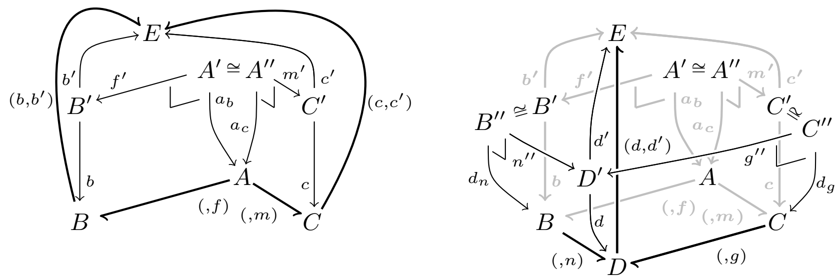
In the left hand diagram in Figure 1, the two arrows $B \leftarrow f - A - m \rightarrow C$ describe a diagram from the three object category $\cdot \leftarrow \cdot \rightarrow \cdot$, and the cospan $B - n \rightarrow D \leftarrow g - C$ gives a cocone for this diagram. That the back faces are pullback squares means that we have a cartesian transformation from $B' \leftarrow f' - A' - m' \rightarrow C$ to $B \leftarrow f - A - m \rightarrow C$.

Adhesive categories are thus precisely categories with pullbacks in which pushouts along monomorphisms exist and are VK-cocones.

3 Van Kampen cocones in a span bicategory

We begin the study of VK-cocones in the span bicategory by explaining roughly how Van Kampen squares induce bipushout squares in the bicategory of spans via the embedding Γ. An illustration of this is given in Figure 2.

At the base of Figure 2(a) is (the image of) a $\mathbb{C}$-span $B \leftarrow f - A - m \rightarrow C$ in $\mathcal{S}pan_{\mathbb{C}}$, i.e. a span of spans. Further, if two spans $(b, b')\colon B \rightharpoonup E$ and $(c, c')\colon C \rightharpoonup E$ are a pseudo-cocone for $B \leftarrow (,f) - A - (,m) \rightarrow C$ then taking pullbacks of b along f and



(a) $\mathcal{S}pan_{\mathbb{C}}$-cocone of the $\mathbb{C}$-span $B \xleftarrow{f} A \xrightarrow{m} C$

(b) Mediating $\mathcal{S}pan_{\mathbb{C}}$-morphism from the $\mathbb{C}$-cocone $B \xrightarrow{n} D \xleftarrow{g} C$

Fig. 2: Cocones and mediating morphisms consisting of spans

c along m (in $\mathbb{C}$) yields isomorphic objects over A, say a_b and a_c; as a result we obtain two pullback squares that will be the back faces of a commutative cube.

Next, let the bottom of Figure 2(b) be (the image of) a commuting $\mathbb{C}$-square, thus yielding another pseudo-cocone of $B \overset{(,f)}{\leftharpoondown} A \overset{(,m)}{\rightharpoonup} C$, namely $B \overset{(,n)}{\rightharpoonup} D \overset{(,g)}{\leftharpoondown} C$. If there is a mediating morphism $(d, d')\colon D \rightharpoonup E$ from the latter cocone to the one of Figure 2(a) ($B \overset{(b,b')}{\rightharpoonup} E \overset{(c,c')}{\leftharpoondown} E$) then pulling back d along n and g results in morphisms which are isomorphic to b and c, respectively, say d_n and d_g; the resulting pullback squares provide the front faces of a cube.

Now, if $B \xrightarrow{n} D \xleftarrow{g} C$ is a VK-cocone of $B \xleftarrow{f} A \xrightarrow{m} C$ then such a mediating morphism can be constructed by taking D' as the pushout of B' and C' over either one of A' or A''. The morphisms $d\colon D' \to E$ and $d'\colon D' \to D$ arise from the universal property of pushouts, everything commutes and the front faces are pullback squares because of the VK-property. Further this mediating morphism is essentially unique, which means that given any other span $(e, e')\colon D \rightharpoonup E$ such that both $(b, b') \cong (e, e') \circ (, n)$ and $(c, c') \cong (e, e') \circ (, g)$ hold, the two spans (e, e') and (d, d') are isomorphic via a unique isomorphism.

Though this sketch lacks relevant technical details, it nevertheless may suffice to convey the flavor of the diagrams that are involved in the proof of the fact that Van Kampen squares in $\mathbb{C}$ induce bipushouts in $\mathcal{S}pan_{\mathbb{C}}$. Moreover, also the converse holds, i.e. if the image of a pushout is a bipushout in $\mathcal{S}pan_{\mathbb{C}}$ then it is a Van Kampen square.

3.1 Span bicolimits

Clearly any diagram in $\mathcal{S}pan_{\mathbb{C}}$ can be "decomposed" into a diagram in $\mathbb{C}$: each arrow in $\mathcal{S}pan_{\mathbb{C}}$ gives two $\mathbb{C}$-arrows from a carrier object; further a 2-cell in $\mathcal{S}pan_{\mathbb{C}}$ is an $\mathbb{C}$-arrow between the carriers satisfying certain commutativity requirements.

We shall start with further observations along these lines. Roughly we are able to "drop a dimension" in the following sense. First, it is easy to see that

$[\mathbb{J},\mathbb{C}]$ inherits a choice of pullbacks from $\mathbb{C}$. In particular, it follows that $\mathcal{S}pan_{[\mathbb{J},\mathbb{C}]}$ is an SU-bicategory. Now, given $\mathcal{F},\mathcal{G}\in[\mathbb{J},\mathbb{C}]$ we note that:

- spans of natural transformations from $\mathcal{F}$ to $\mathcal{G}$ correspond to lax transformations from $\Gamma\mathcal{F}$ to $\Gamma\mathcal{G}$; and
- morphisms of such spans are the counterpart of modifications.

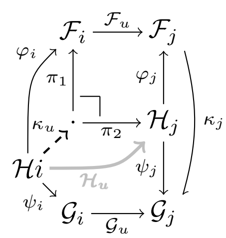
The following lemma makes this precise.

Lemma 14. *There is a strict homomorphism*

$$\mathbf{\Gamma}\colon \mathcal{S}pan_{[\mathbb{J},\mathbb{C}]} \to \mathrm{Hom}_l\,[\mathbb{J},\mathcal{S}pan_{\mathbb{C}}]$$

that takes $\mathcal{F}\in[\mathbb{J},\mathbb{C}]$ to $\Gamma\mathcal{F}$ and is full and faithful on both arrows and 2-cells.

Proof (proof sketch). Here we only give the definition of $\mathbf{\Gamma}$ as checking the details involves tedious calculations.



A span of natural transformations $(\varphi,\psi)\colon \mathcal{F}\rightharpoonup\mathcal{G}$ with carrier $\mathcal{H}$ is mapped to a lax transformation $\mathbf{\Gamma}_{\mathcal{F},\mathcal{G}}(\varphi,\psi)$ as follows: for each $i\in\mathbb{J}$, we put $\kappa_i := (\varphi_i,\psi_i)\colon \mathcal{F}_i\rightharpoonup\mathcal{G}_i$, and for each morphism $u\colon i\to j$ in $\mathbb{J}$, we define a 2-cell $\kappa_u\colon (,\mathcal{G}_u)\circ\kappa_i\to\kappa_j\circ(,\mathcal{F}_u)$ as sketched to the right. More explicitly, using that $\mathcal{F}_u\circ\varphi_i=\varphi_j\circ\mathcal{H}_u$ holds by naturality of φ, the arrow $\kappa_u\colon\mathcal{H}_i\to\mathcal{F}_i\times_{\mathcal{F}_j}\mathcal{H}_j$ is the unique one satisfying $\varphi_i=\pi_1\circ\kappa_u$ and $\mathcal{H}_u=\pi_2\circ\kappa_u$. To check that κ_u is a 2-cell it remains to check the equation $\psi_j\circ\pi_2\circ\kappa_u=\psi_j\circ\mathcal{H}_u=\mathcal{G}_u\circ\psi_i$, which follows by the naturality of ψ.

Further, a 2-cell between spans $(\varphi,\psi),(\varphi',\psi')\colon\mathcal{F}\rightharpoonup\mathcal{G}$ with respective carriers $\mathcal{H},\mathcal{H}'$ is a natural transformation $\xi\colon\mathcal{H}\to\mathcal{H}'$ satisfying both $\varphi'\circledcirc\xi=\varphi$ and $\psi'\circledcirc\xi=\psi$. This induces a modification $\{\xi_i\}_{i\in\mathbb{J}}\colon\mathbf{\Gamma}(\varphi,\psi)\to\mathbf{\Gamma}(\varphi',\psi')$. □

Corollary 15. *For any functor $\mathcal{F}\in[\mathbb{J},\mathbb{C}]$, the strict homomorphism $\mathbf{\Gamma}$ defines a natural isomorphism between the following two functors of type $[\mathbb{J},\mathbb{C}]\to\mathbf{Cat}$:*

$$\mathcal{S}pan_{[\mathbb{J},\mathbb{C}]}(\mathcal{F},\_)\;\cong\;\mathrm{Hom}_l\,[\mathbb{J},\mathcal{S}pan_{\mathbb{C}}](\Gamma\mathcal{F},\Gamma\_).\qquad\square$$

The above lemma and corollary can be adapted to talk about strong transformations instead of lax ones (this will recur when we discuss bicolimits formally). This restriction to strong transformations has a counterpart on the other side of the isomorphism of Corollary 15: we need to restrict to those spans in $\mathcal{S}pan_{[\mathbb{J},\mathbb{C}]}(\mathcal{F},\mathcal{G})$ that have a cartesian transformation from the carrier to $\mathcal{F}$.

Recall that a cartesian transformation between functors is a natural transformation with all naturality squares pullbacks (cf. Definition 8). It is an easy exercise to show that cartesian natural transformations include all natural isomorphisms and are closed under pullback. Hence – in a similar way as one can restrict the span bicategory to all partial map spans, i.e. those with the left leg monic – we let $\mathcal{S}pan^{\Leftarrow}_{[\mathbb{J},\mathbb{C}]}$ be the (non-full) sub-bicategory of $\mathcal{S}pan_{[\mathbb{J},\mathbb{C}]}$ that has all those spans $(\varphi,\psi)\colon\mathcal{F}\rightharpoonup\mathcal{G}$ in $\mathcal{S}pan_{[\mathbb{J},\mathbb{C}]}(\mathcal{F},\mathcal{G})$ as arrows of which the left leg φ is a cartesian transformation. Adapting the proof of Lemma 14, one obtains the following proposition.

Proposition 16. *There is a strict homomorphism* $\mathbf{\Gamma}\colon \mathcal{S}pan^{\Leftarrow}_{[\mathbb{J},\mathbb{C}]} \to \mathrm{Hom}\,[\mathbb{J}, \mathcal{S}pan_{\mathbb{C}}]$ *which is full and faithful on both arrows and 2-cells. For any functor* $\mathcal{F} \in [\mathbb{J}, \mathbb{C}]$, $\mathbf{\Gamma}$ *defines a natural isomorphism between the following functors* $[\mathbb{J}, \mathbb{C}] \to \mathbf{Cat}$*:*

$$\mathcal{S}pan^{\Leftarrow}_{[\mathbb{J},\mathbb{C}]}(\mathcal{F}, \_) \;\cong\; \mathrm{Hom}\,[\mathbb{J}, \mathcal{S}pan_{\mathbb{C}}](\Gamma\mathcal{F}, \Gamma\_). \qquad \square$$

The above lets us pass between diagrams in $\mathcal{S}pan_{\mathbb{C}}$ and $\mathbb{C}$: for example the strong transformations of homomorphisms to $\mathcal{S}pan_{\mathbb{C}}$ are those spans of natural transformations of functors to $\mathbb{C}$ that have a cartesian first leg; the modifications of the former are the morphisms of spans of the latter. This observation will be useful when relating the notion of bicolimit in $\mathcal{S}pan_{\mathbb{C}}$ with the notion of VK-cocone in $\mathbb{C}$.

An elementary definition of bicolimits. For our purposes we need to recall only the definition of (conical) bicolimits [17] for functors with an (ordinary) small category $\mathbb{J}$ as domain. Given a homomorphism $\mathcal{M}\colon \mathbb{J} \to \mathscr{B}$, a bicolimit of $\mathcal{M}$ is an object $\mathrm{bicol}\,\mathcal{M} \in \mathscr{B}$ with a pseudo-cocone $\kappa\colon \mathcal{M} \to \Delta(\mathrm{bicol}\,\mathcal{M})$ such that "pre-composition" with κ gives an equivalence of categories

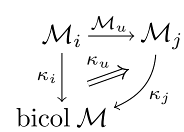
$$\mathscr{B}(\mathrm{bicol}\,\mathcal{M}, X) \simeq \mathrm{Hom}\,[\mathbb{J}, \mathscr{B}](\mathcal{M}, \Delta X) \tag{1}$$

that is natural in X (i.e. the right hand side is essentially representable as a functor $\lambda X.\,\mathrm{Hom}\,[\mathbb{J}, \mathscr{B}](\mathcal{M}, \Delta X)\colon \mathscr{B} \to \mathbf{Cat}$); the pair $\langle \mathrm{bicol}\,\mathcal{M}, \kappa\rangle$ is referred to as *the bicolimit of* $\mathcal{M}$. We will often speak of $\kappa\colon \mathcal{M} \to \Delta\mathrm{bicol}\,\mathcal{M}$ as a bicolimit without mentioning the pair $\langle \mathrm{bicol}\,\mathcal{M}, \kappa\rangle$ explicitly.

To make the connection with the elementary characterisation of Van Kampen cocones in Proposition 10, we shall use the fact that equivalences of categories can be characterised as full, faithful functors that are essentially surjective on objects, and work with the following equivalent, elementary definition.

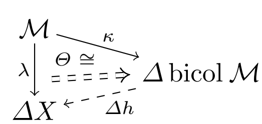
Definition 17 (Bicolimits). Given an SU-bicategory $\mathscr{B}$, a category $\mathbb{J}$ and a strict homomorphism $\mathcal{M}\colon \mathbb{J} \to \mathscr{B}$, a *bicolimit* for $\mathcal{M}$ consists of:

- an object $\mathrm{bicol}\,\mathcal{M} \in \mathscr{B}$;
- a pseudo-cocone $\kappa\colon \mathcal{M} \to \Delta\,\mathrm{bicol}\,\mathcal{M}$: for each object $i \in \mathbb{J}$ an arrow $\kappa_i\colon \mathcal{M}_i \to \mathrm{bicol}\,\mathcal{M}$, and for each $u\colon i \to j$ in $\mathbb{J}$ an invertible 2-cell $\kappa_u\colon \kappa_i \to \kappa_j \circ \mathcal{M}_u$ satisfying the axioms required for κ to be a strong transformation.

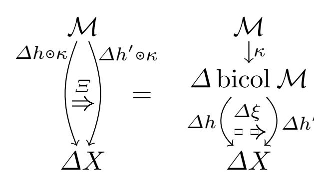


The bicolimit satisfies the following universal properties.

(i) essential surjectivity:
for any pseudo-cocone $\lambda\colon \mathcal{M} \to \Delta X$, there exists $h\colon \mathrm{bicol}\,\mathcal{M} \to X$ in $\mathscr{B}$ and an invertible modification $\Theta\colon \lambda \to \Delta h \circledcirc \kappa$; and



(ii) fullness and faithfullness:
for any $h, h'\colon \mathrm{bicol}\,\mathcal{M} \to X$ in $\mathscr{B}$ and each modification $\Xi\colon \Delta h \circledcirc \kappa \to \Delta h' \circledcirc \kappa$, there is a unique 2-cell $\xi\colon h \to h'$ satisfying $\Xi = \Delta\xi * I_\kappa$ (and hence ξ is invertible iff Ξ is).



The pair $\langle h, \Theta \rangle$ of Condition (i) is called a *mediating cell* from κ to λ.

Condition (ii) of this definition implies that mediating cells from a bicolimit to a pseudo-cocone are *essentially unique*: any two such mediating cells $\langle h, \Theta \rangle$ and $\langle h', \Theta' \rangle$ are isomorphic since $\Theta' \circledcirc \Theta^{-1} \colon \Delta h \circledcirc \kappa \to \Delta h' \circledcirc \kappa$ corresponds to a unique invertible 2-cell $\zeta \colon h \to h'$ such that $\Theta' \circledcirc \Theta^{-1} = \Delta\zeta * I_\kappa$.

To relate the notion of bicolimit with the characterisation of VK-cocones of Proposition 10, we shall reformulate the above elementary definition. Given a pseudo-cocone $\kappa \colon \mathcal{M} \to \Delta C$, a morphism $h \colon C \to D$ will be called *universal for* κ or κ*-universal* if, given any other morphism $h' \colon C \to D$ with a modification $\Xi \colon \Delta h \circledcirc \kappa \to \Delta h' \circledcirc \kappa$, there exists a *unique* 2-cell $\xi \colon h \to h'$ satisfying $\Xi = \Delta\xi * I_\kappa$; further, a mediating cell $\langle h, \Theta \rangle$ is called universal, if the morphism h is universal. The motivation behind this terminology and the slightly redundant statement of the following proposition will become apparent in §4; its proof is straightforward.

Proposition 18. *A pseudo-cocone $\kappa \colon \mathcal{M} \to \Delta C$ from a diagram $\mathcal{M}$ to C is a bicolimit iff both of the following hold:*

(i) for any pseudo cocone $\lambda \colon \mathcal{M} \to \Delta D$ there is a universal mediating cell $\langle h \colon C \to D, \Theta \rangle$ from κ to λ;
(ii) all arrows $h \colon C \to D$ are universal for κ. □

We are interested in bicolimits of strict homomorphisms of the form $\Gamma\mathcal{F}$ where $\mathcal{F} \colon \mathbb{J} \to \mathbb{C}$ is a functor and $\Gamma \colon \mathbb{C} \to \mathcal{S}pan_{\mathbb{C}}$ is the covariant embedding of $\mathbb{C}$. The defining equivalence of bicolimits in (1) specialises as follows:

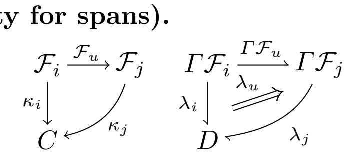
$$\mathcal{S}pan_{\mathbb{C}}(\operatorname{bicol} \Gamma\mathcal{F}, X) \simeq \operatorname{Hom}[\mathbb{J}, \mathcal{S}pan_{\mathbb{C}}](\Gamma\mathcal{F}, \Delta X).$$

Using Proposition 16, this is equivalent to:

$$\mathcal{S}pan_{\mathbb{C}}(\operatorname{bicol} \Gamma\mathcal{F}, X) \simeq \mathcal{S}pan^{\Leftarrow}_{[\mathbb{J},\mathbb{C}]}(\mathcal{F}, \Delta X).$$

We shall exploit working in $\mathcal{S}pan^{\Leftarrow}_{[\mathbb{J},\mathbb{C}]}$ in the following lemma which relates the concepts involved in the elementary definition of bicolimits with diagrams in $\mathbb{C}$. It will serve as the technical backbone of our main theorem.

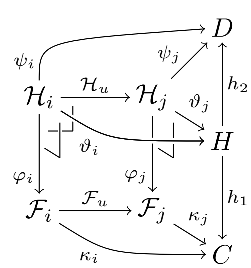
Lemma 19 (Mediating cells and universality for spans).
Let $\kappa \colon \mathcal{F} \to \Delta C$ be a cocone in $\mathbb{C}$ of a diagram $\mathcal{F} \in [\mathbb{J}, \mathbb{C}]$, and let $\lambda \colon \Gamma\mathcal{F} \to \Delta D$ be a pseudo-cocone in $\mathcal{S}pan_{\mathbb{C}}$ where $\lambda_i = (\varphi_i, \psi_i)$ for all $i \in \mathbb{J}$:



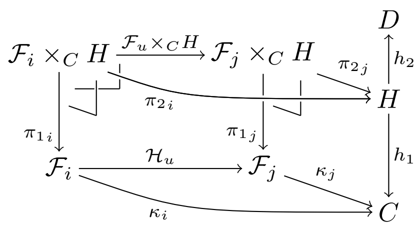
(i) to give a mediating cell

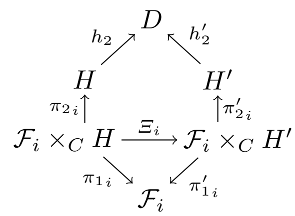
$$\langle C \xleftarrow{h_1} H \xrightarrow{h_2} D, \Theta \colon \lambda \to \Delta(h_1, h_2) \circledcirc \Gamma\kappa \rangle$$

from $\Gamma\kappa$ to λ is to give a cocone $\vartheta \colon \mathcal{H} \to \Delta H$ where $\mathcal{H}$ is the carrier functor of the image of λ in $\mathcal{S}pan^{\Leftarrow}_{[\mathbb{J},\mathbb{C}]}(\mathcal{F}, \Delta D)$ (cf. Proposition 16) such that the resulting three-dimensional diagram (†) in $\mathbb{C}$ (to the right) commutes and its lateral faces ${}^{\mathcal{H}_i}_{\mathcal{F}_i}\!\downarrow\!\overset{\urcorner}{\to}\!\downarrow^{H}_{C}$ are pullbacks;


(†)

(ii) to give a modification $\varXi\colon \Delta(h_1,h_2)\circledcirc \varGamma\kappa \to \Delta(h'_1,h'_2)\circledcirc \varGamma\kappa$ for a pair of spans $(h_1,h_2),(h'_1,h'_2)\colon C \rightharpoonup D$ is to give a cartesian transformation $\varXi\colon \mathcal{F}\times_{\Delta C}\Delta H \to \mathcal{F}\times_{\Delta C}\Delta H'$ such that the two equations $\pi'_1\circ\varXi = \pi_1$ and $(\Delta h'_2)\circ\pi'_2\circ\varXi = (\Delta h_2)\circ\pi_2$ hold.

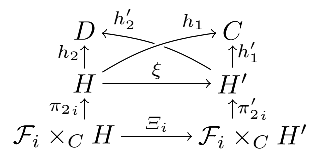




(‡)

Here $\mathcal{F} \xleftarrow{\pi_1} \mathcal{F}\times_{\Delta C}\Delta H \xrightarrow{\pi_2} \Delta H$ is the pullback of $\mathcal{F} \xrightarrow{\kappa} \Delta C \xleftarrow{\Delta h_1} \Delta H$ as sketched in (‡) above and similarly for $h'\colon H' \to C$.

Further, to give a cell $\xi\colon (h_1,h_2)\to(h'_1,h'_2)$ that satisfies $\Delta\xi * I_{\varGamma\kappa} = \varXi$ is to give a $\mathbb{C}$-arrow $\xi\colon H\to H'$ which satisfies the three equations $h'_1\circ\xi = h_1$, $h'_2\circ\xi = h_2$ and $\Delta\xi\circ\pi_2 = \pi'_2\circ\varXi$;



(iii) given a span $(h_1,h_2)\colon C\rightharpoonup D$, if the pullback of κ along h_1 is a colimit, i.e. if $\pi_2\colon \mathcal{F}\times_{\Delta C}\Delta H\to\Delta H$ is a colimit, then (h_1,h_2) is universal for $\varGamma\kappa$;

(iv) conversely, if (h_1,h_2) is universal for $\varGamma\kappa$, then $\pi_2\colon \mathcal{F}\times_{\Delta C}\Delta H\to\Delta H$ is a colimit – provided that some colimit of $\mathcal{F}\times_{\Delta C}\Delta H$ exists in $\mathbb{C}$.

Proof. Both (i) and (ii) are immediate consequences of Proposition 16.

As for (iii), we need to show that every modification $\varXi\colon \Delta(h_1,h_2)\circledcirc\varGamma\kappa \to \Delta(h'_1,h'_2)\circledcirc\varGamma\kappa$ is equal to $\Delta\xi * I_{\varGamma\kappa}$ for a unique $\xi\colon (h_1,h_2)\to(h'_1,h'_2)$. By (ii), $\varXi$ is a natural transformation $\varXi\colon \mathcal{F}\times_{\Delta C}\Delta H\to\mathcal{F}\times_{\Delta C}\Delta H'$. Then, by naturality of $\varXi$, we have that $\pi'_{2j}\circ\varXi_j\circ(\mathcal{F}_u\times_{\Delta C}\Delta H) = \pi'_{2i}\circ\varXi_i$ holds for all $u\colon i\to j$ in $\mathbb{J}$, and since π_2 is a colimit we have a unique $\xi\colon H\to H'$ satisfying $\xi\circ\pi_{2i} = \pi'_{2i}\circ\varXi_i$ for all $i\in\mathbb{J}$. The equations $h_i = h'_i\circ\xi$ follow from the universal property of π_2 (and the properties of $\varXi$). To show uniqueness of ξ, let $\zeta\colon (h_1,h_2)\to(h'_1,h'_2)$ be a 2-cell such that $\varXi = \Delta\zeta * I_{\varGamma_\kappa}$; then using the second statement of Lemma 19(ii), $\Delta\zeta\circ\pi_2 = \pi'_2\circ\varXi$; hence $\zeta = \xi$ follows since π_2 is a colimit. In summary, (h_1,h_2) is universal for $\varGamma\kappa$.

To show (iv), let $\langle H',\vartheta\rangle$ be a colimit of $\mathcal{F}\times_{\Delta C}\Delta H$. Now, it suffices to show that there is a $\mathbb{C}$-morphism $\xi\colon H\to H'$ such that $\vartheta = \Delta\xi\circ\pi_2$.[6] By the universal property of ϑ, we obtain unique $\mathbb{C}$-arrows $h'_1\colon H'\to C$ and $h'_2\colon H'\to D$ such that $\Delta h'_1\circ\vartheta = \kappa\circ\pi_1$ and $\Delta h'_2\circ\vartheta = \Delta h_2\circ\pi_2$. It also follows that the two equations $h_1\circ k = h'_1$ and $h_2\circ k = h'_2$ hold. Pulling back κ along h'_1 yields a span $\mathcal{F} \xleftarrow{\pi'_1} \mathcal{F}\times_{\Delta C}\Delta H' \xrightarrow{\pi'_2} \Delta H'$; we then obtain a natural transformation $\varXi\colon \mathcal{F}\times_{\Delta C}\Delta H\to\mathcal{F}\times_{\Delta C}\Delta H'$ which satisfies $\pi_1 = \pi'_1\circ\varXi$ and $\vartheta = \pi'_2\circ\varXi$, and

[6] The reason is that once such a ξ is provided, there is a unique $k\colon H'\to H$ satisfying $\Delta k\circ\vartheta = \pi_2$, and thus $\xi\circ k = \mathrm{id}_{H'}$ by the universal property of colimits; moreover $k\circ\xi = \mathrm{id}_H$ must hold since (h_1,h_2) is universal for $\varGamma\kappa$.

hence also $\Delta h_2 \circ \pi_2 = \Delta h_2' \circ \vartheta = \Delta h_2' \circ \pi_2' \circ \Xi$. By (ii), this defines a modification $\Xi\colon \Delta(h_1, h_2) \odot \Gamma\kappa \to \Delta(h_1', h_2') \odot \kappa$. Using universality, we get a unique $\xi\colon H \to H'$ such that $h_1' \circ \xi = h_1$, $h_2' \circ \xi = h_2$ and $\Delta\xi \circ \pi_2 = \pi_2' \circ \Xi = \vartheta$. □

4 Van Kampen cocones as span bicolimits

Here we prove the main result of this paper, Theorem 22. Roughly speaking, the conclusion is that (under natural assumptions – existence of pullbacks and enough colimits in $\mathbb{C}$) to be VK in $\mathbb{C}$ is to be a bicolimit in $\mathcal{S}pan_{\mathbb{C}}$. The consequence is that "being VK" is a universal property; in $\mathcal{S}pan_{\mathbb{C}}$ rather than in $\mathbb{C}$.

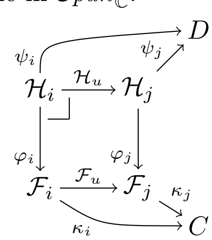
The proof relies on a correspondence between the elementary characterisation of Van Kampen cocones in $\mathbb{C}$ of Proposition 10 and the universal properties of pseudo-cocones in $\mathcal{S}pan_{\mathbb{C}}$ of Proposition 18. More precisely, given a colimit $\kappa\colon \mathcal{M} \to \Delta C$ in $\mathbb{C}$, we shall show that:

- $\Gamma\kappa$-universality *of all* spans $(h_1, h_2)\colon C \rightharpoonup D$ corresponds to the implication (ii) ⇒ (i) of Proposition 10, which is also known as pullback-stability or *universality* of the colimit κ;
- *existence of some* universal mediating cell from $\Gamma\kappa$ to any $\lambda\colon \Gamma\kappa \to \Delta D$ is the counterpart of the implication (i) ⇒ (ii) of Proposition 10, which – for want of a better name – we here refer to as "*converse universality*" of κ;
- thus, $\Gamma\kappa$ is a bicolimit in $\mathcal{S}pan_{\mathbb{C}}$ if and only if the colimit κ is Van Kampen.

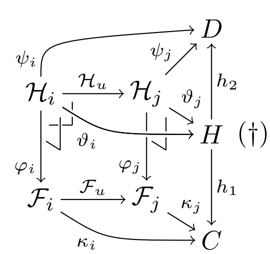
The first two points are made precise by the statements of the following two lemmas. The third point is the statement of the main theorem.

Lemma 20 (Converse universality). *Let $\mathcal{F} \in [\mathbb{J}, \mathbb{C}]$ where $\mathbb{C}$ has pullbacks and for all $(\tau\colon \mathcal{E} \to \mathcal{F}) \in [\mathbb{J}, \mathbb{C}] \Downarrow \mathcal{F}$ a colimit of $\mathcal{E}$ exists. Then $\kappa\colon \mathcal{F} \to \Delta_{\mathbb{J}} C$ satisfies "converse universality" iff given any pseudo-cocone $\lambda\colon \Gamma\mathcal{F} \to \Delta D$, there exists a universal mediating cell $\langle (h_1, h_2), \Theta\rangle$ from $\Gamma\kappa$ to λ in $\mathcal{S}pan_{\mathbb{C}}$.*

Proof. (⇒) Suppose that $\lambda\colon \Gamma\mathcal{F} \to \Delta D$ is a pseudo-cocone in $\mathcal{S}pan_{\mathbb{C}}$.



For $u\colon i \to j$ in $\mathbb{J}$, we obtain a commutative diagram, as illustrated (cf. Proposition 16). Let $\vartheta\colon \mathcal{H} \to \Delta H$ be the colimit of $\mathcal{H}$; thus we obtain $h_1\colon H \to C$ and $h_2\colon H \to D$ making diagram (†) commute. By converse universality, the side faces ${}^{\mathcal{H}_i}_{\mathcal{F}_i}\!\downarrow\!\lrcorner\!\downarrow^{H}_{C}$ are pullback squares; using Lemma 19(ii) we get an invertible modification $\Theta\colon \lambda \to \Delta(h_1, h_2) \odot \Gamma\kappa$. That (h_1, h_2) is universal follows from Lemma 19(iii) since ϑ is a colimit.



(⇐) If in diagram (†) $D = \operatorname{col}\mathcal{H}$ and $\langle D, \psi\rangle$ is the corresponding colimit, we first use the assumption to obtain a universal mediating cell $\langle (h_1, h_2), \Theta\rangle$ from $\Gamma\kappa$ to $\lambda^{(\varphi,\psi)}$ where $\lambda^{(\varphi,\psi)}$ is the pseudo-cocone corresponding to the cartesian transformations $\varphi\colon \mathcal{H} \to \mathcal{F}$ and $\psi\colon \mathcal{H} \to \Delta D$ such that $\lambda_i^{(\varphi,\psi)} = (\varphi_i, \psi_i)$ as in Lemma 19(i); the latter also provides $\vartheta\colon \mathcal{H} \to \Delta H$ such that $h_2 \circ \vartheta_i = \psi_i$ and all ${}^{\mathcal{H}_i}_{\mathcal{F}_i}\!\downarrow\!\lrcorner\!\downarrow^{H}_{C}$ are pullback squares.

It suffices to show that $h_2 = \mathrm{id}_H$. However, by the universal property of the colimit $\langle D, \psi \rangle$, there is an arrow $k\colon D \to h$ such that $k \circ \psi_i = \vartheta_i$. The equation $h_2 \circ k = \mathrm{id}_D$ holds because $\langle D, \psi \rangle$ is a colimit in $\mathbb{C}$, and $k \circ h_2 = \mathrm{id}_H$ follows since (h_1, h_2) is universal for $\Gamma\kappa$. $\square$

Lemma 21 (Universality). *Let $\mathcal{F} \in [\mathbb{J}, \mathbb{C}]$ where $\mathbb{C}$ has pullbacks such that for all $(\tau\colon \mathcal{E} \to \mathcal{F}) \in [\mathbb{J}, \mathbb{C}] \Downarrow \mathcal{F}$, a colimit of $\mathcal{E}$ exists. Then $\kappa\colon \mathcal{F} \to \Delta C$ satisfies universality iff every morphism $(h_1, h_2)\colon C \rightharpoonup D$ in $\mathcal{S}pan_{\mathbb{C}}$ is universal for $\Gamma\kappa$.*

Proof. ($\Rightarrow$) Any morphism (h_1, h_2) leads to a diagram (‡) where all the side-faces are pullbacks. By universality of κ, the cocone π_2 of the top face is a colimit; thus (h_1, h_2) is universal for $\Gamma\kappa$ by Lemma 19(iii).
($\Leftarrow$) Suppose that in diagram (‡) the side faces are all pullbacks. By assumption (h_1, h_2) is universal for $\Gamma\kappa$, thus $\langle H, \pi_2\colon \mathcal{F} \times_{\Delta C} \Delta H \to \Delta H \rangle$ is a colimit by Lemma 19(iv). $\square$

Finally, these two lemmas together with Proposition 18 imply our main result.

Theorem 22. *Let $\mathcal{F} \in [\mathbb{J}, \mathbb{C}]$ where $\mathbb{C}$ has pullbacks and for all cartesian transformations $\tau\colon \mathcal{E} \to \mathcal{F}$, a colimit of $\mathcal{E}$ exists. Then a cocone $\kappa\colon \mathcal{F} \to \Delta C$ is Van Kampen iff $\Gamma\kappa\colon \Gamma\mathcal{F} \to \Delta C$ is a bicolimit in $\mathcal{S}pan_{\mathbb{C}}$.* $\square$

5 Conclusion, related work and future work

We gave a general definition of Van Kampen cocone that captures several previously studied notions in computer science, topology, and related areas, showing that they are instances of the same concept. Moreover, we have provided two alternative characterisations: the first one is elementary, and involves only basic category theoretic notions; the second one exhibits it as a *universal property*: Van Kampen cocones are just those colimits that are preserved by the canonical covariant embedding into the span bicategory.

Although this result is purely category theoretic, there exists closely related work in theoretical computer science. Apart from the references already given in the introduction, we mention the unfolding semantics of Petri nets in terms of coreflections. The latter has been generalised to graph grammars in [1]. Using ω-adhesive categories, i.e. adhesive categories in which colimits of ω-chains of monomorphisms are Van Kampen, it is possible to give such a coreflective unfolding semantics for grammars which rewrite objects of an ω-adhesive category [2]. Moreover, the morphisms between grammars in the latter work have a direct relation to the span-bicategory since they are essentially (a 1-categorical counterpart of) spans which preserve the structure of grammars.

Finally, the definition of Van Kampen cocone allows for several natural variations. For example, one may replace the slice category over the object at the "tip" of cocones by a (full) subcategory of it; this is exactly the step from global descent to $\mathbb{E}$-descent [13] and is closely related to the proposals in [7, 9] for a weakening of the notion of adhesivity. Alternatively, one may start with cocones or diagrams of a particular form. In this way quasi-adhesive categories [21] arise as in the latter only pushouts along *regular* monos are required to be VK; another example is the work of Cockett and Guo [6], where Van Kampen cocones exist for a class of diagrams that naturally arises in their study of join restriction categories. Thus, possibly combining the latter two ideas, several new forms of Van Kampen cocones and diagrams arise as the subject for future research.

Acknowledgment. The authors thank the anonymous referees for their helpful suggestions that have helped to significantly improve this article.

References

1. P. Baldan. *Modelling Concurrent Computations: from Contextual Petri Nets to Graph Grammars.* PhD thesis, Dipartimento di Informatica, Università di Pisa, 2000.
2. P. Baldan, A. Corradini, T. Heindel, B. König, and P. Sobociński. Unfolding grammars in adhesive categories. In *Algebra and Colagebra in Computer Science,* CALCO '09, 2009.
3. J. Bénabou. Introduction to bicategories. In *Midwest Category Seminar*, volume 42 of *Lect. Notes Math.*, pages 1–77. Springer, 1967.
4. R. Brown and G. Janelidze. Van Kampen theorems for categories of covering morphisms in lextensive categories. *J. Pure Appl. Algebra*, 119:255–263, 1997.
5. A. Carboni, S. Lack, and R. F. C. Walters. Introduction to extensive and distributive categories. *J. Pure Appl. Algebra*, 84(2):145–158, February 1993.
6. R. Cockett and X. Guo. Join restriction categories and the importance of being adhesive. Unpublished manuscript, slides from CT'07 presentation available at `http://pages.cpsc.ucalgary.ca/~robin/talks/jrCat.pdf`, 2007.
7. H. Ehrig, A. Habel, J. Padberg, and U. Prange. Adhesive high-level replacement categories and systems. In *International Conference on Graph Transformation,* ICGT *'04*, 2004.
8. H. Ehrig and B. König. Deriving bisimulation congruences in the DPO approach to graph rewriting. In *Foundations of Software Science and Computation Structures,* FoSSaCS *'04*, volume 2987 of *Lect. Notes Comput. Sc.*, pages 151–166. Springer, 2004.
9. H. Ehrig and U. Prange. Weak adhesive high-level replacement categories and systems: a unifying framework for graph and Petri net transformations. In *Essays dedicated to Joseph A. Goguen*, volume 4060 of *Lect. Notes Comput. Sc.*, pages 235–251. Springer, 2006.
10. F. Gadducci, R. Heckel, and M. Llabrés. A bi-categorical axiomatisation of concurrent graph rewriting. In *Category theory and computer science,* CTCS *'99*, volume 29 of *Elect. Notes Theor. Comput. Sc.*, 1999.
11. B. Jacobs. Parameters and parametrization in specification, using distributive categories. *Fundamenta Informaticae*, 24(3):209–250, 1995.
12. G. Janelidze, M. Sobral, and W. Tholen. Beyond barr exactness: Effective descent morphisms. In M. C. Pedicchio and W. Tholen, editors, *Categorical Foundations: Special Topics in Order, Topology, Algebra, and Sheaf Theory.* Cambridge University Press, 2004.
13. G. Janelidze and W. Tholen. Facets of descent, I. *Applied Categorical Structures*, 2(3):245–281, 1994.

14. P. Johnstone, S. Lack, and P. Sobociński. Quasitoposes, quasiadhesive categories and Artin glueing. In *Algebra and Coalgebra in Computer Science,* CALCO '07, volume 4626 of *Lect. Notes Comput. Sc.* Springer, August 2007.
15. P. T. Johnstone. *Sketches of an Elephant: A topos theory compendium, vol 1.* Clarendon Press, 2002.
16. P. Katis, N. Sabadini, and R. F. C. Walters. Bicategories of processes. *J. Pure Appl. Algebra*, 115:141–178, 1997.
17. G. M. Kelly. Elementary observations on 2-categorical limits. *Bull. Austral. Math. Soc.*, 39:301–317, 1989.
18. G. M. Kelly and R. H. Street. Review of the elements of 2-categories. *Lect. Notes Math.*, 420:75–103, 1974.
19. S. Lack. A 2-categories companion. arXiv:math/0702535v1 [math.CT] 19 Feb 2007.
20. S. Lack and P. Sobociński. Adhesive categories. In *Foundations of Software Science and Computation Structures,* FoSSaCS '04, volume 2987 of *Lect. Notes Comput. Sc.*, pages 273–288. Springer, 2004.
21. S. Lack and P. Sobociński. Adhesive and quasiadhesive categories. *RAIRO Theor. Inform. Appl.*, 39(3):511–546, 2005.
22. S. Lack and P. Sobociński. Toposes are adhesive. In *International Conference on Graph Tranformation,* ICGT '06, volume 4178 of *Lect. Notes Comput. Sc.*, pages 184–198. Springer, 2006.
23. F. W. Lawvere. Some thoughts on the future of category theory. In *Category Theory*, volume 1488 of *Lect. Notes Math.*, pages 1–13, Como, 1991. Springer.
24. S. Mac Lane. *Categories for the Working Mathematician.* Springer, 1998.
25. S. Milius. On colimits in categories of relations. *Appl. Categor. Str.*, 11(3):287–312, 2003.
26. E. Robinson and G. Rosolini. Categories of partial maps. *Inf. Comput.*, 79(2):95–130, 1988.
27. V. Sassone and P. Sobociński. Reactive systems over cospans. In *Logic in Computer Science, LiCS '05*, pages 311–320. IEEE Press, 2005.
28. R. F. C. Walters. *Categories and Computer Science.* Carslaw Publications, 1991.