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Logical Completeness for String Diagrams

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Contents

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Abstract

String diagrams can be used as a compositional syntax for different kinds of computational structures. This thesis views string diagrams through a logical/universal algebraic perspective – diagrams represent formulas, allowing the use of diagrammatic reasoning for proofs. The category theoretic backdrop of this playground is given by the concept of Cartesian bicategory – a categorical algebra of relations – that we consider as a relational theory through the lens of functorial semantics. Given a Cartesian bicategory, functorial semantics provides us with an appropriate notion of model as a structure-preserving functor to the category of sets and relations.

Functoriality implies that diagrammatic reasoning is sound. The central question this approach raises and that we shall settle is the question of completeness in the logical sense: does every property that is shared by all models have a diagrammatic proof?

The main result of the thesis is a positive answer to the above question. To show this, we give a combinatorial characterisation of Cartesian bicategories in terms of hypergraphs equipped with interfaces, inspired by a seminal result by Chandra and Merlin. This reduces the problem of logical completeness to a categorical lifting problem that we solve using a transfinite construction and a compactness argument.

Introduction

Category theory and logic share an intimate relationship. Many examples for this can be found in [37] or [29]. The most important one for us is the one established by William Lawvere [31]. He uncovered the close relationship between Cartesian categories – i.e. categories with finite products – on the one hand and equational algebraic reasoning on the other. But there is a reason that categories and logic have not announced their marriage yet – there is somebody else, somebody that is often overlooked. That somebody is combinatorics.

In many cases, categories and logic come together with a combinatorial structure to form what we call the *love triangle*:

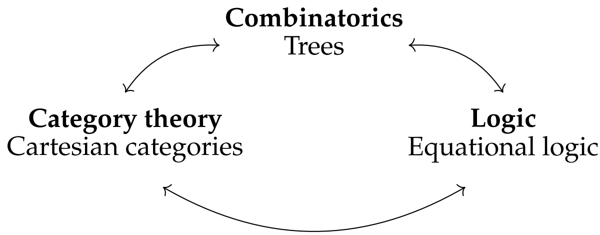
$$\begin{tikzcd}
 & \textbf{Combinatorics} \arrow[dl, bend right, leftrightarrow] \arrow[dr, bend left, leftrightarrow] & \\
\textbf{Category theory} \arrow[rr, bend right, leftrightarrow] & & \textbf{Logic}
\end{tikzcd}$$

There are numerous examples of this in Computer science, a famous one is the Curry-Howard-Lambek correspondence:

$$\begin{tikzcd}
 & \begin{array}{c}\textbf{Combinatorics}\\ \text{Simply typed } \Lambda\text{-calculus}\end{array} \arrow[dl, bend right, leftrightarrow] \arrow[dr, bend left, leftrightarrow] & \\
\begin{array}{c}\textbf{Category theory}\\ \text{Cartesian closed categories}\end{array} \arrow[rr, bend right, leftrightarrow] & & \begin{array}{c}\textbf{Logic}\\ \text{Intuitionistic logic}\end{array}
\end{tikzcd}$$

Admittedly, it is a bit of a stretch to call Lambda calculus a combinatorial structure – the structure is mostly syntactic. Nevertheless, we will consider that vertex of the love triangle the combinatorial one in anticipation of the structures that are to come.

Lawvere's relationship between Cartesian categories and equational logic is no different, again there is someone else. Algebraic reasoning is reasoning on terms, which can be combinatorically described via their syntax trees. This gives Lawvere's love triangle:

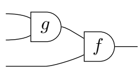


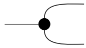
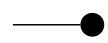
The goal of this thesis is to push the limits of algebra and enrich all three corners of the love triangle:

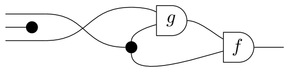
- A logic that is more expressive than equational logic.
- A combinatorial structure more general than trees.
- A categorical structure richer than Cartesian categories.

The way to this enrichment is through *string diagrams* [39], diagrammatic representations of the morphisms in monoidal categories. String diagrams have wide applications in Computer science from quantum computing [1, 20, 35, 28] over concurrency theory [14, 15, 17, 16] and control theory [10, 3] to linguistics [38].

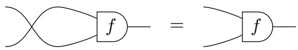
Since Cartesian categories are a special class of monoidal categories, Lawvere's love triangle allows us to translate algebraic terms into string diagrams. A term like $f(g(x_1, x_2), x_3)$, with binary operations f, g, is represented as the diagram



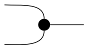
But Cartesian categories have a special property that characterises them among monoidal categories – they allow for copying and discarding [26], which in diagrams are represented by  and . Using this we can render the term $f(g(x_3, x_1), x_1))$ as



where x_2 is discarded because the term ignores it. This allows to describe algebraic properties diagrammatically, for example the diagrammatic equality

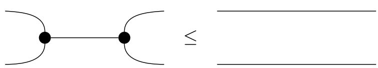


expresses commutativity of f.

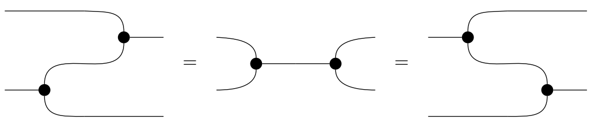
But our hunger for more leads to a natural question: Can we make sense of  and  in some way? This requires a shift of perspective: If we insist to think of our diagrams as functions we are out of luck, but thinking of them as relations we are suddenly in business.

This leads us to consider the notion of Cartesian bicategory, originally introduced in [18]. Cartesian bicategories are to Cartesian categories as Rel, the category of sets and relations, is to Set, the category of sets and functions.

Since relations are ordered via subsets, the definition of Cartesian bicategories axiomatises inequalities between diagrams, such as

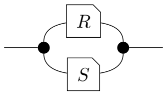


for example. The most important axiom of Cartesian bicategories is the Frobenius law

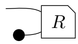


which is found in a surprising variety of applications [1, 20, 14, 10].

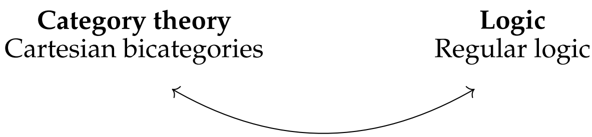
This generalisation from functions to relations ripples through the love triangle to give a more expressive logic. Not only can we talk about implication now, we can even encode some logical connectives.



is an encoding of the conjunction, $R \wedge S$, into diagrams and



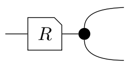
intuitively represents all those x_1 for which there exists x_2 such that $R(x_1, x_2)$ holds, where R is some binary property. In other words, our logic can express conjunction and existential quantification, a fragment of first-order logic known as *regular logic*.



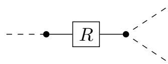
But once again, there is somebody else. This somebody was identified by Chandra and Merlin in their seminal work [19], in which they uncover the connection between regular logic and hypergraphs, a generalisation of graphs where edges are allowed to join more than just two vertices. In this relationship, each formula of regular logic is translated into a hypergraph equipped with an interface and implication of logical formulas corresponds to a graph homomorphism *in the opposite direction*. This makes Chandra and Merlin's insight one edge of a love triangle which we will study in this thesis: The Frobenius love triangle:

$$\begin{tikzcd}[column sep=small]
& \begin{array}{c}\textbf{Combinatorics}\\ \text{Hypergraphs}\end{array} \arrow[dl, bend right, leftrightarrow] \arrow[dr, bend left, leftrightarrow] & \\
\begin{array}{c}\textbf{Category theory}\\ \text{Cartesian bicategories}\end{array} \arrow[rr, bend right, leftrightarrow] & & \begin{array}{c}\textbf{Logic}\\ \text{Regular logic}\end{array}
\end{tikzcd}$$

This means that there is a translation between string diagrams and hypergraphs, which translates, for example



into the hypergraph



The dashed lines are the interface, the graph-analogue of strings being connected to the outside world. This translation has already been given in [13], where it was shown to be a one-to-one correspondence. The first contribution of this thesis goes one step further. Not only do string diagrams correspond to hypergraphs, hypergraph homomorphisms correspond to exactly those inequalities that are provable from the laws of Cartesian bicategories. To be more precise, we define free Cartesian bicategories $\mathbb{CB}_\Sigma$, where Σ is a set of symbols and the morphisms in $\mathbb{CB}_\Sigma$ are terms freely generated from the symbols. Then $\mathbb{CB}_\Sigma$ has a purely combinatorial characterisation.

Theorem 1. *$\mathbb{CB}_\Sigma$ is isomorphic to the Cartesian bicategory of hypergraphs with interfaces.*

But we can take further inspiration from Lawvere's love triangle. In analogy to the notion of a Lawvere theory which identifies an algebraic theory with a Cartesian category, we define the notion of a *Frobenius theory* given by a Cartesian bicategory. Frobenius theories behave like a relational generalisation of Lawvere theories. Every Lawvere theory can be seen as a Frobenius theory by requiring its algebraic operations to be functions, but there are Frobenius theories that are not algebraic such as the theory of partial orders or the theory of equivalence relations.

Additionally, we also have a functorial semantics [32] for Frobenius theories. Just like a model of a Lawvere theory $\mathcal{C}$ is a structure-preserving functor $\mathcal{C} \to \mathsf{Set}$, a model of a Frobenius theory $\mathcal{T}$ is a functor $\mathcal{T} \to \mathsf{Rel}$. This puts syntax and semantics on equal footing – in the case of algebraic theories both are Cartesian categories, in the case of Frobenius theories both are Cartesian bicategories. An immediate consequence of this notion of model is *soundness*, that is any inequality that holds in $\mathcal{T}$ (i.e. anything provable from the axioms of the theory) will hold in any model. Having a model theory therefore gives the important ability to demonstrate that something is *not* a consequence of the axioms by giving a model of the theory that does not satisfy the property in question.

This leads us to the central contribution of this thesis, which regards the question of *logical completeness*. Is there always a model to testify that something does not follow from the axioms? Or, to put it positively, is every property that is shared by all models already provable from the axioms?

Theorem. *Yes.*

To put it less bluntly, the main theorem of this thesis is the following:

Theorem 2. *If A, B are morphisms in a Frobenius theory $\mathcal{T}$ satisfying*

$$\mathcal{M}(A) \subseteq \mathcal{M}(B)$$

for all models $\mathcal{M} \colon \mathcal{T} \to \mathsf{Rel}$. Then $A \leq B$ in $\mathcal{T}$.

We proved a special case of this in [8], the special case of $\mathcal{T} = \mathbb{CB}_\Sigma$ as mentioned above. This had the important consequence that the laws of Cartesian bicategories exactly characterise implication between regular logical formulas. Translating this into the language of databases, which was the original context of Chandra and Merlin's paper [19], gave us an axiomatisation of the inclusion of conjunctive queries, queries given by regular logical formulas, through the laws of Cartesian bicategories. Theorem 2 is a broad generalisation of that result.

In essence, the present thesis will lead us on a sight-seeing tour around the Frobenius love triangle

$$\begin{tikzcd}[column sep=small]
& \begin{array}{c}\textbf{Combinatorics}\\ \text{Hypergraphs}\end{array} \arrow[dl, leftrightarrow, bend right] \arrow[dr, leftrightarrow, bend left] & \\
\begin{array}{c}\textbf{Category theory}\\ \text{Cartesian bicategories}\end{array} \arrow[rr, leftrightarrow, bend right] & & \begin{array}{c}\textbf{Logic}\\ \text{Regular logic}\end{array}
\end{tikzcd}$$

Starting from Chandra and Merlin's translation of regular logic into hypergraphs, we continue from hypergraphs to Cartesian bicategories through their string diagrams. This gives a logical perspective on Cartesian bicategories through Frobenius theories, which raises the problem of logical completeness. To solve this, we push it around the triangle into the realm of combinatorics, where it turns into a categorical question of lifting properties that we answer generally.

Outline of the thesis

- Chapter 1 introduces the main actor of this thesis – Cartesian bicategories, with a slight twist from how they were originally defined in [18]. What we simply call a Cartesian bicategory here was more elaborately called a "Cartesian bicategory of relations" by Carboni and Walters. After discussing some of their basic properties, we will introduce the important notion of a map in a Cartesian bicategory. This generalises (graphs of) functions in Rel to arbitrary Cartesian bicategories, with similar properties. For example, maps always form a Cartesian category. We will then investigate the other direction and construct a Cartesian bicategory $\mathsf{Span}^{\sim}\mathcal{C}$ from a Cartesian category

$\mathcal{C}$, whose morphisms are given as spans $X \leftarrow A \rightarrow Y$. The ordering between spans is given by the existence of a commutative diagram

$$\begin{tikzcd}
& A \arrow[dl] \arrow[dr] \arrow[dd] & \\
X & & Y \\
& B \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

between them. In order to compose spans, $\mathcal{C}$ needs to have pullbacks, a condition we can further relax to mere weak pullbacks. Dually, we define $\mathsf{Cospan}^{\sim}\mathcal{C}$ where morphisms are cospans $X \rightarrow A \leftarrow Y$ and the ordering is reversed – the morphism that makes the diagram commute goes in the opposite direction.

- The problem that guides us in Chapter 2 is that of reconstructing a Cartesian bicategory from its maps, which we will accomplish through a construction similar to $\mathsf{Span}^{\sim}\mathcal{C}$. To that end, we consider *tame* Cartesian bicategories – those Cartesian bicategories that have sufficiently many maps that are well-behaved. This notion generalises the notion of a functionally complete Cartesian bicategory introduced in [18]. For the reconstruction we prove that the ordering in a tame Cartesian bicategory is uniquely determined by the surjective maps, a notion analogous to surjective functions. We use this to give a generalisation of $\mathsf{Span}^{\sim}\mathcal{C}$, denoted $\mathsf{Span}^{\mathcal{S}}\mathcal{C}$, that takes a class $\mathcal{S}$ of morphisms as parameter and the ordering generalises to the existence of a commutative diagram

$$\begin{tikzcd}
& A \arrow[dl] \arrow[dr] & \\
X & A' \arrow[u, "\pi"'] \arrow[d] & Y \\
& B \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

where $\pi \in \mathcal{S}$. In order for this to be a Cartesian bicategory, $\mathcal{S}$ needs to be closed under several constructions such as products and composition, which are similar to closure properties of Grothendieck topologies. Tame Cartesian bicategories are then the ones that satisfy $\mathcal{B} \cong \mathsf{Span}^{\mathcal{S}} \mathrm{Map}(\mathcal{B})$ with $\mathcal{S}$ the class of surjective maps. This nicely extends earlier work we did on the axiom of choice in Cartesian bicategories [9]. The axiom of choice is equivalently stated as "every total relation contains a map", which is readily formalised for the Cartesian bicategory Rel and therefore is an interesting property to consider for any Cartesian bicategory. For tame Cartesian bicategories, satisfying choice implies that surjective maps are exactly split epis, and it turns out that in that case $\mathsf{Span}^{\mathcal{S}} \mathrm{Map}(\mathcal{B})$ agrees with $\mathsf{Span}^{\sim} \mathrm{Map}(\mathcal{B})$, a fact that was also proved in [9]. In particular, the axiom of choice (for Rel) implies that $\mathsf{Rel} \cong \mathsf{Span}^{\sim} \mathsf{Set}$, a description that will be immensely useful for us.

We close the chapter with the dual to $\mathsf{Span}^{\mathcal{S}}\mathcal{C}$, denoted $\mathsf{Cospan}^{\mathcal{W}}\mathcal{C}$, where $\mathcal{W}$ will be referred to as a system of witnesses for reasons that become clear in Chapter 4 (Spoiler: They will witness inequalities in a Frobenius theory).

- Chapter 3 introduces the notion of a monoidal signature Σ and gives the syntactic description of $\mathbb{CB}_\Sigma$, furthermore showing its universal property as the free Cartesian bicategory on Σ. With such a signature we associate a category $\mathcal{C}_\Sigma$ that will be useful for us in Chapter 4 to define hypergraphs as presheaves over it.

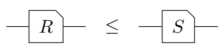
  We then introduce the notion of Frobenius theory as a Cartesian bicategory with natural numbers as objects (a kind of PROP). We can see $\mathbb{CB}_\Sigma$ as a purely syntactic Frobenius theory, one that does not impose any axioms, because $\mathbb{CB}_\Sigma$ is freely generated. This gives the notion of a presentation of a Frobenius theory: Given a signature Σ and a set of inequalities E, we can take $\mathbb{CB}_\Sigma$ modulo the ordering generated from E and obtain a Frobenius theory $\mathbb{CB}_{\Sigma/E}$. Remarkably, all Frobenius theories arise in this way.

  We finish the Chapter by discussing the problem of completeness and showing a larger example of diagrammatic reasoning in action.

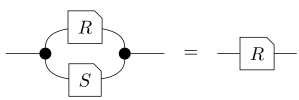
- Chapter 4 will start by taking a closer look at $\mathbb{CB}_{\varnothing}$, the free Cartesian bicategory on empty signature, which is the free Cartesian bicategory on one object. We then prove that $\mathsf{Cospan}^{\sim}\,\mathsf{FinSet}$, where FinSet is (a skeleton of) the category of finite sets has the same universal property, which implies that they are isomorphic. To that end we use a characterisation of FinSet as the free Cocartesian category on one object.

  To extend this to non-empty signature, we introduce the category of Σ-hypergraphs as a presheaf category. Denoting the category of finite hypergraphs as FHyp_{Σ}, we formalise hypergraphs with interfaces as a full subcategory $\mathsf{DiscCospan}^{\sim}\,\mathsf{FHyp}_{\Sigma}$ of $\mathsf{Cospan}^{\sim}\,\mathsf{FHyp}_{\Sigma}$, called the discrete cospans, which are cospans $n \to G \leftarrow m$, where n (respectively m) represents the hypergraph with n (respectively m) vertices without any edges. We then prove the first central result, Theorem 4.2.21 which says that $\mathbb{CB}_{\Sigma} \cong \mathsf{DiscCospan}^{\sim}\,\mathsf{FHyp}_{\Sigma}$ as Cartesian bicategories. Given a morphism $R\colon n \to m$, we denote a corresponding discrete cospan as $n \to \mathcal{G}_R \leftarrow m$ and call $\mathcal{G}_R$ a universal graph of R.

  We can use this combinatorial characterisation to give a similar characterisation for $\mathbb{CB}_{\Sigma/E}$. This introduces an important idea: We identify an inequality

  with the equality

  and the universal graph of the left-hand side, $\mathcal{G}_{R\cap S}$, will turn out to be the result of gluing together $\mathcal{G}_R$ and $\mathcal{G}_S$ along their interfaces. Therefore there is a canonical morphism $\mathcal{G}_R \to \mathcal{G}_{R\cap S}$

that we associate with the inequality $R \leq S$. In such a way, we can translate a set of axioms into a set E of morphisms in FHyp_Σ. It turns out that under this translation, consequences of the axioms exactly correspond to those morphisms that can be generated from E through composition, coproducts and several other categorical constructions. The same constructions appeared in Chapter 2, as the closure properties of systems of witnesses. In fact, this is where systems of witnesses get their name from, because consequences of the axioms are precisely those corresponding to morphisms that are in $\mathcal{W}(E)$, the system of witnesses generated from E. This gives the characterisation of $\mathbb{CB}_{\Sigma/E} \cong \mathsf{DiscCospan}^{\mathcal{W}(E)}\,\mathsf{FHyp}_\Sigma$, which is defined similarly to $\mathsf{DiscCospan}^{\sim}\,\mathsf{FHyp}_\Sigma$ as the full subcategory of $\mathsf{Cospan}^{\mathcal{W}(E)}\,\mathsf{FHyp}_\Sigma$ on discrete cospans.

- In Chapter 5, the translation of axioms of a Frobenius theory into morphisms between hypergraphs opens the door to translating logical properties into categorical ones. The trick here is that any hypergraph G induces a model

  $$\mathcal{M}_G\colon \mathbb{CB}_\Sigma \cong \mathsf{DiscCospan}^{\sim}\,\mathsf{FHyp}_\Sigma \to \mathsf{Span}^{\sim}\,\mathsf{Set} \cong \mathsf{Rel}$$

  through its Hom-functor $\mathrm{Hom}(\_, G)$.

  The Lifting Lemma 5.1.4 establishes an important connection between logical properties of $\mathcal{M}_G$ and categorical lifting properties of G – it is essentially an incarnation of the Yoneda Lemma for Cartesian bicategories. Together with the realisation that all models arise in this way as $\mathcal{M}_G$ for some hypergraph, this allows us to come to a categorical understanding of completeness: A model of $\mathbb{CB}_{\Sigma/E}$ corresponds to a hypergraph that satisfies a lifting property known as E-injectivity. A property shared by all models then corresponds to a morphism that satisfies a lifting property with respect to all E-injective objects – such a morphism is called a cofibration. Since we established in Chapter 4 that all provable consequences of the axioms correspond to morphisms in $\mathcal{W}(E)$, the class of witnesses generated from E, this gives the categorical incarnation of the completeness theorem: Prove that cofibrations are witnesses. At this point, there is no need to restrict our attention to hypergraphs only – this is an interesting categorical question in its own right that deserves general attention. What we do then is establish a sufficient condition for when a cofibration is already a witness, placing some mild assumptions on codomains and domains of morphisms in E and on the category in question.

The proof of the completeness theorem then defines, starting from an object X an E-injective object X'. The trick is to define this X' as a transfinite composition of witnesses. Then, given a cofibration $f\colon X \to Y$, the lifting property makes f factor through X' and a compactness argument then shows that a finite composition suffices. Since the class of witnesses is closed under finite compositions, this makes f itself a witness, which finishes the proof of the categorical incarnation of the completeness theorem.

Relationship of the present thesis to the literature

- Chapter 1: The notion of Cartesian bicategory and results regarding them have first appeared in [18], the major difference being the string diagrammatic language we use here to present them. The notion of $\mathsf{Span}^{\sim}\mathcal{C}$ has appeared before in [25].
- Chapter 3: Frobenius theories and their functorial semantics have been introduced in [7].

Contributions by the author

- Chapter 2: The notion of a Cartesian bicategory with enough maps and the considerations of the axiom of choice in Cartesian bicategories have been published first in [9]. The remainder has not been published before.

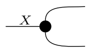
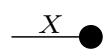
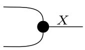
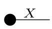
- Chapter 4: The characterisation of $\mathbb{CB}_\Sigma$ via discrete cospans of hypergraphs has been published in [8]. The characterisation of $\mathbb{CB}_{\Sigma/E}$ has not been published before.
- Chapter 5: The completeness result for $\mathbb{CB}_\Sigma$ has been published in [8]. The general completeness theorem is published here for the first time.

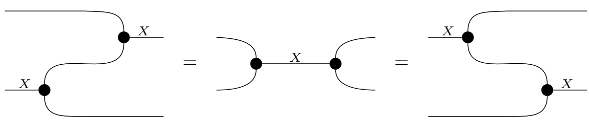
Chapter 1

Cartesian Bicategories

Introduction

Cartesian bicategories were first introduced in [18] as a categorical algebra of relations and as an alternative to Freyd and Scedrov's allegories [24]. RFC Walters had a certain distaste for the approach through allegories; he referred to the modular law of allegories as a *formica mentale*, a "complication which prevents thought" [40].

In addition to copying  and discarding  on every object X, as are available in Cartesian categories, Cartesian bicategories introduce  and , each of which are adjoint to their mirrored version. Together they satisfy the important Frobenius law

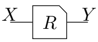
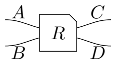


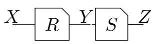
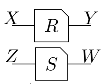
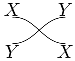
which is ubiquitous in Computer science and has resulted in applications of Cartesian bicategories to the compositional study of various systems, for example in [11].

Outline of the Chapter In Section 1.1 we will state the definition of a Cartesian bicategory and investigate their fundamental properties. The fundamental notion of a map is introduced and further investigated in Section 1.2. There we investigate properties that maps enjoy, introduce the alternative formulation originally used in [18] of maps as right adjoints and prove that maps in a Cartesian bicategory $\mathcal{B}$ form a Cartesian category $\mathrm{Map}(\mathcal{B})$. In Section 1.3 we introduce the $\mathsf{Span}^{\sim}\mathcal{C}$ construction – which was considered in [25], but unfortunately has been largely ignored – that gives an important class of examples of Cartesian bicategories.

1.1 Cartesian bicategories

We will use *string diagrams* as an intuitive graphical notation for what is formally represented as morphisms in a symmetric monoidal category $\mathcal{B}$ with monoidal product $\otimes$ and monoidal unit I, details for this can be found in [39]. Here we would like to give an intuitive way to read string diagrams, one that doesn't require the machinery of category theory.

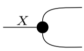
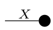
Intuitively, a box  denotes a process R that receives an input of type X and produces output of type Y. We will also write this as $R\colon X \to Y$. It is possible to talk about several inputs and several outputs by stacking wires, for example  receives inputs of type A and B respectively and produces output of type C and D respectively. In other words, a compound type is formed by stacking wires and we will write such a compound type formed from types A and B as $A \otimes B$.

A special process is given by , which is the process that doesn't do anything – the identity on type X. We now have several ways to build larger and more interesting processes: For $R\colon X \to Y$ and $S\colon Y \to Z$, we can form the sequential composition , which in symbols we will denote as $R; S\colon X \to Z$. This is the process that first applies R and then applies S on the result. For $R\colon X \to Y$ and $S\colon Z \to W$, their parallel composition is , in symbols denoted as $R \otimes S\colon X \otimes Z \to Y \otimes W$. This is the process that executes R and S in parallel. It is now possible to iterate these ways of constructing diagrams to form processes of arbitrary complexity. If we want to change the order of inputs or outputs, we can use the symmetries . As a special case, there is also a type I that corresponds to having no wires and an empty diagram, denoted  that represents the process of doing nothing to no input and obtaining no output.

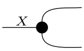
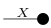
The formal theory of symmetric monoidal categories ensures that we do not need to worry about how our diagrams are constructed. If two diagrams have the same connectivity, they represent the same process. This allows us to use diagrammatic reasoning, that is a formal manipulation of diagrams, that now behave like a two-dimensional analogue of the terms used in algebra.

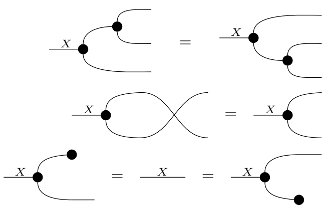
If it is clear from the context how wires are labelled, we will declutter our diagrams and omit labels.

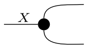
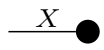
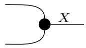
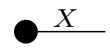
Definition 1.1.1. A Cartesian bicategory is a symmetric monoidal category $(\mathcal{B}, \otimes, I)$ enriched over the category of posets. This means that every Hom-set $\mathrm{Hom}_{\mathcal{B}}(X, Y)$ is equipped with a partial order and that both composition and $\otimes$ are monotonous operations. Every object $X \in \mathcal{B}$ is equipped with morphisms

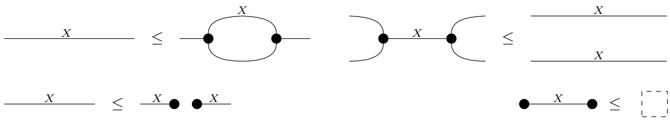
 $: X \to X \otimes X$ and  $: X \to I$

such that

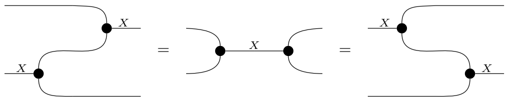
-  and  form a cocommutative comonoid, that is they satisfy



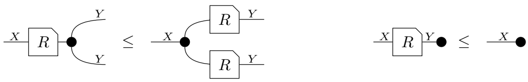
-  and  have right adjoints  and  respectively, that is



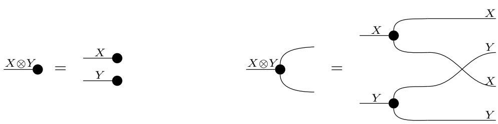
- The Frobenius law holds, that is



- Each morphism $R\colon X \to Y$ is a lax comonoid homomorphism, that is



- The choice of comonoid on every object is coherent with the monoidal structure[1] in the sense that



Remark 1.1.2. Definition 1.1.1 is a slight deviation from the terminology used in [18]. There, the Frobenius law is not part of the definition of a Cartesian bicategory. Instead, Cartesian bicategories that satisfy it are called Cartesian bicategories of relations. Here, we will not consider any Cartesian bicategories that do not satisfy Frobenius, so we simplify the terminology.

Definition 1.1.3. A morphism of Cartesian bicategories is a monoidal functor preserving the ordering and the chosen monoids and comonoids.

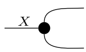
[1] In the original definition of [18] this property is replaced by requiring the uniqueness of the comonoid/monoid. However, as suggested in [36], coherence seems to be the property of primary interest.

The archetypal example of a Cartesian bicategory is the category of sets and relations Rel, with Cartesian product of sets, hereafter denoted by $\times$, as monoidal product and $1 = \{\bullet\}$ as unit I. To be precise, Rel has sets as objects and relations $R \subseteq X \times Y$ as arrows $X \to Y$. Composition and monoidal product are defined as expected:

$$R; S = \{(x, z) \mid \exists y \text{ s.t. } (x, y) \in R \text{ and } (y, z) \in S\},$$

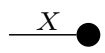
$$R \otimes S = \{\big((x_1, x_2)\,,\,(y_1, y_2)\big) \mid (x_1, y_1) \in R \text{ and } (x_2, y_2) \in S\}.$$

For each set X, the comonoid structure is given by the diagonal function $X \to X \times X$ and the unique function $X \to 1$, considered as relations. That is



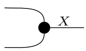
$$= \{\big(x, (x, x)\big) \mid x \in X\}$$

and



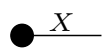
$$= \{(x, \bullet) \mid x \in X\}$$

Their right adjoints are given by their opposite relations:



$$= \{\big((x, x), x\big) \mid x \in X\}$$

and

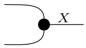
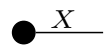


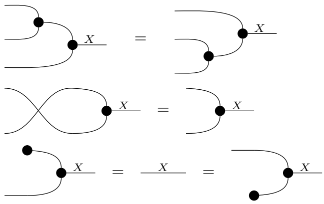
$$= \{(\bullet, x) \mid x \in X\}$$

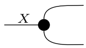
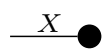
The reader can verify that these are adjoint and that, moreover, the Frobenius law holds. We will investigate the property of every relation being a lax comonoid homomorphism in more detail later. Following the analogy with Rel, we will often call arbitrary morphisms of a Cartesian bicategory relations.

There are many examples of Cartesian bicategories that are somewhat similar to Rel, for instance LinRel, the category of linear relations of vector spaces where the monoidal product is the direct sum of vector spaces, for further details see [12]. Nevertheless, there are examples of Cartesian bicategories that are significantly different, i.e. that are not a form of Rel with additional structure. We will explore some of those examples in Section 1.3.

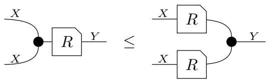
As an immediate consequence of the axioms of Cartesian bicategories one discovers an interesting symmetry; each horizontal reflection of one of the axioms is also a property of Cartesian bicategories.

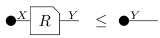
Lemma 1.1.4.  *and*  *form a commutative monoid, that is*



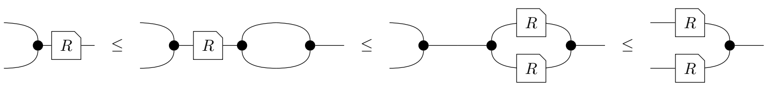
Proof. This follows from  and  being a cocommutative comonoid and the uniqueness of adjoints. □

Lemma 1.1.5. *Every morphism $R\colon X \to Y$ in a Cartesian bicategory is a lax monoid homomorphism, that is*

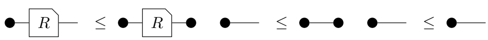




Proof. Both properties follow from the adjunction of the comonoid and the monoid. In detail, we have

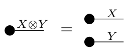


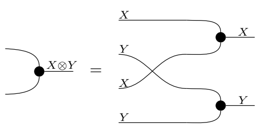
and furthermore



□

Lemma 1.1.6. *The choice of monoid on each object is coherent with the monoidal structure, that is*

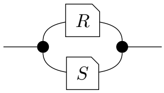




Proof. This follows directly from the coherence of the comonoid and uniqueness of adjoints. $\square$

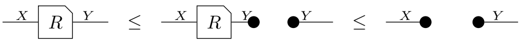
One of the fundamental properties of Cartesian bicategories that follows from the existence of the monoid and comonoid on every object is that every local poset $\mathrm{Hom}_{\mathcal{B}}(X, Y)$ allows to take the intersection of relations and has a top element.

Lemma 1.1.7. *Let $\mathcal{B}$ be a Cartesian bicategory and $X, Y \in \mathcal{B}$. The poset $\mathrm{Hom}(X, Y)$ has a top element given by*  *and the meet of relations $R, S\colon X \to Y$ is given by*

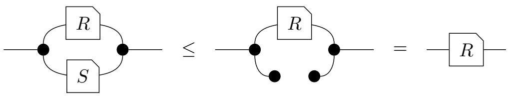


We will also write this meet as $R \cap S$.

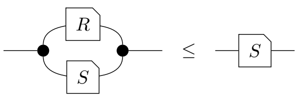
Proof.
- For any relation $R\colon X \to Y$ we have



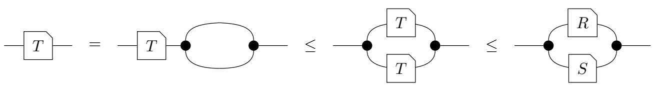
- Let $R, S\colon X \to Y$. Then we have

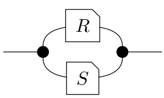


and in the same way



If furthermore $T \leq R$ and $T \leq S$, then we have

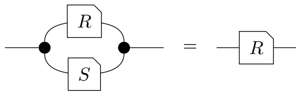


and therefore  is the meet of R and S.

$\square$

The existence of meets allows us to characterise inequalities through equalities. It is generally true in a poset with meets that $x \leq y$ if and only if $x \wedge y = x$, where $\wedge$ denotes the meet. This can for example be found in [22].

Proposition 1.1.8. *Let $\mathcal{B}$ be a Cartesian bicategory and $R, S \colon X \to Y$ morphisms. We have $R \leq S$ if and only if*



Lemma 1.1.9. *A morphism $F \colon \mathcal{B} \to \mathcal{B}'$ of Cartesian bicategories is faithful, in the sense of reflecting equality between morphisms, if and only if it reflects the ordering.*

Proof. A morphism that reflects the ordering is faithful because the ordering is reflexive and antisymmetric. Conversely a faithful morphism reflects the ordering by Proposition 1.1.8. $\square$

To appreciate the property that every morphism in a Cartesian bicategory is a lax comonoid homomorphism, it is useful to spell out its meaning in Rel: in the first inequality, the left and the right-hand side are, respectively, the relations

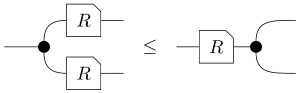
$$\{\big(x, (y, y)\big) \mid (x, y) \in R\} \quad \text{and} \quad \{\big(x, (y, z)\big) \mid (x, y) \in R \text{ and } (x, z) \in R\}, \tag{1.1.1}$$

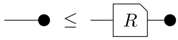
while in the second inequality, they are the relations

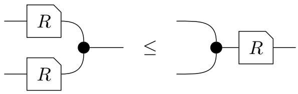
$$\{(x, \bullet) \mid \exists y \in Y \text{ s.t. } (x, y) \in R\} \quad \text{and} \quad \{(x, \bullet) \mid x \in X\}. \tag{1.1.2}$$

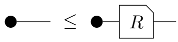
It is immediate to see that the two left-to-right inclusions hold for any relation $R \subseteq X \times Y$, while the right-to-left inclusions hold exactly when R is a function: a relation which is single valued and total. This observation justifies the following definition.

Definition 1.1.10. Let R be a morphism in a Cartesian bicategory. We call R

single valued if 

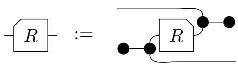
total if 

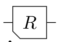
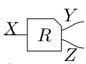
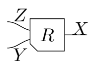
injective if 

surjective if 

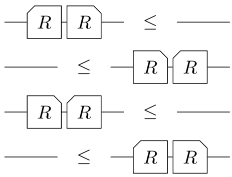
By translating the last two inequalities in Rel, similarly to what we have shown in (1.1.1) and (1.1.2), the reader can immediately check that these correspond to the usual properties of injectivity and surjectivity for relations. Moreover, since the converses of these inequalities hold in Cartesian bicategories, the four inequalities are actually equalities.

We can characterise all the notions of Definition 1.1.10 equivalently in terms of *opposite morphisms* R^{op} which are defined for any morphism R as follows:



Remark 1.1.11. It would be justified to denote the opposite morphism as  and it also might seem to align better with the graphical intuition. However, matters become subtle when considering multiple wires. Rotating the box  would give , where $Y \otimes Z$ and $Z \otimes Y$ are different, though isomorphic, types. For our purpose it is appropriate to have the opposite morphism go in the opposite direction, without mediation of a reflection. But it is interesting that the graphical calculus would be capable of more, even though we do not use these capabilities here.

Proposition 1.1.12. *Let R be a morphism in a Cartesian bicategory.*



iff R is single valued.

iff R is total.

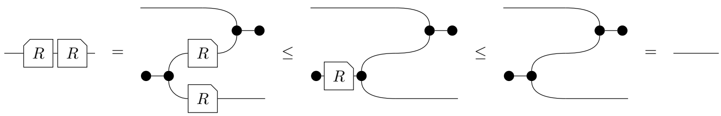
iff R is injective.

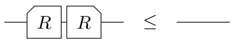
iff R is surjective.

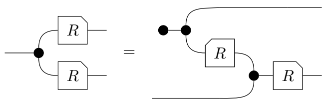
In particular, R is surjective iff R^{op} is total and R is injective iff R^{op} is single valued.

Proof. We show the proofs for single valued and total. The proofs for injectivity and surjectivity are analogous. The last statement follows from the others and the fact that $(R^{\mathrm{op}})^{\mathrm{op}} = R$

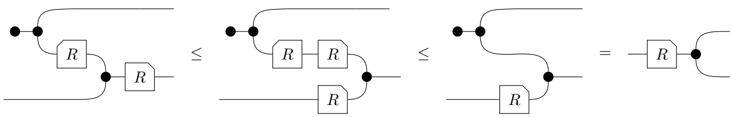
- Let R be single valued. Then



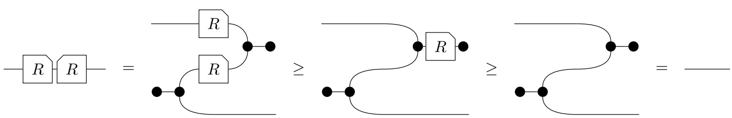
Conversely, if  , then by the Frobenius law one gets

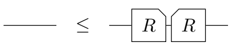


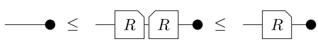
and from there



- Let R be total. Then



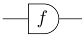
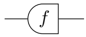
Conversely, if  , then



$\square$

1.2 Maps

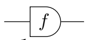
Definition 1.2.1. A map in a Cartesian bicategory is a relation f that is a comonoid homomorphism, i.e. is single valued and total.

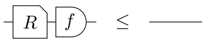
We will write  to denote a map f and  for its opposite, which we will also refer to as a comap. Note that we use lower-case letters for maps and upper-case letters for arbitrary morphisms.

Example 1.2.2. Let us consider some examples of maps.

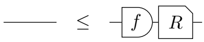
- Maps in Rel are exactly (graphs of) functions.
- In LinRel, maps are exactly linear maps between vector spaces.

The original treatment of Cartesian bicategories in [18] introduces maps as those morphisms that admit a right adjoint. We show below that this amounts to the same notion.

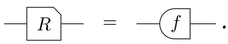
Proposition 1.2.3. *A morphism*  *is a map if and only if it has a right adjoint – a morphism R such that*

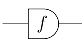
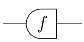


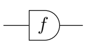
and

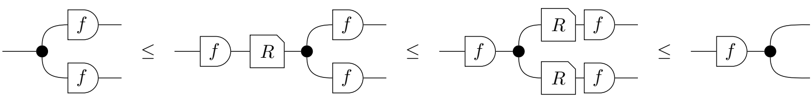


In that case, necessarily

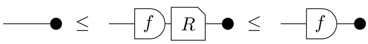


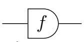
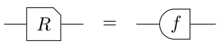
Proof. If  is a map, then  is a right adjoint by Proposition 1.1.12.

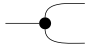
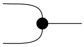
On the other hand, if  has a right adjoint R, then it is a map since

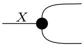
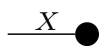


and



Therefore,  is indeed a map and  by uniqueness of adjoints. □

Since by definition of Cartesian bicategories  has right adjoint  and  has right adjoint , we get that they are in fact maps.

Proposition 1.2.4. *Let $\mathcal{B}$ be a Cartesian bicategory and $X \in \mathcal{B}$ an object. Then*  *and*  *are maps.*

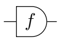
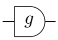
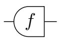
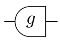
The identity is a map, and maps are easily shown to be closed under composition, so they constitute a category.

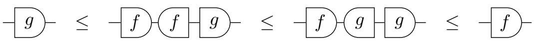
Definition 1.2.5. Given a Cartesian bicategory $\mathcal{B}$, we define its category of maps, $\mathrm{Map}(\mathcal{B})$ to have the same objects as $\mathcal{B}$ and as morphism the maps of $\mathcal{B}$. Dually, we define its category of comaps, $\mathrm{Comap}(\mathcal{B})$ to have the same objects as $\mathcal{B}$ and as morphisms the comaps of $\mathcal{B}$.

Remark 1.2.6. Clearly $\mathrm{Comap}(\mathcal{B}) \cong \mathrm{Map}(\mathcal{B})^{\mathrm{op}}$, by taking a map to its opposite.

By the following proposition, the ordering of $\mathcal{B}$ becomes trivial when restricted to maps.

Proposition 1.2.7. *Let f, g be maps such that $f \leq g$. Then $f = g$.*

Proof. Since  $\leq$ , also  $\leq$ . Therefore



$\square$

By analogy with Rel, we think of the comonoid on any object in a Cartesian bicategory as giving a way to copy and discard. By Definition, maps respect these operations. Therefore, $\mathrm{Map}(\mathcal{B})$, which inherits the monoidal product from $\mathcal{B}$, has what in [26] is called uniform copying and deleting. This has the consequence that $\otimes$ induces a categorical product on $\mathrm{Map}(\mathcal{B})$.

Lemma 1.2.8. *For a Cartesian bicategory $\mathcal{B}$, the monoidal product $\otimes$ induces a product on $\mathrm{Map}(\mathcal{B})$. The monoidal unit I becomes a terminal object in $\mathrm{Map}(\mathcal{B})$. In other words, $(\mathrm{Map}(\mathcal{B}), \otimes, I)$ is a Cartesian category.*

Proof. Follows from Theorem 6.13 in [26]. $\square$

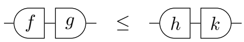
Since it is useful, we will also state the dual of this lemma.

Lemma 1.2.9. *For a Cartesian bicategory $\mathcal{B}$, the monoidal product $\otimes$ induces a coproduct on* $\mathrm{Comap}(\mathcal{B})$*. The monoidal unit I becomes an initial object in* $\mathrm{Comap}(\mathcal{B})$*, so* $(\mathrm{Comap}(\mathcal{B}), \otimes, I)$ *is Cocartesian category.*

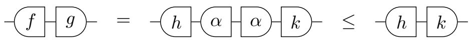
Interestingly, commutative diagrams of maps give rise to inequalities in a straightforward manner.

Lemma 1.2.10. *Let $\mathcal{B}$ be a Cartesian bicategory and*

$$\begin{tikzcd}
& A \arrow[dl, "f"'] \arrow[dr, "g"] \arrow[dd, "\alpha"] & \\
B & & C \\
& D \arrow[ul, "h"] \arrow[ur, "k"'] &
\end{tikzcd}$$

a commutative diagram of maps. Then  .

Proof.



□

Lemma 1.2.10 motivates us to synthesise Cartesian bicategories where all inequalities arise in this way. This will be the driving force in Definition 1.3.3.

1.3 Spans

Definition 1.3.1 (Span)**.** Let $\mathcal{C}$ be a finitely complete category. A *span* from X to Y is a pair of arrows $X \leftarrow A \rightarrow Y$ in $\mathcal{C}$.

A morphism $\alpha\colon (X \leftarrow A \rightarrow Y) \Rightarrow (X \leftarrow B \rightarrow Y)$ is an arrow $\alpha\colon A \rightarrow B$ in $\mathcal{C}$ s.t. the following diagram commutes:

$$\begin{tikzcd}
& A \arrow[dl] \arrow[dr] \arrow[dd, "\alpha"] & \\
X & & Y \\
& B \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

Two spans $X \leftarrow A \rightarrow Y$ and $X \leftarrow B \rightarrow Y$ are *isomorphic* if α is an isomorphism. For $X \in \mathcal{C}$, the *identity span* is $X \xleftarrow{\mathrm{id}_X} X \xrightarrow{\mathrm{id}_X} X$. The composition of spans $X \leftarrow A \xrightarrow{f} Y$ and $Y \xleftarrow{g} B \rightarrow Z$ is $X \leftarrow A \times_{f,g} B \rightarrow Z$, obtained by taking the pullback $A \times_{f,g} B$ of f and g. This data defines the bicategory [6] $\mathsf{Span}\,\mathcal{C}$: the objects are those of $\mathcal{C}$, the arrows are spans and 2-cells are homomorphisms. Finally, $\mathsf{Span}\,\mathcal{C}$ has monoidal product given by the product in $\mathcal{C}$, with unit the final object $1 \in \mathcal{C}$.

To avoid the complications that come with bicategories, such as composition being associative only up to isomorphism, it is common to consider a *category* of spans, where isomorphic spans are equated: let $\mathsf{Span}^{\leq}\mathcal{C}$ be the monoidal category that has isomorphism classes of spans as arrows. Note that, when going from bicategory to category, after identifying isomorphic arrows it is usual to simply discard the 2-cells. Differently, we consider $\mathsf{Span}^{\leq}\mathcal{C}$ to be locally preordered with $(X \leftarrow A \rightarrow Y) \leq (X \leftarrow B \rightarrow Y)$ if there exists a morphism $\alpha\colon (X \leftarrow A \rightarrow Y) \Rightarrow (X \leftarrow B \rightarrow Y)$. It is an easy exercise to verify that this (pre)ordering is well-defined and compatible with composition and monoidal product. Note that, in general, $\leq$ is a genuine preorder: i.e. it is possible that $(X \rightarrow A \leftarrow Y) \leq (X \rightarrow B \leftarrow Y) \leq (X \rightarrow A \leftarrow Y)$ without the spans being isomorphic.

Since $\mathsf{Span}^{\leq}\mathcal{C}$ is preorder enriched, rather than poset enriched, it is *not* a Cartesian Bicategory. However, one can transform a preorder enriched category into a poset enriched one with a simple construction: for $\mathsf{Span}^{\leq}\mathcal{C}$, one first defines $\sim \; = \; \leq \cap \geq$, namely $(X \leftarrow A \rightarrow Y) \sim (X \leftarrow B \rightarrow Y)$ iff there exists $\alpha\colon (X \leftarrow A \rightarrow Y) \Rightarrow (X \leftarrow B \rightarrow Y)$ and $\beta\colon (X \leftarrow B \rightarrow Y) \Rightarrow (X \leftarrow A \rightarrow Y)$, and then one takes equivalence classes of morphisms of $\mathsf{Span}^{\leq}\mathcal{C}$ modulo $\sim$. It is worth observing that pullbacks are no longer necessary to compose $\sim$-equivalence classes of spans: weak pullbacks are sufficient, since non-isomorphic weak pullbacks of the same diagram all belong to the same $\sim$-equivalence class.

Definition 1.3.2. A diagram

$$\begin{tikzcd} P \arrow[r] \arrow[d] & A \arrow[d] \\ B \arrow[r] & C \end{tikzcd}$$

in a category is called a *weak pullback* diagram – and P is called a *weak pullback* – if for any commutative diagram

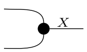
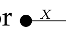
$$\begin{tikzcd} T \arrow[dr, dashed] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\ & P \arrow[r] \arrow[d] & A \arrow[d] \\ & B \arrow[r] & C \end{tikzcd}$$

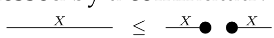
of solid arrows there exists a (not necessarily unique) dashed arrow.

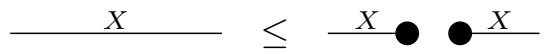
Definition 1.3.3 ($\mathsf{Span}^{\sim}$)**.** Let $\mathcal{C}$ be a category with finite products and weak pullbacks. The posetal category $\mathsf{Span}^{\sim}\mathcal{C}$ has the same objects as $\mathcal{C}$ and as morphisms $\sim$-equivalence classes of spans. The order is defined as in $\mathsf{Span}^{\leq}\mathcal{C}$. Composition is given by weak pullbacks in $\mathcal{C}$. Identities, monoidal product and unit are as in $\mathsf{Span}\,\mathcal{C}$.

Proposition 1.3.4. *Let $\mathcal{C}$ be a category with finite products and weak pullbacks. Then $\mathsf{Span}^{\sim}\mathcal{C}$ is a Cartesian bicategory.*

Proof. It is straightforward to check that the identity span is the identity for composition via weak pullbacks. Associativity of composition is surprisingly almost the same as associativity of composition in $\mathsf{Span}\,\mathcal{C}$. Associativity in the latter is proved using the universal property of pullbacks, therefore showing that both ways of composing amount to isomorphic spans. In the same way, for weak pullbacks the two composites satisfy the same weak universal property, thus the two composites agree up to mutual morphisms. In other words, they are $\sim$-equivalent spans and therefore the same morphism in $\mathsf{Span}^{\sim}\mathcal{C}$.

For  we take the span $X \times X \leftarrow X \rightarrow X$ and for  we take $1 \leftarrow X \rightarrow X$.

With this information, one has only to check that the inequalities in Definition 1.1.1 hold: each of them is witnessed by a commutative diagrams in $\mathcal{C}$. As an example, we illustrate  .

The left hand side is the span $X \xleftarrow{\mathrm{id}_X} X \xrightarrow{\mathrm{id}_X} X$. The right hand side is the composition of $X \xleftarrow{\mathrm{id}_X} X \xrightarrow{!} 1$ and $1 \xleftarrow{!} X \xrightarrow{\mathrm{id}_X} X$. Since the product $X \xleftarrow{\pi_1} X \times X \xrightarrow{\pi_2} X$ is a pullback of $X \xrightarrow{!} 1 \xleftarrow{!} X$, the composition turns out to be exactly the span $X \xleftarrow{\pi_1} X \times X \xrightarrow{\pi_2} X$. Now the diagonal $\Delta\colon X \to X \times X$ makes the following diagram in $\mathcal{C}$ commute. Therefore Δ witnesses the inequality .

$$\begin{tikzcd}
 & X \arrow[dl, "\mathrm{id}_X"'] \arrow[dr, "\mathrm{id}_X"] \arrow[dd, "\Delta"] & \\
X & & X \\
 & X \times X \arrow[ul, "\pi_1"] \arrow[ur, "\pi_2"'] &
\end{tikzcd}$$

$\square$

Example 1.3.5.

- We will show in Corollary 2.2.10 that $\mathsf{Span}^{\sim}\,\mathsf{Set} \cong \mathsf{Rel}$.
- Similarly, for Vect the category of vector spaces, the Cartesian bicategory of linear relations is given as $\mathsf{LinRel} = \mathsf{Span}^{\sim}\mathsf{Vect}$.

The dual of spans, called cospans, are similarly useful for us, so we can make similar definitions in that case.

Definition 1.3.6. Let $\mathcal{C}$ be a category with finite coproducts and weak pushouts. Then we define

$$\mathsf{Cospan}^{\sim}\mathcal{C} = \mathsf{Span}^{\sim}\mathcal{C}^{\mathrm{op}}$$

Explicitly, $\mathsf{Cospan}^{\sim}\mathcal{C}$ has the same objects as $\mathcal{C}$ and a morphism $X \to Y$ is a cospan $X \to A \leftarrow Y$. We have $X \to A \leftarrow Y \leq X \to B \leftarrow Y$ if and only if there is a morphism *in the opposite direction*, i.e. a morphism $\alpha\colon B \to A$ such that

$$\begin{tikzcd}
 & B \arrow[dd, "\alpha"] & \\
X \arrow[ur] \arrow[dr] & & m \arrow[ul] \arrow[dl] \\
 & A &
\end{tikzcd}$$

Chapter 2

Spans and the axiom of choice

Introduction

Maps are a very important class of morphisms in a Cartesian bicategory. As we have seen, for the category of relations, the maps are just functions. This example is an important one, because functions uniquely determine relations – it is possible to reconstruct Rel from its maps. The problem of how to reconstruct Cartesian bicategories has been studied in [18] and a sufficient condition for this, called functional completeness, was established along with a recipe for how to do the reconstruction. We want to continue and expand those results here. We extend the class of functionally complete Cartesian bicategories to the more general notion of a tame Cartesian bicategory and prove more generally that every tame Cartesian bicategory can be reconstructed from its category of maps via Theorem 2.1.18.

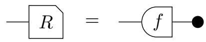
In more detail, it will turn out that the class of surjective maps plays an important role in this reconstruction. We use this observation to synthesise the definition of a category equipped with a notion of *cover* in such a way that maps in a Cartesian bicategory equipped with surjective maps are an example. We can then construct a generalisation of $\mathsf{Span}^{\sim}\mathcal{C}$, written $\mathsf{Span}^{\mathcal{S}}\mathcal{C}$, which allows to specify a class of covers as additional parameter. A Cartesian bicategory $\mathcal{B}$ can then be reconstructed as $\mathsf{Span}^{\mathcal{S}}\,\mathrm{Map}(\mathcal{B})$ where $\mathcal{S}$ is the class of surjective maps.

We use this to give a new perspective on the study of Cartesian bicategories that satisfy the axiom of choice, which we started in [9]. We show that Cartesian bicategories with choice are often tame – under rather mild conditions – and that for a tame Cartesian bicategory to satisfy the axiom of choice is equivalent to surjective maps splitting, which means that $\mathcal{S}$ above coincides with the class of split epis. We then prove that in the special case of $\mathcal{S}$ consisting of split epis, the general $\mathsf{Span}^{\mathcal{S}}\mathcal{C}$ reduces to $\mathsf{Span}^{\sim}\mathcal{C}$ again. This allows to reproduce the classification of Cartesian bicategories with choice that was Theorem 30 in [9].

Outline of the chapter In Section 2.1, we will start by considering those Cartesian bicategories that have enough maps, a property that implies that every relation can be expressed as a span of maps. It turns out that for Rel, weak pullbacks of its maps – functions – play an important role which motivates us to introduce the notion of a tame Cartesian bicategory. For such a tame Cartesian bicategory, the ordering is uniquely determined by the surjective maps, which satisfy a number of closure properties such as closure under composition, products and weak pullbacks. We synthesise the situation, by introducing the notion of a category with covers, a category with finite products and weak pullbacks equipped with a class of morphisms with the same closure properties. Using this we define $\mathsf{Span}^{\mathcal{S}}\mathcal{C}$ as a generalisation of $\mathsf{Span}^{\sim}\mathcal{C}$, where the latter is the special case of $\mathsf{Span}^{\mathcal{S}}\mathcal{C}$ for $\mathcal{S}$ the class of split epis. This allows to reconstruct a tame Cartesian bicategory $\mathcal{B}$ from its category of maps as $\mathsf{Span}^{\mathcal{S}}\,\mathrm{Map}(\mathcal{B})$. Section 2.2 applies this to Cartesian bicategories that satisfy the axiom of choice as also considered in [9]. Lastly, Section 2.3 develops the dual of the theory of categories with covers, called categories with witnesses, the name stemming from their property of witnessing inequalities in Chapter 4. Here we also introduce some technicalities that will be useful later on.

2.1 Cartesian bicategories and maps

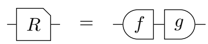
Definition 2.1.1. We say a Cartesian bicategory *has enough maps* if for every morphism $R\colon X \to I$ there is a map $f\colon Z \to X$ such that



Remark 2.1.2. A similar notion to Definition 2.1.1 is considered in [18] under the name of functional completeness. The important difference is that f is required to be injective. It is therefore true that any functionally complete Cartesian bicategory has enough maps, but the converse is false. In other words, having enough maps is a generalisation of functional completeness. All categories of the form $\mathsf{Span}^{\sim}\mathcal{C}$ have enough maps, but for example the category $\mathsf{Cospan}^{\sim}\,\mathsf{FinSet}$ of cospans of finite sets is not functionally complete. This category will play an important role in Section 4.1 as the free Cartesian bicategory on one object.

Having enough maps has the important consequence that every morphism can be written in terms of maps as follows:

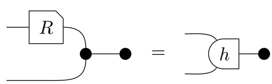
Lemma 2.1.3. *A Cartesian bicategory has enough maps if and only if for every morphism $R\colon X \to Y$, there are maps $f\colon Z \to X$ and $g\colon Z \to Y$ such that*

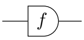
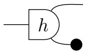
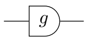
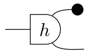


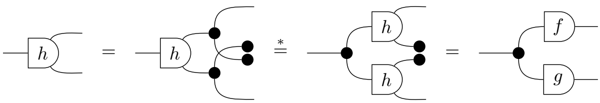
We call this a comap-map factorisation *of R.*

Proof.

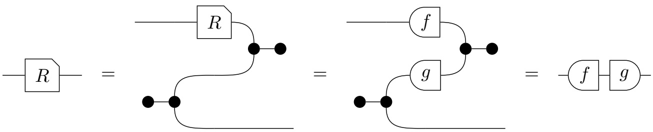
- If a Cartesian bicategory has comap-map factorisations it has enough maps, because that is what the condition specialises to in the case of $Y = I$.
- Conversely, if there are enough maps, there is a map $h\colon Z \to X \otimes Y$ such that



Let  =  and  = , the projection of h onto X and Y respectively. Because $\mathrm{Map}(\mathcal{B})$ is a Cartesian category by Lemma 1.2.8, h can be reconstructed as $\langle f, g\rangle$. In the following we show the explicit computation, because it is very instructive:



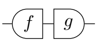
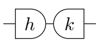
where the step marked $*$ uses coherence of the comonoid and the fact that h is a map. Therefore we have



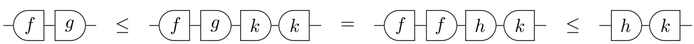
$\square$

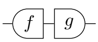
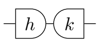
Lemma 2.1.4. *Let $\mathcal{B}$ be a Cartesian bicategory and consider the following diagram in* $\mathrm{Map}(\mathcal{B})$.

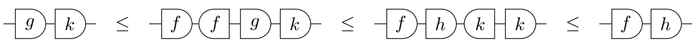
$$\begin{tikzcd} & A \arrow[dl, "f"'] \arrow[dr, "g"] & \\ B \arrow[dr, "h"'] & & C \arrow[dl, "k"] \\ & D & \end{tikzcd} \tag{2.1.1}$$

Then  $\leq$  *if and only if the diagram commutes.*

Proof. If the diagram commutes, then by general properties of adjoints we have



Conversely, if  $\leq$ , then



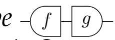
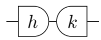
and since both sides are maps, they are equal by Proposition 1.2.7.

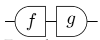
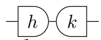
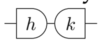
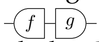
$\square$

That means, the category of maps can “see” inequalities like the one in Lemma 2.1.4 as commutative squares. We are interested in those Cartesian bicategories where maps can furthermore identify equalities. In Rel, equalities correspond to weak pullbacks of functions, an observation going back to [5].

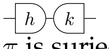
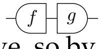
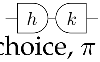
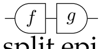
Proposition 2.1.5. *Let*

$$\begin{tikzcd} & A \arrow[dl, "f"'] \arrow[dr, "g"] & \\ B \arrow[dr, "h"'] & & C \arrow[dl, "k"] \\ & D & \end{tikzcd}$$

be a commutative diagram in Set. *Then we have*  $=$  *in* Rel *if and only if the diagram is a weak pullback in* Set.

Proof. As Lemma 2.1.4 shows,  $\leq$  comes from the commutativity of the diagram. Let's unpack the reverse inequality in Rel: If  $=$ , this means that whenever $h(b) = k(c)$, there is an $a \in A$ such that $b = f(a)$ and $c = g(a)$. Now, f, g induce a morphism $\pi\colon A \to P$ where P is the pullback in Set of

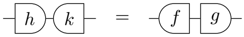
$$\begin{tikzcd} & P \arrow[dl] \arrow[dr] & \\ B \arrow[dr, "h"'] & & C \arrow[dl, "k"] \\ & D & \end{tikzcd}$$

By Proposition A.2.2, the diagram is a weak pullback if and only if π is a split epi. The pullback P in Set is the subset of $B \times C$ of those (b, c) that satisfy $h(b) = k(c)$. We have $\pi(a) = (f(a), g(a))$. Therefore, if A is a weak pullback, π is a split epi and therefore surjective and hence  $=$ . Conversely,  $=$  implies that π is surjective, so by the axiom of choice, π is a split epi, hence A is a weak pullback. $\square$

We will now restrict our attention to those Cartesian bicategories that have an interplay between relations and maps that is similar to that of Rel.

Definition 2.1.6. Call a Cartesian bicategory $\mathcal{B}$ *tame* if it satisfies the following two conditions:

- $\mathcal{B}$ has enough maps.

- We have  if and only if the diagram

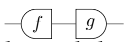
$$\begin{tikzcd}[row sep=small, column sep=small] & A \arrow[dl, "f"'] \arrow[dr, "g"] & \\ B \arrow[dr, "h"'] & & C \arrow[dl, "k"] \\ & D & \end{tikzcd}$$

  is a weak pullback diagram in $\mathrm{Map}(\mathcal{B})$.

Remark 2.1.7. We have already seen that functionally complete Cartesian bicategories have enough maps. It furthermore follows from Corollary (3.4) in [18] that every functionally complete Cartesian bicategory is an example of a tame one.

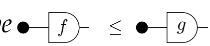
Now we want to characterise the ordering in a tame Cartesian bicategory.

Proposition 2.1.8. *For a tame Cartesian bicategory $\mathcal{B}$, there is a morphism of Cartesian bicategories $F\colon \mathsf{Span}^{\sim}\,\mathrm{Map}(\mathcal{B}) \to \mathcal{B}$. This F is identity on objects and full.*

Proof. Let $F(X \xleftarrow{f} A \xrightarrow{g} B) =$ . Since $\mathcal{B}$ is tame, $\mathrm{Map}(\mathcal{B})$ has well-behaved weak pullbacks and therefore F preserves composition. It is furthermore monoidal because products in $\mathrm{Map}(\mathcal{B})$ are induced by the monoidal product in $\mathcal{B}$ and it preserves the ordering by Lemma 1.2.10.

□

The only thing that prevents $F\colon \mathsf{Span}^{\sim}\,\mathrm{Map}(\mathcal{B}) \to \mathcal{B}$ from being an isomorphism is that it is in general not faithful. The reason for that is that the ordering in $\mathcal{B}$ might consist of more than just the inequalities that are mediated by morphisms as in Lemma 1.2.10. We therefore need to characterise the inequalities in $\mathcal{B}$ that we are currently overlooking.

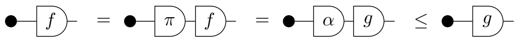
Lemma 2.1.9. *In a tame Cartesian bicategory $\mathcal{B}$ we have* 

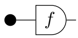
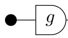
if and only if there is a commutative diagram

$$\begin{tikzcd}
& A \arrow[dl, "f"'] \\
X & P \arrow[l] \arrow[u, "\pi"'] \arrow[d, "\alpha"] \\
& B \arrow[ul, "g"]
\end{tikzcd}$$

with π a surjective map.

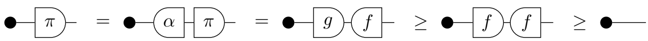
Proof. Assume we have α and π such that $\pi \ ; \ f = \alpha \ ; \ g$ with π surjective. Then



Conversely, if  $\leq$ , let

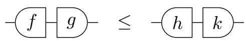
$$\begin{tikzcd}
& P \arrow[dl, "\alpha"'] \arrow[dr, "\pi"] & \\
A \arrow[dr, "g"'] & & B \arrow[dl, "f"] \\
& X &
\end{tikzcd}$$

be a weak pullback. This makes the required diagram commute and π is surjective, because



$\square$

Corollary 2.1.10. *In a tame Cartesian bicategory we have*

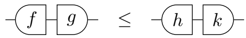


if and only if there is a commutative diagram of maps

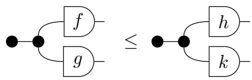
$$\begin{tikzcd}
& A \arrow[dl, "f"'] \arrow[dr, "g"] & \\
B & A' \arrow[u, "\pi"'] \arrow[d, "\alpha"] & C \\
& D \arrow[ul, "h"] \arrow[ur, "k"'] &
\end{tikzcd}$$

with π surjective.

Proof. This follows from Lemma 2.1.9 by bending around the wires. More precisely,



is equivalent to



and applying Lemma 2.1.9 to the latter gives an equivalent diagram of the form

$$\begin{tikzcd} & A \arrow[dl] \\ B \times C & A' \arrow[l] \arrow[u, "\pi"'] \arrow[d, "\alpha"] \\ & D \arrow[ul] \end{tikzcd}$$

By unpacking the universal property of the product, this is easily seen to be equivalent to the diagram above. ☐

Therefore, a tame Cartesian bicategory is uniquely determined by its category of maps and the knowledge of which maps are surjective. The class of surjective maps has a number of interesting properties:

Proposition 2.1.11. *Let $\mathcal{B}$ be a tame Cartesian bicategory. Let $\mathcal{S}$ be the class of surjective maps in $\mathcal{B}$.*

- *$\mathcal{S}$ is closed under composition.*
- *$\mathcal{S}$ is closed under weak pullback.*
- *$\mathcal{S}$ is closed under binary products.*
- *If $f \,;\, \pi \in \mathcal{S}$ for a map f, then $\pi \in \mathcal{S}$.*

In fact, these properties characterise categories of maps and the classes of surjectives that can arise in a tame Cartesian bicategory. We therefore synthesise this definition and then prove a correspondence.

Definition 2.1.12. Let $\mathcal{C}$ be a category with finite products and weak pullbacks. A class of morphisms $\mathcal{S}$ in $\mathcal{C}$ is called a system of covers if

- $\mathcal{S}$ contains identities.
- $\mathcal{S}$ is closed under composition.

- $\mathcal{S}$ is closed under weak pullback, more precisely if $\pi \in \mathcal{S}$ and

$$\begin{tikzcd} & P \arrow[dl, "\pi'"'] \arrow[dr] & \\ A \arrow[dr] & & B \arrow[dl, "\pi"] \\ & X & \end{tikzcd}$$

  is a weak pullback diagram, then $\pi' \in \mathcal{S}$.

- $\mathcal{S}$ is closed under binary products.

- If $f \,;\, \pi \in \mathcal{S}$, then $\pi \in \mathcal{S}$. We call this closure under post-splits.

A pair $(\mathcal{C}, \mathcal{S})$ where $\mathcal{C}$ has finite products and weak pullbacks and $\mathcal{S}$ is a system of covers is called a category with covers. A morphism of categories with covers is a functor that preserves finite products, weak pullbacks and covers.

Example 2.1.13.
- Set equipped with the class of split epis is a category with covers.

- Let R be a ring and $R{-}\mathsf{mod}$ the category of R-modules. Recall that an R-module P is called projective if $\mathrm{Hom}(P, \_)\colon R{-}\mathsf{mod} \to \mathsf{Ab}$ preserves epis. The full subcategory $\mathcal{C}$ of $R{-}\mathsf{mod}$ on projective modules has finite products, given by direct sums of the modules, but in general only weak pullbacks. If for example we take $R = \mathbb{Z}/4$, then the diagram

$$\begin{tikzcd} & \{0,2\} \arrow[dl] \arrow[dr] & \\ 0 \arrow[dr] & & \mathbb{Z}/4 \arrow[dl, "\cdot 2"] \\ & \mathbb{Z}/4 & \end{tikzcd}$$

  is a pullback diagram in $R{-}\mathsf{mod}$, but $\{0, 2\}$ is not a projective module. However, weak pullbacks exist in $\mathcal{C}$ because any projective cover of the pullback in $R{-}\mathsf{mod}$ will do and every R-

module has a projective cover. Hence

$$\begin{tikzcd}
 & \mathbb{Z}/4 \arrow[dl] \arrow[dr, "\cdot 2"] & \\
0 \arrow[dr] & & \mathbb{Z}/4 \arrow[dl, "\cdot 2"] \\
 & \mathbb{Z}/4 &
\end{tikzcd}$$

is an example of a weak pullback in $\mathcal{C}$. It is not a pullback because

$$\begin{tikzcd}
 & \mathbb{Z}/4 \arrow[d, "\cdot(-1)"] \arrow[ddl, bend right] \arrow[ddr, bend left, "\cdot 2"] & \\
 & \mathbb{Z}/4 \arrow[dl] \arrow[dr, "\cdot 2"] & \\
0 \arrow[dr] & & \mathbb{Z}/4 \arrow[dl, "\cdot 2"] \\
 & \mathbb{Z}/4 &
\end{tikzcd}$$

is a commutative diagram, but negation is not the identity.

If we endow both $\mathcal{C}$ and $R-\mathsf{mod}$ with split epis as their systems of covers, then the inclusion functor $\mathcal{C} \to R-\mathsf{mod}$ is a morphism of categories with covers that does not preserve pullbacks.

Proposition 2.1.14. *If $\mathcal{B}$ is a tame Cartesian bicategory then* $\mathrm{Map}(\mathcal{B})$ *is a category with covers in a canonical way, equipped with the class of surjective maps.*

Remark 2.1.15. The need to endow $\mathrm{Map}(\mathcal{B})$ with the additional structure of a category with covers arises from the fact that $\bullet\!\!-\!\!-$ is not a map, therefore $\mathrm{Map}(\mathcal{B})$ cannot internalise the notion of surjective that $\mathcal{B}$ has.

Conversely, given a category with covers $(\mathcal{C}, \mathcal{S})$, we can define a tame Cartesian bicategory from it, in a way that is inspired from Corollary 2.1.10.

The posetal category $\mathsf{Span}^{\mathcal{S}}\mathcal{C}$ is defined in analogy to $\mathsf{Span}^{\sim}\mathcal{C}$, only with a different ordering.

Definition 2.1.16. Let $(\mathcal{C}, \mathcal{S})$ be a category with covers. Define an ordering on spans via $(X \leftarrow A \rightarrow Y) \leq (X \leftarrow B \rightarrow Y)$ if there is a commutative diagram

$$\begin{tikzcd}
& A \arrow[dl] \arrow[dr] & \\
X & A' \arrow[u, "\pi"'] \arrow[d, "\alpha"'] & Y \\
& B \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

with π a cover. We define $\sim$ as the equivalence relation $s_1 \sim s_2$ if $s_1 \leq s_2 \leq s_1$. Now $\mathsf{Span}^{\mathcal{S}}\mathcal{C}$ has the same objects as $\mathcal{C}$ and as morphisms $\sim$-equivalence classes of spans. These equivalence classes are ordered via $\leq$. Composition is given by weak pullbacks in $\mathcal{C}$. Identities, monoidal product and unit are as in $\mathsf{Span}\,\mathcal{C}$.

Lemma 2.1.17. *If $(\mathcal{C}, \mathcal{S})$ is a category with covers, then $\mathsf{Span}^{\mathcal{S}}\mathcal{C}$ is a Cartesian bicategory.*

Proof. Since the ordering in $\mathsf{Span}^{\mathcal{S}}\mathcal{C}$ is coarser than that in $\mathsf{Span}^{\sim}\mathcal{C}$, it suffices to prove that the former is indeed an ordering and compatible with composition and monoidal product of spans. The axioms of Cartesian bicategories then follow from the fact that $\mathsf{Span}^{\sim}\mathcal{C}$ is a Cartesian bicategory.

- Let us first check that the defined ordering is indeed reflexive and transitive. It is reflexive because $\mathcal{S}$ contains identities, it is transitive because $\mathcal{S}$ is closed under weak pullback.

- The ordering is compatible with the monoidal product of spans because the product of covers is a cover. Let us check that it is also compatible with composition. Given spans $A, B \colon X \rightarrow Y$, $C \colon Y \rightarrow Z$ such that $A \leq B$, we will prove $A \mathbin{;} C \leq B \mathbin{;} C$. The proof that given $C \leq D$, also $B \mathbin{;} C \leq B \mathbin{;} D$ is similar.

  So assume $X \leftarrow A \rightarrow Y \leq X \leftarrow B \rightarrow Y$ and $Y \leftarrow C \rightarrow Z$ are given, we want to show that also $A \mathbin{;} C \leq B \mathbin{;} C$. By assumption, we have a commutative diagram

$$\begin{tikzcd}
& A \arrow[dl] \arrow[dr] & \\
X & A' \arrow[l] \arrow[r] \arrow[u, "\pi"'] \arrow[d] & Y \\
& B \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

with $\pi \in \mathcal{S}$. Now form the respective composites with $Y \leftarrow C \rightarrow Z$, so let

$$\begin{tikzcd}
& & P \arrow[dl] \arrow[dr] & & \\
& A \arrow[dl] \arrow[dr] & & C \arrow[dl] \arrow[dr] & \\
X & & Y & & Z
\end{tikzcd}$$

and

$$\begin{tikzcd}
& & Q \arrow[dl] \arrow[dr] & & \\
& B \arrow[dl] \arrow[dr] & & C \arrow[dl] \arrow[dr] & \\
X & & Y & & Z
\end{tikzcd}$$

with weak pullbacks P and Q. We need to construct a diagram

$$\begin{tikzcd}
& P \arrow[dl] \arrow[dr] & \\
X & P' \arrow[l] \arrow[r] \arrow[u, "\pi'"'] \arrow[d, "\alpha"'] & Y \\
& Q \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

Let P' be a weak pullback

$$\begin{array}{ccc}
P' & \xrightarrow{\pi'} & P \\
\downarrow & & \downarrow \\
A' & \xrightarrow{\pi} & A
\end{array}$$

Then $\pi' \in \mathcal{S}$ since covers are closed under weak pullback. This fits into a larger commutative diagram

$$\begin{tikzcd}
P' \arrow[r, "\pi'"] \arrow[d] & P \arrow[r] \arrow[d] & C \arrow[dd] \\
A' \arrow[r, "\pi"] \arrow[d] & A \arrow[dr] & \\
B \arrow[rr] & & Y
\end{tikzcd}$$

so there exists an induced morphism $\alpha\colon P' \to Q$ by the weak universal property of Q. It is straightforward to check that the diagram

$$\begin{tikzcd}
& P \arrow[dl] \arrow[dr] & \\
X & P' \arrow[l] \arrow[r] \arrow[u, "\pi'"] \arrow[d, "\alpha"] & Y \\
& Q \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

commutes.

□

Theorem 2.1.18. *Let $\mathcal{B}$ be a tame Cartesian bicategory. Let $\mathcal{S}$ be the class of surjective maps. Then*

$$\mathsf{Span}^{\mathcal{S}}\,\mathrm{Map}(\mathcal{B}) \cong \mathcal{B}$$

Proof. This follows from Proposition 2.1.8 and Corollary 2.1.10. □

Remark 2.1.19. There is a close connection between categories with covers as defined here and sites as defined in sheaf theory, i.e. categories equipped with a Grothendieck topology [34]. The important difference is that a Grothendieck topology consists of families of morphisms with common codomain, while a cover in our sense is just a single morphism. However, the axioms that we require for covers somewhat mirror the axioms for Grothendieck topologies.

It is tempting to say that $\mathsf{Span}^{\sim}\mathcal{C}$ is $\mathsf{Span}^{\mathcal{S}}\mathcal{C}$ with $\mathcal{S}$ consisting only of identity morphisms. However, identities do not form a system of covers, because $h\ ;\ \pi$ an identity, doesn't necessarily imply π an identity, but a split epi. In fact, split epis are the smallest system of covers possible.

Lemma 2.1.20. *Split epis are a system of covers. They are furthermore the smallest such system in the sense that it is contained in every other system of covers.*

Proof. Let $\pi\colon A \to B$ be a split epi, so there is $f\colon B \to A$ such that $f \mathbin{;} \pi = \mathrm{id}_B$.

- Let $\mathcal{S}$ be any system of covers, then $f \mathbin{;} \pi = \mathrm{id}_B \in \mathcal{S}$ and therefore also $\pi \in \mathcal{S}$.
- It therefore remains to prove that split epis themselves form a system of covers. Almost all axioms are straightforward to check so we will only verify the closure under weak pullbacks here. Let $g\colon X \to B$ be any morphism and P a weak pullback of π and g. We get a commutative diagram

$$\begin{tikzcd}
B \arrow[r, "f"] \arrow[rr, bend left, "\mathrm{id}_B"] & A \arrow[r, "\pi"] & B \\
& P \arrow[u, "\overline{g}"] \arrow[r, "\overline{\pi}"] & X \arrow[u, "g"] \\
X \arrow[uu, "g"] \arrow[ur, dashed, "h"] \arrow[urr, bend right, "\mathrm{id}_X"'] & &
\end{tikzcd}$$

  where h exists by the weak universal property of P, hence $\overline{\pi}$ is a split epimorphism.

□

Lemma 2.1.21. *Let $\mathcal{C}$ be a category with products and weak pullbacks and let $\mathcal{S}$ be the class of split epis. Then*

$$\mathsf{Span}^{\sim}\mathcal{C} \cong \mathsf{Span}^{\mathcal{S}}\mathcal{C}$$

Proof. Since the two categories have the same objects and only differ in the way the ordering is defined, it suffices to prove that the orderings agree. If there is a commutative diagram

$$\begin{tikzcd}
& A \arrow[dl] \arrow[dr] \arrow[dd, "\alpha"] & \\
X & & Y \\
& B \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

we immediately get a diagram

$$\begin{tikzcd}
 & A \arrow[dl] \arrow[dr] & \\
X & A \arrow[l] \arrow[r] \arrow[u, "\mathrm{id}"'] \arrow[d, "\alpha"'] & Y \\
 & B \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

Conversely, given a diagram

$$\begin{tikzcd}
 & A \arrow[dl] \arrow[dr] & \\
X & A' \arrow[l] \arrow[r] \arrow[u, "\pi"'] \arrow[d, "\alpha"'] & Y \\
 & B \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

where $\pi\colon A' \to A$ is a split epi, there is then a $g\colon A \to A'$ such that $g \mathbin{;} \pi = \mathrm{id}_A$. Let $\alpha' = g \mathbin{;} \alpha$. This makes the diagram

$$\begin{tikzcd}
 & A \arrow[dl] \arrow[dr] \arrow[dd, "\alpha'"] & \\
X & & Y \\
 & B \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

commute. $\square$

Example 2.1.22.

1. Any category with finite products and weak pullbacks can be seen as a category with covers by equipping it with the class of split epis, or by equipping it with all morphisms. The former is initial, the latter is final.

2. A regular category becomes a category with covers equipped with its class of regular epis.

2.2 Choice

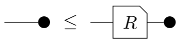
In this section we will show the essence of our paper [9], recast in a different light, using the results from the previous section.

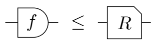
One of the many equivalent formulations of the axiom of choice in set theory is

Every total relation contains a map.

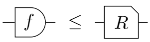
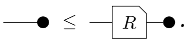
In a total relation every element in the domain is related to at least one element in the codomain. A map is obtained by choosing one element in the codomain for each element in the domain. This can be stated in the language of Cartesian bicategories.

Definition 2.2.1 (Choice)**.** Let $\mathcal{B}$ be a Cartesian bicategory. We say that $\mathcal{B}$ satisfies the axiom of choice (AC), or that $\mathcal{B}$ has choice, if the following holds for any morphism $R\colon X \to Y$:

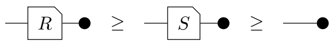
 (R is total) implies

$\exists$ map $f\colon X \to Y$ such that  (AC)

Observe that the converse implication holds in any Cartesian bicategory.

Lemma 2.2.2. *If*  *then* .

Proof. If S is total and $S \leq R$, then R is total:



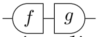
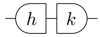
□

Example 2.2.3. The usual axiom of choice implies that Rel satisfies (AC).

Choice has the interesting consequence that for every inequality between spans there exists a map to fill the square.

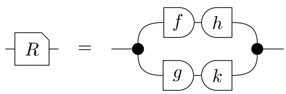
Lemma 2.2.4. *Let $\mathcal{B}$ be a Cartesian bicategory with choice and*

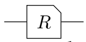
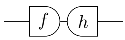
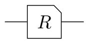
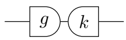
$$\begin{tikzcd} & A \arrow[dl, "f"'] \arrow[dr, "g"] & \\ B & & C \\ & D \arrow[ul, "h"] \arrow[ur, "k"'] & \end{tikzcd}$$

a diagram of maps such that  $\leq$ *. Then there is a map* $\omega\colon A \to D$ *such that the following diagram commutes.*

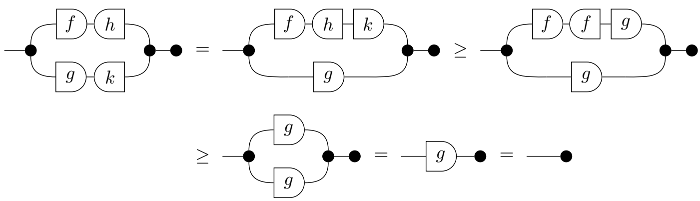
$$\begin{tikzcd}
 & A \arrow[dl, "f"'] \arrow[dr, "g"] \arrow[dd, "\omega"] & \\
B & & C \\
 & D \arrow[ul, "h"] \arrow[ur, "k"'] &
\end{tikzcd}$$

Proof. Consider $R\colon A \to D$ given by

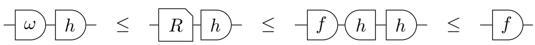


hence  $\leq$  and  $\leq$ .

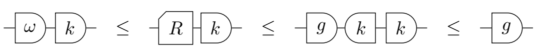
R is total, since



so by the axiom of choice, there is a map $\omega \leq R$. This satisfies



and



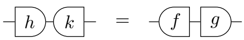
and since both side are maps we have equality by Proposition 1.2.7. $\square$

Remark 2.2.5. The converse of Lemma 2.2.4 holds in any Cartesian bicategory by Lemma 1.2.10.

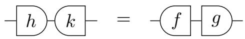
Lemma 2.2.6. *Let $\mathcal{B}$ be a Cartesian bicategory with enough maps and choice. Then $\mathcal{B}$ is tame.*

Proof. Consider the following commutative diagram in $\mathrm{Map}(\mathcal{B})$:

$$
\begin{tikzcd}
 & A \arrow[dl, "f"'] \arrow[dr, "g"] & \\
B \arrow[dr, "h"'] & & C \arrow[dl, "k"] \\
 & D &
\end{tikzcd}
\tag{2.2.1}
$$

We need to prove that  if and only if (2.2.1) is a weak pullback.

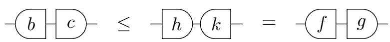
- Assume

  We want to show that (2.2.1) is a weak pullback. Given a commutative diagram of solid arrows below,

  $$
  \begin{tikzcd}
   & T \arrow[ddl, bend right, "b"'] \arrow[ddr, bend left, "c"] \arrow[d, dotted, "\omega"] & \\
   & A \arrow[dl, "f"'] \arrow[dr, "g"] & \\
  B \arrow[dr, "h"'] & & C \arrow[dl, "k"] \\
   & D &
  \end{tikzcd}
  $$

  we need to construct the dotted arrow. By Lemma 2.1.4, we get

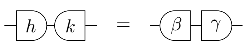
  

  and therefore by Lemma 2.2.4 we get $\omega \colon T \to A$ as desired.

- Let now (2.2.1) be a weak pullback diagram. We want to prove that

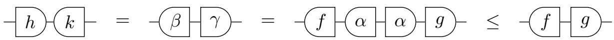
  By Lemma 2.1.3, take

  to be a factorisation with $\beta \colon T \to B$ and $\gamma \colon T \to C$. By Lemma 2.1.4, the external square of the following diagram commutes.

$$
\begin{tikzcd}
& T \arrow[d, dotted, "\alpha"] \arrow[ddl, bend right, "\beta"'] \arrow[ddr, bend left, "\gamma"] & \\
& A \arrow[dl, "f"'] \arrow[dr, "g"] & \\
B \arrow[dr, "h"'] & & C \arrow[dl, "k"] \\
& D &
\end{tikzcd}
$$

Since (2.2.1) is a weak pullback, there is $\alpha\colon T \to A$ making the above commute. With this we get

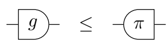


□

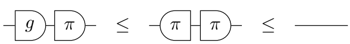
Another common formulation of the axiom of choice in set theory is the assertion that every surjective function $\pi\colon X \to Y$ splits, namely, there exists a function $\rho\colon Y \to X$ such that $\rho \mathbin{;} \pi = \mathrm{id}_Y$. For general Cartesian bicategories, splitting surjectives is a weaker condition than choice, however for tame Cartesian bicategories they will turn out to agree.

Lemma 2.2.7. *Surjective maps split in any Cartesian bicategory with choice.*

Proof. Let $\pi\colon X \to Y$ be a surjective map. Therefore, $\pi^{\mathrm{op}}\colon Y \to X$ is a total relation, so by (AC) there is a map $g\colon Y \to X$ such that



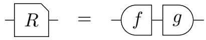
Now we have

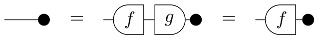


and since both the left hand side and the right hand side of that inequality are maps, we have by Proposition 1.2.7 that $g \mathbin{;} \pi = \mathrm{id}_Y$. □

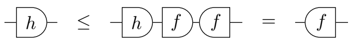
In a tame Cartesian bicategory, the converse of Lemma 2.2.7 is true:

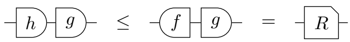
Proposition 2.2.8. *A tame Cartesian bicategory satisfies* (AC) *iff surjective maps split.*

Proof. By Proposition 2.2.7, it suffices to prove that (AC) holds if surjective maps split. So let $R\colon X \to Y$ be a total relation and take a comap-map factorisation  with maps f, g. Since R is total,



so f is surjective. Since surjective maps split, there exists a map h that is a pre-inverse of f, so $h \mathbin{;} f = \mathrm{id}$. Then



and therefore , so R contains a map. $\square$

We can use this to classify all tame Cartesian bicategories satisfying the axiom of choice.

Theorem 2.2.9. *A tame Cartesian bicategory satisfies the axiom of choice if and only if it is isomorphic to* $\mathsf{Span}^{\sim}\mathcal{C}$ *for some category* $\mathcal{C}$ *with products and weak pullbacks.*

Proof. We have $\mathsf{Span}^{\sim}\mathcal{C} \cong \mathsf{Span}^{\mathcal{S}}\mathcal{C}$ where $\mathcal{S}$ is the class of split epis. It is therefore a tame Cartesian bicategory where surjectives split and by Proposition 2.2.8, it satisfies (AC).

Conversely, if $\mathcal{B}$ is a tame Cartesian bicategory satisfying choice, we get $\mathcal{B} \cong \mathsf{Span}^{\mathcal{S}}\,\mathrm{Map}(\mathcal{B})$ where $\mathcal{S}$ is the class of surjectives, which by Proposition 2.2.8 agrees with the class of split epis and therefore

$$\mathcal{B} \cong \mathsf{Span}^{\mathcal{S}}\,\mathrm{Map}(\mathcal{B}) \cong \mathsf{Span}^{\sim}\,\mathrm{Map}(\mathcal{B})$$

$\square$

Assuming that we work in a Set theory where the axiom of choice holds, Rel is a tame Cartesian bicategory that satisfies choice. We can therefore apply the above classification and obtain the following:

Corollary 2.2.10.

$$\mathsf{Rel} \cong \mathsf{Span}^{\sim}\,\mathsf{Set}$$

2.3 Witnesses

The whole theory of categories with covers has a surprisingly useful dual. We therefore spell out the details here and they will be used extensively later on.

Definition 2.3.1. Let $\mathcal{C}$ be a category having coproducts and weak pushouts. A class of witnesses on $\mathcal{C}$ is a class of morphisms $\mathcal{W}$ (each of which called a witness) that is

- closed under composition and contains identities
- closed under (binary) coproducts, that is if $w\colon X \to Y$ and $w'\colon X' \to Y'$ are witnesses, so is $w + w'\colon X + X' \to Y + Y'$.
- closed under weak pushouts, that is if $w\colon X \to Y$ is a witness and
$$\begin{array}{ccc} X & \xrightarrow{w} & Y \\ \downarrow & & \downarrow \\ X' & \xrightarrow{\overline{w}} & Y' \end{array}$$
is a weak pushout, then $\overline{w}$ is a witness as well.
- closed under pre-splits, that is if $w\colon X \to Y$ and $g\colon Y \to Z$ have the property that $w \ ; \ g \in \mathcal{W}$, then $w \in \mathcal{W}$. We call this closure under pre-splits.

A category with witnesses is a tuple $(\mathcal{C}, \mathcal{W})$, where $\mathcal{C}$ is a category with coproducts and weak pushouts and $\mathcal{W}$ is a class of witnesses on $\mathcal{C}$. For a set of morphisms E let $\mathcal{W}(E)$ be the smallest class of witnesses containing E.

Clearly, $(\mathcal{C}, \mathcal{W})$ is a category with witnesses if and only if $(\mathcal{C}^{\mathrm{op}}, \mathcal{W}^{\mathrm{op}})$ is a category with covers, where $\mathcal{W}^{\mathrm{op}}$ is the same class as $\mathcal{W}$ just formally reversed to see it as a class of morphisms in $\mathcal{C}^{\mathrm{op}}$. In analogy to Definition 1.3.6, we can therefore dualise Definition 2.1.16

Definition 2.3.2. Let $(\mathcal{C}, W)$ be a category with witnesses, then we define
$$\mathsf{Cospan}^{\mathcal{W}}\mathcal{C} = \mathsf{Span}^{\mathcal{W}^{\mathrm{op}}}\mathcal{C}^{\mathrm{op}}$$

In Chapter 4 witnesses will be useful and we will particularly need to work with $\mathcal{W}(E)$, which we will want to construct in stages. It is therefore interesting to see which closure properties remain stable when we close with respect to another. For example, starting from a class that is closed under coproducts and weak pushouts, we can close under composition without losing this property.

Lemma 2.3.3. *If $\mathcal{X}$ is a class of morphisms that contains identities and is closed under coproducts and weak pushouts, so is the closure of $\mathcal{X}$ under composition.*

Proof. Let $\mathcal{X}'$ be the closure of $\mathcal{X}$ under composition. It is easy to check that $\mathcal{X}'$ is closed under coproducts, we only show the more interesting case of weak pushouts: The closure of $\mathcal{X}$ under composition is obtained by taking composites of sequences of morphisms in $\mathcal{X}'$. We can argue by induction on the length of those sequences. For trivial sequences, we need to show that weak pushouts along identities are in $\mathcal{X}$ but that follows from the assumptions on $\mathcal{X}$. Let now $x \in \mathcal{X}'$ and $y \in \mathcal{X}$. For the sake of induction we may assume that weak pushouts along x are in $\mathcal{X}'$ again and we have to show that also weak pushouts along $x \,;\, y$ remain in $\mathcal{X}'$. Let

$$\begin{tikzcd} A \arrow[r, "x"] \arrow[d, "f"] & B \arrow[r, "y"] & C \arrow[d] \\ A' \arrow[rr, "z"] & & C' \end{tikzcd}$$

be a weak pushout. Taking a weak pushout of f along x yields a diagram

$$\begin{tikzcd} A \arrow[r, "x"] \arrow[d, "f"] & B \arrow[r, "y"] \arrow[d] & C \arrow[d] \\ A' \arrow[r, "\overline{x}"] & B' & C' \end{tikzcd}$$

By the weak universal property of the weak pushout B', there is a morphism $\overline{y}$ such that $z = \overline{x} \,;\, \overline{y}$. Therefore

$$\begin{tikzcd} A \arrow[r, "x"] \arrow[d, "f"] & B \arrow[r, "y"] \arrow[d] & C \arrow[d] \\ A' \arrow[r, "\overline{x}"] & B' \arrow[r, "\overline{y}"] & C' \end{tikzcd}$$

is a commutative diagram. It is easy to see that C' being a weak pushout of the outer rectangle implies that the right square is a weak pushout, too. By assumption on x we have $\overline{x} \in \mathcal{X}'$ and since $\mathcal{X}$ is closed under weak pushouts we have $\overline{y} \in \mathcal{X}$, therefore $z = \overline{x} \,;\, \overline{y} \in \mathcal{X}'$ as desired. $\square$

This means that starting from a class closed under coproducts and weak pushouts, we can generate a system of witnesses in a straightforward manner:

Lemma 2.3.4. *If $\mathcal{X}$ contains identities and is closed under coproducts and weak pushouts, let $\mathcal{X}'$ be the closure of $\mathcal{X}$ under composition. Then*

$$\mathcal{W}(\mathcal{X}) = \{w \mid \exists g : w \,;\, g \in \mathcal{X}'\}$$

Proof. Let $W = \{w \mid \exists g : w \,;\, g \in \mathcal{X}'\}$.

- To show $\mathcal{W}(\mathcal{X}) \subseteq W$ it suffices to see that W is a system of witnesses, since by definition $\mathcal{W}(\mathcal{X})$ is the smallest one that contains $\mathcal{X}$. By Lemma 2.3.3, $\mathcal{X}'$ contains identities, and is closed under coproducts, weak pushouts and composition. It suffices to see that all these properties carry over to W and that W has the splitting property.
  - Let $w_1, w_2 \in W$. We want to prove that $w_1 \,;\, w_2 \in W$. Let g_i be such that $w_i \,;\, g_i \in \mathcal{X}'$. Let $x = w_2 \,;\, g_2$ and consider a weak pushout
$$\begin{array}{ccc} A & \xrightarrow{x} & B \\ \downarrow{\scriptstyle g_1} & & \downarrow{\scriptstyle \overline{g_1}} \\ C & \xrightarrow{\overline{x}} & P \end{array}$$
    and since $x \in \mathcal{X}'$, we have $\overline{x} \in \mathcal{X}'$. This gives us
$$w_1 \,;\, w_2 \,;\, g_2 \,;\, \overline{g_1} = w_1 \,;\, x \,;\, \overline{g_1} = w_1 \,;\, g_1 \,;\, \overline{x} \in \mathcal{X}'$$
    and therefore $w_1 \,;\, w_2 \in W$.
  - Since $\mathcal{X}'$ is closed under coproducts, so is W.

- Let $w \in W$ and

$$\begin{array}{ccc} A & \xrightarrow{w} & B \\ \downarrow f & & \downarrow \overline{f} \\ C & \xrightarrow{\overline{w}} & P \end{array}$$

  be a weak pushout. We need to show $\overline{w} \in W$. Let g be a morphism such that $w\ ;\ g \in \mathcal{X}'$. We can add this to the above diagram and take another weak pushout Q to obtain

$$\begin{array}{ccccc} A & \xrightarrow{w} & B & \xrightarrow{g} & D \\ \downarrow f & & \downarrow & & \downarrow \\ C & \xrightarrow{\overline{w}} & P & \xrightarrow{\overline{g}} & Q \end{array}$$

  This makes Q also a weak pushout of the outer diagram and therefore we have $\overline{w}\ ;\ \overline{g} \in \mathcal{X}'$, hence $\overline{w} \in W$.

- Lastly, the splitting property is easy to check, because if $w\ ;\ h \in W$, then there is g such that $w\ ;\ h\ ;\ g = w\ ;\ (h\ ;\ g) \in \mathcal{X}'$, so by definition, $w \in W$.

This exhibits W as a system of witnesses and clearly $\mathcal{X} \subseteq W$, hence $\mathcal{W}(\mathcal{X}) \subseteq W$.

- Conversely, we want to show $W \subseteq \mathcal{W}(\mathcal{X})$. Clearly, $\mathcal{X} \subseteq \mathcal{W}(\mathcal{X})$ and since $\mathcal{W}(\mathcal{X})$ is closed under composition, we get $\mathcal{X}' \subseteq \mathcal{W}(\mathcal{X})$. But $\mathcal{W}(\mathcal{X})$ is furthermore closed under splits and therefore $W \subseteq \mathcal{W}(\mathcal{X})$.

$\square$

Chapter 3

Frobenius Theories

Introduction

In his PhD thesis [31], William Lawvere introduced his beautiful categorical perspective on algebraic theories and how to see the semantics of such a theory as a suitable functor, aptly called *functorial semantics*. Lawvere's idea was roughly as follows: Given an algebraic theory, such as the theory of groups, we can construct a category called its syntactic category $\mathcal{C}$ as follows: The objects of $\mathcal{C}$ are natural numbers, a morphism $n \to 1$ is a term in the algebraic theory with n variables and more generally a morphism $n \to m$ is a collection of m such terms. In other words, addition of natural numbers induces a categorical product. The composition in $\mathcal{C}$ works via substitution. Importantly, we see two morphisms $n \to m$ as equal if the corresponding terms are equal modulo the laws of the algebraic theory. For example, for group theory the syntactic category is constructed from multiplication, a morphism $2 \to 1$, the unit of multiplication, a morphism $0 \to 1$, and inversion, a morphism $1 \to 1$. The associative law implies then that the two morphisms $3 \to 1$ that can be constructed from multiplication are equal in the syntactic category.

Lawvere's interesting insight now was this: A model of the algebraic theory, for example a group for the theory of groups, is nothing but a finite product-preserving functor $\mathcal{M}\colon \mathcal{C} \to \mathsf{Set}$. Such a functor picks out a set $S = \mathcal{M}(1)$, which will be the underlying set of of the algebraic structure. For $\mathcal{M}$ to be product-preserving forces $\mathcal{M}(n) = S^n$. Furthermore, every term $t\colon n \to 1$ will be interpreted by a function $\mathcal{M}(t)\colon S^n \to S$, the function that given an assignment of elements in S to all the variables will evaluate the term. For the group theory example, the term $x_1 \cdot x_2$ will be interpreted by the multiplication function. Since in the syntactic category of group theory, the terms $(x_1 \cdot x_2) \cdot x_3$ and $x_1 \cdot (x_2 \cdot x_3)$ are considered equal, this forces $\mathcal{M}$ to interpret multiplication by an associative function.

This perspective on algebraic theories is important because it puts syntax and semantics on equal footing. The syntax of the theory is encoded into a category with the same structure as the semantic category Set has. But as important as this perspective is, for computer science applications it comes with the serious drawback that all interpretations need to be functional. For that reason, the authors of [7] introduced Frobenius theories as a relational analogue to Lawvere theories.

Where Lawvere theories use categories with finite-products, Frobenius theories use Cartesian bicategories. This allows us to use Rel, the archetypal Cartesian bicategory, in place of Set as the semantic category. Whereas morphisms in a Lawvere theory were seen as algebraic terms, which can be represented via syntax trees, the syntactical analogue for Frobenius theories is string diagrams.

Outline of the Chapter In Section 3.1 we will introduce the notion of a monoidal signature, which can be thought of as a set of basic boxes to be used in string diagrams. Given such a signature Σ, we will construct a Cartesian bicategory $\mathbb{CB}_\Sigma$ whose morphisms are the string diagrams that can be built from the basic boxes and that only satisfy those inequalities that are consequences of the laws of Cartesian bicategories. As a Frobenius theory this is to be thought of as mere syntax with no additional properties, like an algebraic theory that imposes no equations.

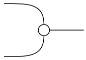
In Section 3.2 we will then introduce Frobenius theories $\mathcal{T}$ as Cartesian bicategories that have natural numbers as their objects. A model of such a Frobenius theory (in Rel) is then just a morphism $\mathcal{T} \to \mathsf{Rel}$. Furthermore, there is a convenient notion of morphism between models that works as expected. It is possible to define a Frobenius theory by giving generators and relations, so one can specify a signature Σ and a set of inequalities E and construct a Frobenius theory $\mathbb{CB}_{\Sigma/E}$ from it. We call (Σ, E) a presentation of a Frobenius theory. Importantly, every Frobenius theory allows such a presentation.

In Section 3.3 we will introduce the problem of completeness, that will be solved in Chapter 5 with Theorem 5.4.28. Every Frobenius theory allows for the use of diagrammatic reasoning, that is given two string diagrams $R, S \colon n \to m$ in $\mathbb{CB}_{\Sigma/E}$ with $R \leq S$ there is a diagrammatic proof for it that uses the laws of Cartesian bicategories and the axioms of the theory. Since models are by definition morphisms of Cartesian bicategories, they respect the ordering so whenever $R \leq S$, we also have $\mathcal{M}(R) \subseteq \mathcal{M}(S)$ for all models $\mathcal{M} \colon \mathbb{CB}_{\Sigma/E} \to \mathsf{Rel}$. The interesting question of completeness is the converse of this: Given any property that is valid in all possible models, can we find a diagrammatic proof for it? The answer is positive, as we will see in Chapter 5.

3.1 Signatures

The idea of a signature is to define the non-logical symbols that a theory uses. For example $+$ in a formal theory of natural numbers would be such a symbol. Classically, for so-called relational signatures, one fixes a set of symbols and specifies an arity for each one. A symbol R having arity $n \in \mathbb{N}$ means that it needs to be interpreted by an n-ary relation. Addition would be a 3-ary symbol, because it will be interpreted as $\{(x, y, x + y) \mid x, y \in \mathbb{N}\}$.

Since we want to draw string diagrams, a mere arity is not enough, because we need to decide how many wires to draw on the left and on the right. This gives rise to the notion of a monoidal signature –

a signature making this additional, more fine-grained choice. In this setting, addition would be represented by a symbol with arity 2 and coarity 1, usually drawn as  .

Definition 3.1.1 (Monoidal signature)**.** A monoidal signature[1] Σ is a set of symbols partitioned into classes $\Sigma_{n,m}$, one for every $n, m \in \mathbb{N}$. For every $\sigma \in \Sigma_{n,m}$, we call n the arity and m the coarity.

A monoidal signature is a concise description of a formal language. The symbols in Σ give the different relationships between entities the language allows to talk about. As this is an abstract notion, let us have a look at different examples.

Example 3.1.2.

- Group theory typically uses a signature with the following symbols: Mult $\in \Sigma_{2,1}$, $1 \in \Sigma_{0,1}$, inverse $\in \Sigma_{1,1}$.
- Zermelo-Fraenkel style set theories typically use a signature with only one symbol $\in$, standing for set membership and living in $\Sigma_{1,1}$.
- To formalise family relationships, one would use a signature with a single symbol $P \in \Sigma_{1,1}$, where $P(x, y)$ is interpreted as x being a parent of y.

Thinking of a signature as a kind of interface, it naturally gives rise to the question what an implementation of that interface might be.

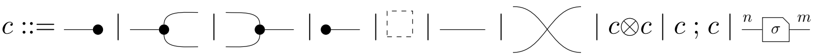
Definition 3.1.3. Let Σ be a monoidal signature. An interpretation of Σ in a Cartesian bicategory $\mathcal{B}$, denoted $F\colon \Sigma \to \mathcal{B}$ consists of a choice of object $F(1) = X \in \mathcal{B}$ and for every $\sigma \in \Sigma_{n,m}$ a choice of morphism $F(\sigma)\colon X^{\otimes n} \to X^{\otimes m}$. Given a morphism $G\colon \mathcal{B} \to \mathcal{B}'$ we can compose them in the obvious way to form an interpretation $F \mathbin{;} G\colon \Sigma \to \mathcal{B}'$.

What paves the way to a functorial semantics for relational theories is that Cartesian bicategories can internalise monoidal signatures

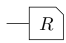
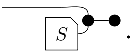
[1] also known under the name monoidal graphs

and their interpretations. More precisely, starting from a monoidal signature, we can construct a Cartesian bicategory that has as morphisms all the string diagrams that can be constructed from the symbols in the signature. This category will turn out to satisfy a universal property with respect to interpretations.

Definition 3.1.4. Let Σ be a monoidal signature. The Cartesian bicategory $\mathbb{CB}_\Sigma$, called the syntactic Cartesian bicategory for Σ, has as objects the natural numbers and morphisms are constructed as follows: For every $\sigma \in \Sigma_{n,m}$, there is a basic morphism $\sigma\colon n \to m$ in $\mathbb{CB}_\Sigma$. A general morphism in $\mathbb{CB}_\Sigma$ is then a string diagram constructed from those basic morphisms and the connectives of Cartesian bicategories. More precisely, morphisms in $\mathbb{CB}_\Sigma$ are terms generated from the following grammar



where $\overset{n}{\text{———}}$ is a shorthand for n stacked wires. The laws of Cartesian bicategories induce a preorder on these terms. Quotienting this preorder to obtain a partial order turns $\mathbb{CB}_\Sigma$ into a Cartesian bicategory. Of course, we will write morphisms of $\mathbb{CB}_\Sigma$ as string diagrams rather than as terms of this grammar.

Remark 3.1.5. Even if Σ and Σ' are different, it is possible to have $\mathbb{CB}_\Sigma \cong \mathbb{CB}_{\Sigma'}$. For example, let $\Sigma = \Sigma_{1,0} = \{R\}$ and $\Sigma' = \Sigma'_{0,1} = \{S\}$, then $\mathbb{CB}_\Sigma \cong \mathbb{CB}_{\Sigma'}$ where the isomorphism identifies  with .

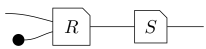
Proposition 3.1.6. *Given a monoidal signature Σ, $\mathbb{CB}_\Sigma$ is the free Cartesian bicategory on Σ in the following sense: There is an obvious interpretation $\Sigma \to \mathbb{CB}_\Sigma$ that satisfies the following universal property: For any other Cartesian bicategory $\mathcal{B}$ with interpretation $F\colon \Sigma \to \mathcal{B}$ there is a unique morphism of Cartesian bicategories $G\colon \mathbb{CB}_\Sigma \to \mathcal{B}$ such that*

$$\begin{tikzcd} \Sigma \arrow[r] \arrow[dr, "F"'] & \mathbb{CB}_\Sigma \arrow[d, "G"] \\ & \mathcal{B} \end{tikzcd}$$

The above universal property is just a concise categorical way of expressing that $\mathbb{CB}_\Sigma$ has as morphisms all string diagrams that can be constructed from the symbols in Σ and the primitives of Cartesian bicategories modulo the Cartesian bicategory laws.

In logical terms, an interpretation $\Sigma \to \mathsf{Rel}$ is known as a Σ-structure, a set that comes equipped with interpretations of the symbols as relations on it. Σ-structures on their own are only mildly interesting, it is more interesting to single out structures that satisfy additional properties, for example singling out abelian groups among all structures that can interpret the same signature. But we now have a way of doing exactly that. If we take $\mathbb{CB}_\Sigma$ as a starting point and refine the ordering on its morphisms, functors will have to respect the new inclusions that we added. In other words, this is a convenient way to specify axioms for a logical theory and is the starting point of our study of Frobenius theories.

Remark 3.1.7. There is a strong connection between $\mathbb{CB}_\Sigma$ and *regular logic*, that is the fragment of first-order logic consisting of existential quantification and finite conjunctions. That connection can be made precise as follows: Consider the following category $\mathcal{L}$, where objects are finite sets and a morphism $X \to Y$ in $\mathcal{L}$ is an equivalence class of regular logical formulae using the relation symbols in Σ with free variables in $X+Y$, modulo logical equivalence. Composition of morphisms $\phi\colon X \to Y$ and $\psi\colon Y \to Z$ is defined to be $\exists y_1 \dots \exists y_n : \phi \wedge \psi$, where $Y = \{y_1, \dots, y_n\}$. $\mathcal{L}$ has a canonical local ordering via implication of formulas. One can see that $\mathcal{L}$ is a Cartesian bicategory and in fact we proved in [8] that $\mathcal{L}$ is equivalent to $\mathbb{CB}_\Sigma$. Under this equivalence, the regular formula $\exists x_2, y_1 : R(x_1, x_2, y_1) \wedge S(y_1, z_1)$ – with free variables x_1, z_1 – corresponds to the string diagram



3.2 Frobenius theories

Frobenius theories were first introduced in [7] as a relational generalisation for Lawvere's approach to algebraic theories.

Definition 3.2.1. A Frobenius theory is a Cartesian bicategory $\mathcal{T}$ with objects the natural numbers and monoidal product given by $+$.

Definition 3.2.2. A presentation of a Frobenius theory consists of a choice of monoidal signature Σ together with a set of inequalities E, where each inequality in E has the form $A \leq B$ for morphisms $A, B \colon n \to m$ in $\mathbb{CB}_\Sigma$. Let $\leq$ be the smallest preorder generated by the laws of Cartesian bicategories and E. Let $\mathbb{CB}_{\Sigma/E}$ be the Cartesian bicategory with objects the natural numbers, where morphisms are equivalence classes of morphisms of $\mathbb{CB}_\Sigma$, with $R, S \colon n \to m$ being equivalent if $R \leq S \leq R$. On these equivalence classes $\leq$ induces a partial order which turns $\mathbb{CB}_{\Sigma/E}$ into a Cartesian bicategory. We call (Σ, E) a *presentation* of $\mathbb{CB}_{\Sigma/E}$.

Proposition 3.2.3. *There is an evident morphism $\mathbb{CB}_\Sigma \to \mathbb{CB}_{\Sigma/E}$, which gives $\mathbb{CB}_{\Sigma/E}$ a universal property: A morphism $\mathcal{M} \colon \mathbb{CB}_\Sigma \to \mathcal{B}$ factors uniquely through $\mathbb{CB}_{\Sigma/E}$ as*

$$\begin{tikzcd}
\mathbb{CB}_\Sigma \arrow[r] \arrow[dr, "\mathcal{M}"'] & \mathbb{CB}_{\Sigma/E} \arrow[d, "\mathcal{M}'"] \\
& \mathcal{B}
\end{tikzcd}$$

if and only if for any inequality $A \leq B$ in E we have $\mathcal{M}(A) \leq \mathcal{M}(B)$. In this case we say $\mathcal{M}$ satisfies E.

Proof. Clearly, if $\mathcal{M}$ factors, it satisfies $\mathcal{M}(A) \leq \mathcal{M}(B)$ because those inequalities hold in $\mathbb{CB}_{\Sigma/E}$. Conversely, note that the morphisms in $\mathbb{CB}_{\Sigma/E}$ are equivalence classes of morphisms modulo the equivalence relation obtained from the axioms. If $\mathcal{M} \colon \mathbb{CB}_\Sigma \to \mathcal{B}$ satisfies E, this means that it is well-defined on those equivalence classes and the induced morphism $\mathcal{M}' \colon \mathbb{CB}_{\Sigma/E} \to \mathcal{B}$ is a morphism of Cartesian bicategories. $\square$

Remark 3.2.4. Another possibility of defining what a Frobenius theory *is* would be to identify it with a presentation (Σ, E). In that case, the associated Cartesian bicategory would be called the syntactic category of the Frobenius theory, which parallels the use of that term for, say, symmetric monoidal theories, or even geometric theories.

Definition 3.2.5. A morphism between Frobenius theories is a morphism between Cartesian bicategories.

In line with [7], we see morphisms of Cartesian bicategories as giving models of one theory in another.

Definition 3.2.6. Let $\mathcal{T}$ be a Frobenius theory and $\mathcal{B}$ a Cartesian bicategory. A model of $\mathcal{T}$ in $\mathcal{B}$ is a morphism of Cartesian bicategories $\mathcal{M}\colon \mathcal{T} \to \mathcal{B}$. A morphism $\mathcal{M} \to \mathcal{N}$ between models is an oplax monoidal natural transformation, that is a monoidal family of morphisms $\eta_X\colon \mathcal{M}(X) \to \mathcal{N}(X)$ in $\mathcal{C}$ such that for any morphism $R\colon X \to Y$ in $\mathcal{B}$, the typical naturality square

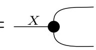
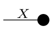
$$\begin{tikzcd} \mathcal{M}(X) \arrow[r, "\mathcal{M}(R)"] \arrow[d, "\eta_X"] & \mathcal{M}(Y) \arrow[d, "\eta_Y"] \\ \mathcal{N}(X) \arrow[r, "\mathcal{N}(R)"] & \mathcal{N}(Y) \end{tikzcd}$$

commutes *oplaxly* in that $\mathcal{M}(R) \,;\, \eta_Y \leq \eta_X \,;\, \mathcal{N}(R)$.

In the following we will let $\mathcal{B}$ default to Rel, so we will speak of models of $\mathcal{T}$ in Rel simply as models of $\mathcal{T}$.

The definition of morphisms of models as oplax natural transformations is a bit surprising at first, but it turns out to yield the right notion. As a first indicator of this, morphisms of models are always maps.

Proposition 3.2.7. *Let $\mathcal{M}, \mathcal{N}$ be models of $\mathcal{B}$ in $\mathcal{C}$ and $\eta\colon \mathcal{M} \to \mathcal{N}$ be a morphism between models. Then every component η_X is a map, i.e. a comonoid homomorphism.*

Proof. The definition of Cartesian bicategory makes η_X a lax comonoid homomorphism. The missing inclusions come from oplax naturality of η with $R =$  and $R =$ . $\square$

Remark 3.2.8. The choice of what to call lax and what to call oplax is not made consistently in the literature. But since Proposition 3.2.7 shows that the condition on morphisms between models gives the opposite inclusion of the lax comonoid homomorphism, it makes sense to call it oplax.

Remark 3.2.9. Since every component η_X of a morphism η between models is a map, it has a right-adjoint η_X^{op}. These right-adjoints form a *lax* monoidal natural transformation themselves, instead of an oplax one. Conversely, it is straightforward to check that, dual to Proposition 3.2.7, a lax monoidal natural transformation is a family of comaps. The left-adjoints of those comaps then form a morphism of models. In other words, if we define a comorphism between models to be a lax monoidal natural transformation, then a morphism $\mathcal{M} \to \mathcal{N}$ is equivalent to a comorphism $\mathcal{N} \to \mathcal{M}$. Lax or oplax natural transformation therefore define essentially the same notion of morphism with only a difference in direction. Somewhat arbitrarily we pick the direction of the map as the direction of the morphism (and therefore stick with *oplax* natural transformations).

Models of $\mathbb{CB}_\Sigma$ in Rel are particularly important and therefore get their own name.

Definition 3.2.10. A Σ-structure is a model $\mathbb{CB}_\Sigma \to \mathsf{Rel}$.

Remark 3.2.11. A Σ-structure $S\colon \mathbb{CB}_\Sigma \to \mathsf{Rel}$ is, via the universal property of $\mathbb{CB}_\Sigma$, the same thing as an interpretation of Σ in Rel. In other words, a Σ-structure consists of an underlying set $S(1)$ and, for every $\sigma \in \Sigma_{n,m}$ a relation $S(\sigma)\colon S(n) = S(1)^n \to S(1)^m = S(m)$. A morphism $f\colon S \to T$ between Σ-structures comes with functions $f_n\colon S(n) \to T(n)$, which, because f is monoidal, are uniquely determined by f_1. That f is an oplax natural transformation implies that if

$$(x, y) \in S(\sigma) \subseteq S(n) \times S(m)$$

then

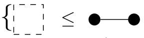
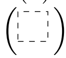
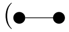
$$(f_n(x), f_m(y)) \in T(\sigma)$$

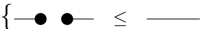
In fact, this is an equivalent characterisation, so that morphisms between Σ-structures are equivalently functions between the underlying sets that preserve all relations.

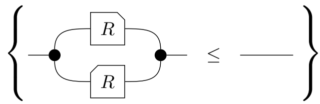
Remark 3.2.12. By the universal property of $\mathbb{CB}_{\Sigma/E}$, a model of $\mathbb{CB}_{\Sigma/E}$ is the same as a Σ-structure that satisfies E, which is what one would expect a model of the Frobenius theory (Σ, E) to be. This shows the fundamental importance of Σ-structures: Every model of (Σ, E) has an underlying Σ-structure and every Σ-structure gives rise to at most one model of $\mathbb{CB}_{\Sigma/E}$. In other words, being a model of $\mathbb{CB}_{\Sigma/E}$, i.e. factoring through $\mathbb{CB}_{\Sigma/E}$, is a property on Σ-structures. It will be indispensably useful for us to think of models of $\mathbb{CB}_{\Sigma/E}$ in terms of their underlying Σ-structures satisfying the axioms specified by E.

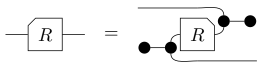
Remark 3.2.13. Since the ordering in Cartesian bicategories is antisymmetric, we can encode equalities into inequalities, so considering only inequalities is not a restriction.

Example 3.2.14. Let us consider examples of Frobenius theories and their models (in Rel).

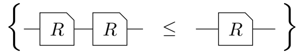
1. Let $\Sigma = \emptyset$ and $E = \{$$\}$, where the left-hand side is the empty diagram representing $\mathrm{id}_0 \colon 0 \to 0$ in $\mathbb{CB}_\Sigma$. Let S be any Σ-structure, since $\Sigma = \emptyset$, S can be identified with the set $S(1)$. Now, $S(0) = \{\bullet\}$ is a one element set, regardless of S (think of it as the set of 0-tuples in $S(1)$). There are only two relations $S(0) \to S(0)$, namely the identity and the empty relation. S  is the identity regardless of S, but S  is empty if and only if $S(1)$ is. Therefore models that satisfy E are precisely non-empty sets. Morphisms between those are just functions.

2. Let $\Sigma = \emptyset$ and $E = \{$$\}$. The interpretation of the left-hand side is the total relation, so this inequality says that equality is total. There are therefore only two models of this theory, the one-element model or the empty one.

3. Let $\Sigma = \{R \colon 1 \to 1\}$ and $E =$ ,

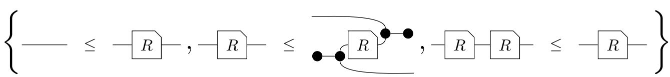
   where . Models of this theory are sets equipped with an antisymmetric relation R. Morphisms between those are functions that preserve the relation.

4. Let $\Sigma = \{R \colon 1 \to 1\}$ and

   $E =$ .

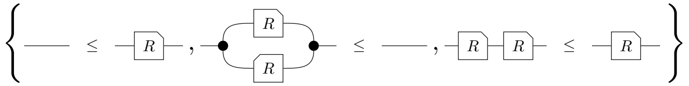
   Models of this theory are are sets equipped with a transitive relation R, morphisms preserve this relation.

5. Let $\Sigma = \{R \colon 1 \to 1\}$ and

$E =$ .

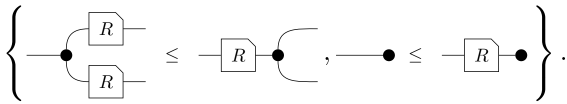
Models of this theory are sets equipped with an equivalence relation, morphisms need to preserve it.

6. Let $\Sigma = \{R\colon 1 \to 1\}$ and

$E =$ 

Then models of this theory are exactly posets, that is sets equipped with a reflexive, antisymmetric and transitive relation. Morphisms are exactly morphisms of posets.

7. Let $\Sigma = \{R\colon 1 \to 1\}$ and

$E =$ 

A Σ-structure is a set together with a binary relation R. The axioms E force R to be a comonoid homomorphism, in other words a function. Therefore, models of this Frobenius theory are sets equipped with an endofunction. This example shows why it is not necessary to distinguish functions from relations in the signature.

Remark 3.2.15. Since maps are comonoid homomorphisms, one can force any symbol to act as a map by postulating the two axioms seen in Example 3.2.14 (7). Since morphisms of Cartesian bicategories preserve maps, this forces the interpretation of those symbols to be maps. This shows that Frobenius theories generalise Lawvere's algebraic theories.

Lemma 3.2.16. *Every Frobenius theory has a presentation.*

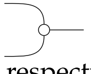
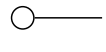
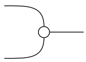
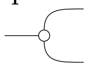
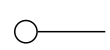
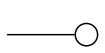
Proof. Given a Frobenius theory $\mathcal{T}$, define a signature Σ with $\Sigma_{n,m} = \mathrm{Hom}_{\mathcal{T}}(n, m)$, that is for every morphism in $\mathcal{T}$ the signature contains a separate symbol. We get a morphism of Cartesian bicategories $\pi\colon \mathbb{CB}_\Sigma \to \mathcal{T}$ that interprets each symbol in Σ as the morphism it represents. Let now E be the set of all inequalities $A \leq B$ where $\pi(A) \leq \pi(B)$ holds in $\mathcal{T}$. It is easy to check that $\mathbb{CB}_{\Sigma/E} \cong \mathcal{T}$. $\square$

3.3 The problem of Completeness

An important question regarding Frobenius theories and their models is the question of completeness. Frobenius theories allow for diagrammatic reasoning, that is $\mathbb{CB}_{\Sigma/E}$ has a string diagrammatic calculus that we can use to prove implications. So, given string diagram $R, S\colon n \to m$ in $\mathbb{CB}_{\Sigma}$, we can use the laws of Cartesian bicategories and the axioms in E to prove $R \leq S$ in $\mathbb{CB}_{\Sigma/E}$. This will then imply that all models of the Frobenius theory respect that proof and satisfy $\mathcal{M}(R) \subseteq \mathcal{M}(S)$ for all $\mathcal{M}\colon \mathbb{CB}_{\Sigma/E} \to \mathsf{Rel}$. This immediately raises the question if we can derive all properties that are shared by all models diagrammatically, which is the question of completeness. Formally, if $\mathcal{M}(R) \subseteq \mathcal{M}(S)$ for all models $\mathcal{M}\colon \mathbb{CB}_{\Sigma/E} \to \mathsf{Rel}$, is it necessarily true that $R \leq S$ in $\mathbb{CB}_{\Sigma/E}$, that is can we necessarily find a string diagrammatic proof of that fact?

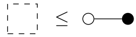
The answer to this question turns out to be positive, which we will prove in Theorem 5.4.28. This means that for all properties that string diagrams are able to express, that is all properties that involve existential quantification and conjunction, diagrammatic reasoning is exactly enough to prove all consequences of a theory.

We will now give an example of this diagrammatic reasoning in action for a Frobenius theory that is not too trivial.

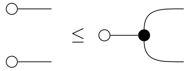
Example 3.3.1. We will prove, diagrammatically, that monoids that enjoy a certain cancellation property can be ordered, similar to how the ordering on natural numbers is defined. For that, we first consider the Frobenius theory of commutative monoids, which is a theory over the signature Σ with  $\in \Sigma_{2,1}$ and  $\in \Sigma_{0,1}$, which we interpret as addition and 0 respectively. For convenience, we will draw the opposite of  as  and the opposite of  as . The theory of commutative monoids has the following axioms: Note that we write some axioms as equalities, which is shorthand for the two axioms corresponding to both inclusions.

- 0 is a map, that is it respects copying and discarding:

- First of all, 0 exists, which corresponds to the following inequality (0 is total):

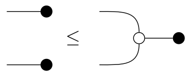


- Secondly, 0 is unique, which corresponds to the following inequality (0 is single-valued):

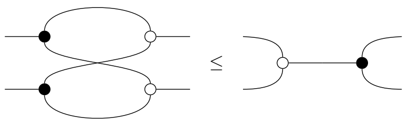


Note that the converse inequalities are already in the laws of Cartesian bicategories, because every morphism is a lax comonoid homomorphism. So, both are even equalities.

- Addition is a map, that is:
  - For any pair of elements, their sum exists ($+$ is total):

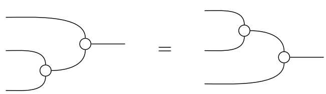


  - This sum is unique, that is ($+$ is single-valued):

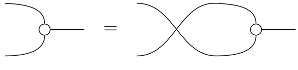


Again, these axioms are really equalities because the reverse inequalities come from the laws of Cartesian bicategories.

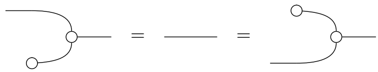
- Addition is associative, that is



- Addition is commutative, that is

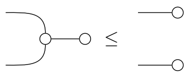


- 0 is the unit for addition, that is

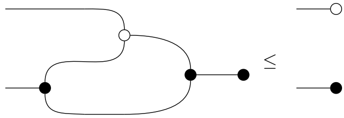


We now add two more axioms to this theory, which are satisfied e.g. by the natural numbers and which are characteristic of positive cancellative monoids:

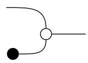
- If $x + y = 0$, then already $x = y = 0$



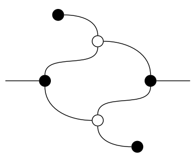
- If $x + y = y$, then $x = 0$



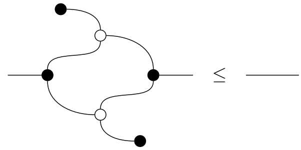
Using these axioms is is possible to reason about cancellative monoids in string diagrams. For example, it is possible to prove that they can be ordered in the same way that natural numbers are ordered via $x \leq y$ if and only if $\exists z : x + z = y$, in diagrams



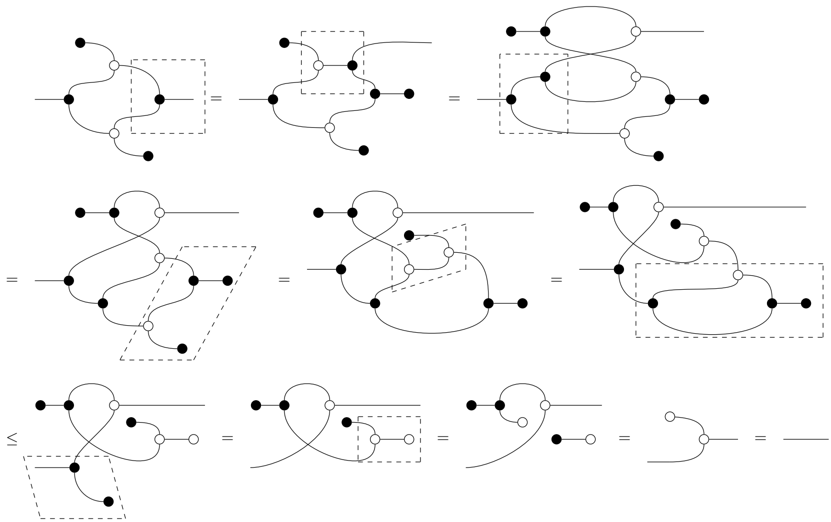
It is straightforward to check that this ordering is reflexive and transitive. We will now prove graphically that it is also antisymmetric, that is that $x \leq y$ and $y \leq x$ imply $x = y$. The starting point of antisymmetry is to assume that we have x, y such that there exists n, m satisfying $x + n = y$ and $y + m = x$. The set of those (x, y) is encoded as the following string diagram:



For antisymmetry we want to prove such pairs to be equal, so



This can be seen via the following derivation. For ease of reading we will in each step mark the part of the diagram that is transformed.



Multisorted signatures

So far, the string diagrams we have been considering in $\mathbb{CB}_\Sigma$ only have a single type of string, they therefore cannot distinguish between different types of inputs and outputs. Logically, these types are called sorts and support for the coexistence of different sorts is given by multisorted signatures. Typically, they tend to be avoided, because they add another layer of complication and although things don't actually become that much harder, they become harder to talk about precisely. But we feel that Cartesian bicategories give a nice layer of abstraction that hide the difficulties that usually come with multisorted signatures. We will briefly describe the analogues of the previous definitions for the multisorted case here. The remainder of the thesis can be read with the multi-sorted case in mind with minor adjustments. In particular, the completeness Theorem 5.4.28 is still true, even for multisorted Frobenius theories.

Definition 3.3.2 (Monoidal signature)**.** Let S be a set of sorts and let S^* denote the free monoid over S. A monoidal signature Σ over S is a set of symbols partitioned into classes $\Sigma_{s,t}$, one for every $s, t \in S^*$. For every $\sigma \in \Sigma_{s,t}$, we call s the arity and t the coarity.

Definition 3.3.3. Let Σ be a monoidal signature over S and $\mathcal{B}$ a strict Cartesian bicategory. An interpretation $F\colon \Sigma \to \mathcal{B}$ consists of

- a map $f\colon S \to \mathrm{Ob}(\mathcal{B})$, which by the universal property of the free monoid extends to a unique monoid morphism $\overline{f}\colon S^* \to \mathrm{Ob}(\mathcal{B})$,
- for every $\sigma \in \Sigma_{s,t}$ a morphism $F(\sigma)\colon \overline{f}(s) \to \overline{f}(t)$ in $\mathcal{B}$.

Definition 3.3.4. Let Σ be a monoidal signature over set of sorts S. The Cartesian bicategory $\mathbb{CB}_\Sigma$, called the syntactic Cartesian bicategory for Σ, has as objects the free monoid over S, S^*, morphisms are constructed as follows: For every $\sigma \in \Sigma_{s,t}$, there is a basic morphism $\sigma\colon s \to t$ in $\mathbb{CB}_\Sigma$. A general morphism in $\mathbb{CB}_\Sigma$ is then a string diagram constructed from those basic morphisms and the connectives of Cartesian bicategories, modulo the laws of Cartesian bicategories.

Definition 3.3.5. Let Σ be a monoidal signature over set of sorts S. The Cartesian bicategory $\mathbb{CB}_\Sigma$ is the free Cartesian bicategory on Σ and has the following universal property: There is an interpretation $\Sigma \to \mathbb{CB}_\Sigma$ and for any other Cartesian bicategory $\mathcal{B}$ with interpretation $F\colon \Sigma \to \mathcal{B}$ there is a unique morphism of Cartesian bicategories $G\colon \mathbb{CB}_\Sigma \to \mathcal{B}$ such that

$$\begin{tikzcd} \Sigma \arrow[r] \arrow[dr, "F"'] & \mathbb{CB}_\Sigma \arrow[d, "G"] \\ & \mathcal{B} \end{tikzcd}$$

Definition 3.3.6 (Multisorted Frobenius theory)**.** A Frobenius theory over set of sorts S is a Cartesian bicategory $\mathcal{T}$ with objects the free monoid over S.

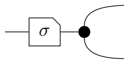
From here on, it is not necessary to make any adjustments to what was presented in Section 3.2. For example, presentations for Frobenius theories work for multisorted theories in the same way they did for single-sorted ones. Since the Completeness Theorem 5.4.28 is proved in categorical form, the proof is agnostic to whether the hypergraphs are coloured or not.

Chapter 4

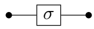
Combinatorial Characterisations

Introduction

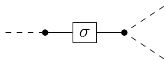
As we have seen in Remark 3.1.7, $\mathbb{CB}_\Sigma$ is tightly connected to regular logic, that is the fragment of first order logic consisting of existential quantification and conjunction. In database theory, where logical formulas are interpreted as queries to relational databases, regular logic corresponds to so called conjunctive queries. In their celebrated paper [19], Chandra and Merlin showed how to efficiently handle conjunctive queries to a database in a way that is provably very close to optimal. And even though *what* their paper shows is important, for our purposes the *how* is even more important. They gave a translation from conjunctive queries to certain combinatorial structures, hypergraphs with interfaces, such that implication between queries (also known as inclusion) corresponds to a morphism between the corresponding structures. With the correspondence between $\mathbb{CB}_\Sigma$ and regular logic in mind, we took this as a starting point, to prove a similar correspondence between string diagrams and hypergraphs with interfaces in [8]. As an example, consider the following string diagram



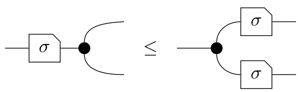
which represents a morphism $1 \to 2$ in $\mathbb{CB}_\Sigma$. If we see this as a graph where the σ box represents a labelled edge the result can be visualised as follows:



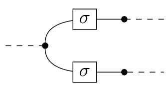
This graph alone does not capture all the information of the string diagram, because the copying on the right is lost. To capture this on the graphs, we endow them with an interface, which can be visualised like so



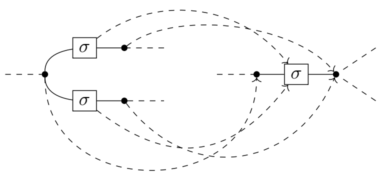
This fairly straightforward translation has a surprising property: Every inequality in $\mathbb{CB}_\Sigma$ corresponds to a morphism between the corresponding graphs that respects the interface. We will prove this assertion in Theorem 4.2.21, but let us consider an example here. If every inequality has a graphical correspondent, that must in particular be true of the axioms of Cartesian bicategories. Consider, for example,



one part of the axiom that σ is a lax comonoid homomorphism. Translating the right-hand side into a graph with interface yields



And we can see that there is a straightforward morphism from the rendering of the right-hand side to that of the left-hand side as follows:



It is important to note that the morphism goes from the right-hand side to the left-hand side, opposite to the direction of the inequality. We have seen this ordering with a morphism in the opposite direction before in Definition 1.3.6, the definition of $\mathsf{Cospan}^{\sim}\mathcal{C}$. In fact, it will turn out that hypergraphs with interfaces can be modelled as a full subcategory of $\mathsf{Cospan}^{\sim}\,\mathsf{FHyp}_{\Sigma}$, where FHyp_{Σ} denotes the category of finite hypergraphs with edges labelled in Σ. To be more precise, it is the full subcategory on cospans $n \to G \leftarrow m$, where n and m are discrete, that is they consist only of vertices with no edges. We call these cospans discrete and denote that full subcategory as $\mathsf{DiscCospan}^{\sim}\,\mathsf{FHyp}_{\Sigma}$. The insight of Chandra and Merlin then translates into Theorem 4.2.21 which proves that $\mathbb{CB}_{\Sigma}$ and $\mathsf{DiscCospan}^{\sim}\,\mathsf{FHyp}_{\Sigma}$ are isomorphic Cartesian bicategories. This correspondence between the string diagrams in $\mathbb{CB}_{\Sigma}$ and discrete cospans of hypergraphs first appeared in [13]. In [8] we proved it to furthermore respect the ordering, making it an isomorphism. To be self-contained, we will construct this isomorphism here from first principles, but it will turn out to be the same one.

Outline of the Chapter In Section 4.1, we will first construct an isomorphism in the special case of $\Sigma = \varnothing$, where hypergraphs are essentially just sets. Since every set is discrete, $\mathsf{DiscCospan}^{\sim}\,\mathsf{FHyp}_{\Sigma}$ agrees with $\mathsf{Cospan}^{\sim}\,\mathsf{FinSet}$, the category of cospans of finite sets. Since Proposition 3.1.6 gives a universal property of $\mathbb{CB}_{\Sigma}$, it suffices to prove that $\mathsf{Cospan}^{\sim}\,\mathsf{FinSet}$ satisfies the same universal property. This can be seen as coming from a nice universal property of FinSet as the free Cocartesian category generated from a single object.

In Section 4.2, we will introduce the category of Σ-hypergraphs which can be seen as the presheaf category over $\mathcal{C}_{\Sigma}$, which will be introduced in Definition 4.2.1. Intuitively, $\mathcal{C}_{\Sigma}$ consists of loose vertices and edges, which are then glued together to form hypergraphs. We can then use a construction that is closely related to the Coyoneda-Lemma, which takes a hypergraph apart into its components. In categorical language, we take the presheaf apart into a coproduct of the representable presheaves that form it. For hypergraphs this means that we associate a hypergraph G with another hypergraph $\mathrm{D}(G)$, which we call a disconnected hypergraph, that is a coproduct of the hyperedges and vertices of G, forgetting about how they were connected. We can then use the results of the previous section to glue $\mathrm{D}(G)$ back together to obtain G, a construction that only involves vertices and not edges. Taking this together with a technical lemma (Lemma 4.2.20) that relates morphisms between disconnected hypergraphs to inequalities in $\mathbb{CB}_\Sigma$, allows us to prove the main Theorem 4.2.21.

In Section 4.3 we will then expand on this result by considering a presentation (Σ, E) of the Frobenius theory $\mathbb{CB}_{\Sigma/E}$. We will show that the inequalities in E can be translated into morphisms of hypergraphs and from E, seen in this way, we can construct a class of morphisms $\mathcal{W}(E)$ in FHyp_Σ that is a system of witnesses. It will turn out that $\mathbb{CB}_{\Sigma/E} \cong \mathsf{DiscCospan}^{\mathcal{W}(E)}\,\mathsf{FHyp}_\Sigma$. This is of great importance for the completeness proof in Chapter 5 as it gives a categorical characterisation of the properties that are provable in $\mathbb{CB}_{\Sigma/E}$ as those that correspond to a morphism in $\mathcal{W}(E)$, that is those morphisms that are constructible from E via coproducts, composition, pushout and presplits.

4.1 Empty signature

In the special case of $\Sigma = \varnothing$, we want to give a simple description of $\mathbb{CB}_\varnothing$. Let FinSet denote the skeleton of the category of finite sets: FinSet has as objects the natural numbers and a morphism $n \to m$ is a function $\{0, \ldots, n-1\} \to \{0, \ldots, m-1\}$. Addition of natural numbers induces a monoidal product on FinSet with monoidal unit 0. This monoidal category is easily seen to be symmetric.

It is well-known that FinSet is the free category with finite coproducts generated by object 1. This translates into the following universal property for FinSet:

Theorem 4.1.1. *For any category $\mathcal{C}$ with finite coproducts and object $X \in$*

$\mathcal{C}$ *there is a unique coproduct-preserving functor* $F\colon \mathsf{FinSet} \to \mathcal{C}$ *with* $F(1) = X$.

Since for any Cartesian bicategory $\mathcal{B}$, the monoidal product on $\mathcal{B}$ induces coproducts in $\mathrm{Comap}(\mathcal{B})$, we can apply Theorem 4.1.1 to obtain a functor $\mathsf{FinSet} \to \mathrm{Comap}(\mathcal{B})$ for any object $X \in \mathcal{B}$. We can use this to show that $\mathsf{Cospan}^{\sim}\,\mathsf{FinSet}$ is the free Cartesian bicategory on one object, which is the same universal property as $\mathbb{CB}_{\varnothing}$ has. They are therefore isomorphic.

This gives a new perspective on the relationship between morphisms of $\mathbb{CB}_{\varnothing}$ and cospans of finite sets that was already presented in [13].

In [8] we improved on that bijection and proved that it in fact gives an isomorphism of Cartesian bicategories. We will obtain the same isomorphism here via the universal property.

Theorem 4.1.2. *For every strict Cartesian bicategory* $\mathcal{B}$ *and object* $X \in \mathcal{B}$, *there is a unique morphism of Cartesian bicategories* $\mathcal{F}\colon \mathsf{Cospan}^{\sim}\,\mathsf{FinSet} \to \mathcal{B}$ *such that* $F(1) = X$, *that is* $\mathsf{Cospan}^{\sim}\,\mathsf{FinSet}$ *is the free Cartesian bicategory on one object.*

Proof. Let $X \in \mathcal{B}$. Consider $\mathrm{Comap}(\mathcal{B})$ as a Cocartesian category. Then by Theorem 4.1.1, there is a unique coproduct-preserving functor $F\colon \mathsf{FinSet} \to \mathrm{Comap}(\mathcal{B})$, with $F(1) = X$.

This extends to a unique morphism $\mathcal{F}\colon \mathsf{Cospan}^{\sim}\,\mathsf{FinSet} \to \mathcal{B}$, as we will show now.

- To show existence of such an $\mathcal{F}$, let

  $$\mathcal{F}(n \xrightarrow{f} k \xleftarrow{g} m) = F(f) \mathbin{;} F(g)^{\mathrm{op}}$$

  We want to prove that this $\mathcal{F}$ is indeed a morphism of Cartesian bicategories. Since F preserves coproducts, $\mathcal{F}$ is monoidal and it is easy to see that it preserves identities. It furthermore preserves the ordering by Lemma 1.2.10. The only remaining point is to show that $\mathcal{F}$ preserves composition.

  Composition in $\mathsf{Cospan}^{\sim}\,\mathsf{FinSet}$ is defined via any weak pushout, but FinSet even allows for pushouts. In the presence of pushouts, every weak pushout allows for mutual morphisms from and to the pushout, which $\mathsf{Cospan}^{\sim}\,\mathsf{FinSet}$ sees as equivalent. It therefore suffices to prove that if

$$\begin{tikzcd}[column sep=small, row sep=small] & p & \\ n \arrow[ur, "h"] & & m \arrow[ul, "j"'] \\ & k \arrow[ul, "f"] \arrow[ur, "g"'] & \end{tikzcd}$$

is a pushout in FinSet, then $F(h)\,;\,F(j)^{\mathrm{op}} = F(f)^{\mathrm{op}}\,;\,F(g)$. We can consider the fibers over every element of p separately and since we can take coproducts, we may assume $p = 1$. In that case, $F(h)\,;\,F(j)^{\mathrm{op}}$ is known as a spider, it is a composition of $n-1$ multiplications followed by $m-1$ comultiplications. That $F(h)\,;\,F(j)^{\mathrm{op}} = F(f)^{\mathrm{op}}\,;\,F(g)$ is then an instance of the Spider Theorem, found as Theorem 6.11 in [21].

This shows that $\mathcal{F}$ is a morphism of Cartesian bicategories.

- To see that $\mathcal{F}$ is necessarily unique, let $\mathcal{G}\colon \mathsf{Cospan}^{\sim}\,\mathsf{FinSet} \to \mathcal{B}$ be another morphism with $\mathcal{G}(1) = X$. Let

  $$G\colon\ \mathsf{FinSet} \to \mathrm{Comap}(\mathcal{B})$$

  be defined on objects as $G(n) = \mathcal{G}(n)$ and on morphisms as

  $$G(f) = \mathcal{G}(n \xrightarrow{f} m \xleftarrow{\mathrm{id}} m)$$

  Since $\mathcal{G}$ preserves composition, it is easy to see that $\mathcal{G}$ is uniquely determined by G as

  $$\mathcal{G}(n \xrightarrow{f} k \xleftarrow{g} m) = G(f)\,;\,G(g)^{\mathrm{op}}$$

  But G preserves coproducts, because $\mathcal{G}$ is monoidal. Therefore, G satisfies the conditions of Theorem 4.1.1 and we have $G = F$, hence $\mathcal{G} = \mathcal{F}$.

Therefore, there is a unique such morphism $\mathcal{F}\colon \mathsf{Cospan}^{\sim}\,\mathsf{FinSet} \to \mathcal{B}$. □

Corollary 4.1.3. $\mathbb{CB}_{\varnothing} \cong \mathsf{Cospan}^{\sim}\,\mathsf{FinSet}$

The isomorphism $\mathbb{CB}_{\varnothing} \cong \mathsf{Cospan}^{\sim}\,\mathsf{FinSet}$ is more explicitly spelled out in Figure 1. Since it respects the symmetric monoidal structure and composition, this data defines it uniquely.

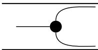
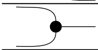
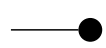
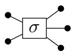
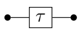
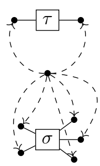
$\mathbb{CB}_\varnothing$	Cospan$^\sim$ FinSet
	$0 \to 0 \leftarrow 0$
	$1 \to 1 \leftarrow 1$
	$1 \to 1 \leftarrow 2$
	$2 \to 1 \leftarrow 1$
	$1 \to 1 \leftarrow 0$
	$0 \to 1 \leftarrow 1$

Figure 1: The isomorphism from Corollary 4.1.3

4.2 String diagrams and hypergraphs

In line with [19] and like we explored in [8], we will construct an isomorphism between $\mathbb{CB}_\Sigma$ and cospans of Σ-hypergraphs. In a hypergraph, edges are generalised to hyperedges that are allowed to join an arbitrary number of vertices. In a Σ-hypergraph, there is one type of hyperedge for each $\sigma \in \Sigma$. We will visualise a vertex in a hypergraph as • and, for example, $\sigma \in \Sigma_{2,3}$ and $\tau \in \Sigma_{1,1}$ would be visualised as  and  respectively. The vertex • together with these boxes can be combined into a category that we call $\mathcal{C}_\Sigma$, that can be visualised as follows (only depicting non-identity morphisms):



This category $\mathcal{C}_\Sigma$ captures the essence of the signature. Formally, we give it one designated object $*$ (representing the vertex) and one object for each $\sigma \in \Sigma$. The only non-identity morphisms have type $* \to \sigma$ and for $\sigma \in \Sigma_{n,m}$ there are $n + m$ of them as in the example above.

Definition 4.2.1. To every monoidal signature we can associate a locally finite category $\mathcal{C}_\Sigma$ with set of objects $\{*\} \cup \Sigma$, and if $\sigma \in \Sigma_{n,m}$

then there are $n + m$ morphisms $* \to \sigma$ in $\mathcal{C}_\Sigma$. The only other morphisms in $\mathcal{C}_\Sigma$ are identity morphisms. Note that there is no pair of composable morphisms that doesn't involve an identity, so associativity of composition is vacuously true.

Remark 4.2.2. The category $\mathcal{C}_\Sigma$ contains less information then Σ, most importantly the division between which wires to draw on the left and which to draw on the right is lost. Intuitively that information is not essential, because we can bend wires.

Remark 4.2.3. If Σ is a multisorted monoidal signature over S, $\mathcal{C}_\Sigma$ is defined to have objects $S \cup \Sigma$ and if $\sigma \in \Sigma_{s,t}$ then there is a morphism $x \to \sigma$ for each occurrence of x in s and t. You can think of $s \in S$ as a coloured vertex.

The category of Σ-hypergraphs is now obtained by taking copies of these basic building blocks that $\mathcal{C}_\Sigma$ defines and gluing them together. Categorically this act of gluing corresponds to colimits and it turns out that the result from freely adjoining colimits to a category $\mathcal{C}$ is the presheaf category $[\mathcal{C}^{\mathrm{op}}, \mathsf{Set}]$, as discussed e.g. in [23]. The representable presheaf corresponding to $*$ will be the vertex, the other representable presheaves will be hyperedges.

Definition 4.2.4. Let Σ be a monoidal signature. The category of Σ-labelled hypergraphs, denoted Hyp_Σ is the presheaf category $[\mathcal{C}_\Sigma^{\mathrm{op}}, \mathsf{Set}]$. A hypergraph is called finite if it consists of finitely many vertices and finitely many hyperedges. We denote the full subcategory on finite hypergraphs as FHyp_Σ.

Remark 4.2.5. For a multisorted signature Σ, a presheaf over $\mathcal{C}_\Sigma$ is a vertex-coloured hypergraph – one colour for each sort.

Let us unpack Definition 4.2.4 a bit. A Σ-hypergraph $G\colon \mathcal{C}_\Sigma^{\mathrm{op}} \to \mathsf{Set}$ consists of $G(*) \in \mathsf{Set}$ which we call the vertex set of G and $G(\sigma)$ for $\sigma \in \Sigma$ which we call the set of σ-hyperedges. Since there are $n+m$ morphisms $* \to \sigma$ in $\mathcal{C}_\Sigma$, we have $n + m$ morphisms $G(\sigma) \to G(*)$, which give the $n + m$ vertices connected to a hyperedge. Usually, these maps are grouped together as a morphism $G(\sigma) \to G(*)^{n+m}$ or as a span $G(*)^n \leftarrow G(\sigma) \to G(*)^m$, as we did in [8]. Here the left arrow maps a hyperedge to the source vertices and the right arrow maps it to the target vertices. Furthermore, a morphism $f\colon G \to H$ between hypergraphs, i.e. a natural transformation between the presheaves, comes with functions $f_*\colon G(*) \to H(*)$ and $f_\sigma\colon G(\sigma) \to H(\sigma)$ such that

$$\begin{tikzcd}
G(*)^n \arrow[d, "f_*^n"] & G(\sigma) \arrow[l] \arrow[r] \arrow[d, "f_\sigma"] & G(*)^m \arrow[d, "f_*^m"] \\
H(*)^n & H(\sigma) \arrow[l] \arrow[r] & H(*)^m
\end{tikzcd}$$

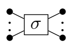
is a commutative diagram of functions.

We will use the result about empty signatures to prove a relationship between $\mathbb{CB}_\Sigma$ and Hyp_Σ.

Definition 4.2.6. The Cartesian bicategory of discrete cospans of finite Hypergraphs, denoted as $\mathsf{DiscCospan}^\sim\,\mathsf{FHyp}_\Sigma$ is the full subcategory of $\mathsf{Cospan}^\sim\,\mathsf{FHyp}_\Sigma$ on objects the natural numbers, seen as hypergraphs with no edges. For a cospan $n \xrightarrow{\iota} G \xleftarrow{\omega} m$, we call ι and ω the interface.

Remark 4.2.7. In the special case of $\Sigma = \varnothing$, the category $\mathsf{Hyp}_\varnothing$ simplifies. Since $\mathcal{C}_\varnothing$ is the trivial one-object category, we have $\mathsf{Hyp}_\varnothing = \mathsf{Set}$ and therefore $\mathsf{DiscCospan}^\sim\,\mathsf{FHyp}_\varnothing = \mathsf{Cospan}^\sim\,\mathsf{FinSet}$.

Definition 4.2.8. For a hypergraph G, we can see the set $G(*)$ as a hypergraph with no edges and seen like this we will denote it as $V(G)$. This gives a functor $V\colon\ \mathsf{Hyp}_\Sigma \to \mathsf{Hyp}_\Sigma$.

As usual for presheaf categories, there is a functor $Y\colon \mathcal{C}_\Sigma \to \mathsf{Hyp}_\Sigma$, that maps $X \in \mathcal{C}_\Sigma$ to the functor $\mathrm{Hom}(\_, X)\colon \mathcal{C}_\Sigma^{\mathrm{op}} \to \mathsf{Set}$, called the Yoneda-embedding. In our case, the Yoneda-embedding is intuitive, we have $Y(*) = \bullet = 1$ the vertex and $Y(\sigma) =$  an isolated σ-hyperedge. Note that each of these naturally comes with an interface: It is natural to equip $Y(*) = 1$ with the interface $1 \to 1 \leftarrow 1$ and for $\sigma \in \Sigma_{n,m}$, it is natural to choose the n vertices we draw to the left as the left interface and the other ones as the right, so $n \to Y(\sigma) \leftarrow m$. We call these canonical cospans.

Lemma 4.2.9. *There is a morphism $\pi\colon \mathbb{CB}_\Sigma \to \mathsf{DiscCospan}^\sim\,\mathsf{FHyp}_\Sigma$ that extends the correspondence of Figure 1 by mapping $\sigma \in \Sigma_{n,m}$ to $n \to Y(\sigma) \leftarrow m$. This morphism is one-to-one on objects.*

Proof. Immediate by the universal property of $\mathbb{CB}_\Sigma$ in Proposition 3.1.6. $\square$

Remark 4.2.10. To fix notation, given $R\colon n \to m$ in $\mathbb{CB}_\Sigma$, we will denote a representative of the $\sim$-equivalence class of $\pi(R)$ as $n \xrightarrow{\iota_R} \mathcal{G}_R \xleftarrow{\omega_R} m$. We will call such a $\mathcal{G}_R$ a universal graph of R. It is important to keep in mind that $\mathcal{G}_R$ is not unique, not even in the usual sense of the word in category theory as unique up to isomorphism. Instead, $\mathcal{G}_R$ is only unique up to mutual morphism. Nevertheless, the properties of $\mathcal{G}_R$ that are of interest to us here, do not depend on the particular choice of representative.

If, for some hypergraph G, $n \to G \leftarrow m$ happens to be a representative of $\pi(R)$, we will also write $\pi(R) = n \to G \leftarrow m$ taking the equivalence class as implicit.

We want to prove this morphism π to be an isomorphism of Cartesian bicategories. For that we need to prove that it is full and faithful, so by Lemma 1.1.9 it suffices to prove fullness and order-reflection.

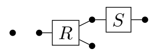
To this end we use Corollary 4.1.3, seeing $\mathsf{Cospan}^\sim\,\mathsf{FinSet}$ as a subcategory of $\mathsf{DiscCospan}^\sim\,\mathsf{FHyp}_\Sigma$ of those cospans $n \to k \leftarrow m$ where k consists of only vertices with no edges. But we can also see $\mathbb{CB}_\varnothing$ as a subcategory of $\mathbb{CB}_\Sigma$ in a straightforward manner. Corollary 4.1.3 then tells us that for hypergraphs with no edges, π is already full and faithful. Intuitively, this means that we only have to concern ourselves with edges and we can let Corollary 4.1.3 handle the vertices.

More precisely, we will first show that every discrete cospan of hypergraphs arises from one that is disconnected, i.e. one where different edges don't have vertices in common. So, for an arbitrary hypergraph G, we construct the disconnected hypergraph on its edges, then glue the vertices back together to obtain G. Since this gluing only involves vertices, it is the jurisdiction of Corollary 4.1.3, so we can restrict our attention to disconnected cospans.

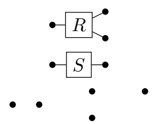
Definition 4.2.11. A disconnected cospan is one that can be written as a sum composed of $n \to Y(\sigma) \leftarrow m$ for $\sigma \in \Sigma_{n,m}$ and $1 \to 1 \leftarrow 1$. In other words, a disconnected hypergraph is one that arises as a coproduct of hypergraphs in the image of Y, that is a coproduct of hyperedges and isolated vertices.

Starting from a hypergraph G, we want to take it apart into the basic components it is made of, to glue them back together later. We will explain that construction with an example before looking at the formal definition.

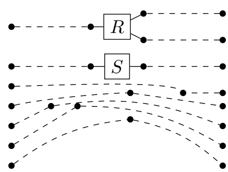
Example 4.2.12. Let $\Sigma_{1,2} = \{R\}$ and $\Sigma_{1,1} = \{S\}$. Consider the following Σ-hypergraph G:



We have $G(R) = 1$, $G(S) = 1$ and $G(*) = 5$. Drawing these components individually gives



This is what we will call the disconnect $\mathrm{D}(G)$. It contains the vertex set of G and – independently – a disconnected copy of every hyperedge in G. Each of these components is equipped with a canonical interface, so taking all of them together gives $7 \to \mathrm{D}(G) \leftarrow 8$ visualised as



Definition 4.2.13. Let G be a Σ-hypergraph. The disconnect of G, denoted $\mathrm{D}(G)$, is the coproduct[1]

$$\coprod_{r \in \mathcal{C}_\Sigma, x \in G(r)} Y(r)$$

In words, $\mathrm{D}(G)$ has a disconnected edge for every edge of G and additionally as many isolated vertices as G has vertices. Since every $Y(r)$ for $r \in \mathcal{C}_\Sigma$ has a canonical interface, so does $\mathrm{D}(G)$ and we get a canonical cospan $\overline{n} \to \mathrm{D}(G) \leftarrow \overline{m}$.

[1]The pairs (r, x) with $r \in \mathcal{C}_\Sigma$ and $x \in G(r)$ that this coproduct ranges over are known as the elements of the functor G

Lemma 4.2.14. *The functor* $\mathrm{D}\colon \mathsf{Hyp}_\Sigma \to \mathsf{Hyp}_\Sigma$ *preserves pushouts and for discrete hypergraphs* n*, we have* $\mathrm{D}(n) = n$*.*

Proof.

- To see that D is indeed a functor, note that any morphism $f\colon G \to H$ induces a morphism $f_r\colon G(r) \to H(r)$ for every $r \in \mathcal{C}_\Sigma$. But these are the index sets in the definition of D, so each f_r defines a morphism

$$\coprod_{x \in G(r)} Y(r) \to \coprod_{x \in H(r)} Y(r)$$

which then combine into a morphism $\mathrm{D}(G) \to \mathrm{D}(H)$.

- That D preserves pushouts follows from the fact that in presheaf categories colimits are computed pointwise.
- The last assertion that $\mathrm{D}(n) = n$ follows immediately from the definition.

□

Remark 4.2.15. If G is equipped with an interface as $n \to G \leftarrow m$, we will write the corresponding disconnected cospan as $\overline{n} \to \mathrm{D}(G) \leftarrow \overline{m}$ even though $\overline{n}$ and $\overline{m}$ do *not* depend on n, m. They are only dependent on G. Despite that, this notation will turn out not to be so bad, because $\overline{n}, \overline{m}$ are in some sense universal – every other interface can find itself in there. We will make this precise in Proposition 4.2.16.

As hinted at already, G can be reconstructed from $\mathrm{D}(G)$, by gluing the vertices back together. More precisely, every discrete cospan of hypergraphs $n \to G \leftarrow m$ can be obtained from $\overline{n} \to \mathrm{D}(G) \leftarrow \overline{m}$, by composing with cospans of sets from both sides, as follows:

Proposition 4.2.16. *Given a discrete cospan* $n \to G \leftarrow m$*, take the canonical cospan* $\overline{n} \to \mathrm{D}(G) \leftarrow \overline{m}$*. Since* n, m *are discrete, they give rise to morphisms* $n \to V(G)$ *and* $m \to V(G)$*. Furthermore there are canonical morphisms* $\overline{n} \to V(G)$ *and* $\overline{m} \to V(G)$ *that assign each vertex of* $\mathrm{D}(G)$ *to the vertex in* G *it originated from. We can then reconstruct* $n \to G \leftarrow m$ *as the composite*

$$\begin{tikzcd}[column sep=small] & V(G) & & \mathrm{D}(G) & & V(G) & \\ n \arrow[ur] & & \overline{n} \arrow[ul] \arrow[ur] & & \overline{m} \arrow[ul] \arrow[ur] & & m \arrow[ul] \end{tikzcd}$$

Proof. Let C be the colimit of

$$\begin{tikzcd}
& & C & & \\
V(G) \arrow[urr] & & \mathrm{D}(G) \arrow[u] & & V(G) \arrow[ull] \\
& \overline{n} \arrow[ul] \arrow[ur] & & \overline{m} \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

This colimit can be obtained as two pushouts, one on the left and one on the right. There is an induced morphism $\alpha\colon C \to G$ and we want to prove α to be an isomorphism. Since $\mathrm{D}(G)$ has the same edges as G, α is a bijection on edges. So it suffices to look at vertices. But taking the pushouts along $\overline{n}$ and $\overline{m}$ identifies each vertex in $\mathrm{D}(G)$ with the underlying vertex in G from which it came. Therefore α is an isomorphism. $\square$

Since every discrete cospan of hypergraphs arises from a disconnected one, it suffices to check fullness of π for disconnected cospans. But that is fairly straightforward.

Proposition 4.2.17. *For any disconnected cospan $n \to E \leftarrow m$ there is a morphism $R\colon n \to m$ in $\mathbb{CB}_\Sigma$ such that $\pi(R) = n \to E \leftarrow m$.*

Proof. Since monoidal product and unit in $\mathbb{CB}_\Sigma$ translate to coproduct and initial object under π, it suffices to check fullness on the basic disconnected cospans. But, by definition of π, we have $\pi(\text{——}) = 1 \to 1 \leftarrow 1$ and $\pi(\text{-}\boxed{\sigma}\text{-}) = n \to Y(\sigma) \leftarrow m$ for $\sigma \in \Sigma_{n,m}$. $\square$

We can combine Proposition 4.2.16 and Proposition 4.2.17 to obtain fullness of π.

Lemma 4.2.18. *The functor π is full.*

Proof. Let $n \to G \leftarrow m$ be a discrete cospan of hypergraphs. Then take the decomposition according to Proposition 4.2.16 to obtain

$$\begin{tikzcd}
& V(G) & & \mathrm{D}(G) & & V(G) & \\
n \arrow[ur] & & \overline{n} \arrow[ul] \arrow[ur] & & \overline{m} \arrow[ul] \arrow[ur] & & m \arrow[ul]
\end{tikzcd}$$

Since the leftmost and rightmost cospans do not involve hyperedges, they are in the image of π by Corollary 4.1.3 and by Proposition 4.2.17

therefore also $n \to G \leftarrow m$ is in the image of π. Therefore, π is full. $\square$

What we have left to prove therefore is that π also reflects the ordering. That is we need to prove that given $R, S \colon n \to m$ in $\mathbb{CB}_\Sigma$, whenever there is a commutative diagram

$$\begin{tikzcd}
& \mathcal{G}_S \arrow[dd, "\alpha"] & \\
n \arrow[ur, "\iota_S"] \arrow[dr, "\iota_R"'] & & m \arrow[ul, "\omega_S"'] \arrow[dl, "\omega_R"] \\
& \mathcal{G}_R &
\end{tikzcd}$$

then already $R \leq S \in \mathbb{CB}_\Sigma$. To that end we will first prove that $\mathbb{CB}_\Sigma$ allows to decompose its morphism in a way that is analogous to Proposition 4.2.16.

Lemma 4.2.19. *Let $R \colon n \to m$ be a morphism in $\mathbb{CB}_\Sigma$. Then we can pick a representative $n \to \mathcal{G}_R \leftarrow m$ of $\pi(R)$ and a decomposition*

$$-\boxed{R}- \;=\; -\boxed{\alpha}-\boxed{R'}-\boxed{\beta}-$$

of R such that

- *$\pi(R') = \overline{n} \to \mathrm{D}(\mathcal{G}_R) \leftarrow \overline{m}$ the canonical cospan on the disconnect of $\mathcal{G}_R$.*

- *$\pi(-\boxed{\alpha}-) = n \to V(\mathcal{G}_R) \leftarrow \overline{n}$ and likewise, $\pi(-\boxed{\beta}-) = \overline{m} \to V(\mathcal{G}_R) \leftarrow m$.*

Proof. We can prove this by induction on the structure of R. The basic building blocks are readily checked. Given such decompositions on R_1 and R_2 it is straightforward to construct one on $R = R_1 \otimes R_2$. The remaining case is that of composition, which follows by Proposition 4.2.14. $\square$

The relevance of Lemma 4.2.19 is that it allows us to reduce the proof of order-reflection to disconnected cospans. Given a commutative diagram

$$\begin{tikzcd}
 & \mathcal{G}_S \arrow[dd, "\alpha"] & \\
n \arrow[ur, "\iota_S"] \arrow[dr, "\iota_R"'] & & m \arrow[ul, "\omega_S"'] \arrow[dl, "\omega_R"] \\
 & \mathcal{G}_R &
\end{tikzcd}$$

we get an induced diagram

$$\begin{tikzcd}
\overline{n} \arrow[r] \arrow[d, "f"'] & \mathrm{D}(\mathcal{G}_S) \arrow[d, "\mathrm{D}(\alpha)"] & \overline{m} \arrow[l] \arrow[d, "g"] \\
\overline{n'} \arrow[r] & \mathrm{D}(\mathcal{G}_R) & \overline{m} \arrow[l]
\end{tikzcd}$$

where f and g map the left and right interfaces of every component of $\mathrm{D}(\mathcal{G}_S)$ as $\mathrm{D}(\alpha)$ specifies.

We will first prove that diagrams like this give rise to inequalities in $\mathbb{CB}_\Sigma$, then prove the general case.

Lemma 4.2.20. *Let $n \to E \leftarrow m$ and $n' \to E' \leftarrow m'$ be disconnected cospans and*

$$\begin{tikzcd}
n \arrow[r] \arrow[d, "f"'] & E \arrow[d, "h"] & m \arrow[l] \arrow[d, "g"] \\
n' \arrow[r] & E' & m' \arrow[l]
\end{tikzcd}$$

a commutative diagram. Let $R\colon n' \to m'$ and $S\colon n \to m$ be morphisms in $\mathbb{CB}_\Sigma$ such that

$$\pi(R) = n' \to E' \leftarrow m'$$

and

$$\pi(S) = n \to E \leftarrow m$$

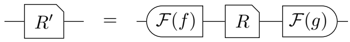
Let $\mathcal{F}\colon \mathsf{Cospan}^\sim\,\mathsf{FinSet} \to \mathbb{CB}_\Sigma$ be the morphism from Theorem 4.1.2, so that f and g give rise to comaps in $\mathbb{CB}_\Sigma$, more precisely let

$$\mathcal{F}(n \xrightarrow{f} n' \xleftarrow{\mathrm{id}} n') = -\boxed{\mathcal{F}(f)}-$$

and

$$\mathcal{F}(m \xrightarrow{g} m' \xleftarrow{\mathrm{id}} m') = -\boxed{\mathcal{F}(g)}-$$

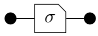
Let now

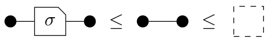


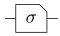
Then $R' \leq S$.

Proof. Since both cospans are disconnected, f and g are uniquely determined by h. Since E' is a coproduct of $Y(*)$ and $Y(\sigma)$, we can consider the fiber for each summand separately. If $E' = Y(*)$, then also E has no edges, so the result follows from Corollary 4.1.3

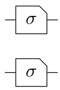
We can therefore assume $E' = Y(\sigma)$ for some $\sigma \in \Sigma_{i,j}$, therefore $n' = i$ and $m' = j$. Since h preserves labels, E must be the sum of a number, say k, of copies of σ. We need only consider the cases of $k = 0, 1, 2$, all other cases reduce to these base cases.

- If $k = 0$, then $S =$  and $R' =$ . We get

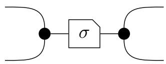


- If $k = 1$, then h is the identity, and we get $R = R' = S =$ , so there is nothing to show.

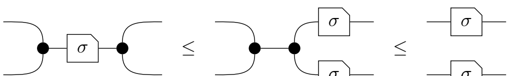
- If $k = 2$, then

$S =$ 

and

$R' =$ 

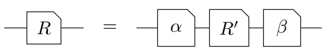
We get



□

Theorem 4.2.21 (Graphical Theorem)**.** $\pi \colon \mathbb{CB}_\Sigma \to \mathsf{DiscCospan}^{\sim}\,\mathsf{FHyp}_\Sigma$ *is an isomorphism of Cartesian bicategories.*

Proof. Thanks to Lemma 4.2.18, it remains to prove that π reflects the ordering. Let $R, S \colon n \to m$ be morphisms in $\mathbb{CB}_\Sigma$ such that $\pi(R) \leq \pi(S)$. Choose $\mathcal{G}_R$ and $\mathcal{G}_S$ as in Lemma 4.2.19. Let



and

$$-\boxed{S}- \quad = \quad -\boxed{\delta}-\boxed{S'}-\boxed{\gamma}-$$

be the corresponding decompositions. Since $\pi(R) \leq \pi(S)$, we get a diagram

$$\begin{tikzcd}
 & \mathcal{G}_S \arrow[dd, "\phi"] & \\
n \arrow[ur, "\iota_S"] \arrow[dr, "\iota_R"'] & & m \arrow[ul, "\omega_S"'] \arrow[dl, "\omega_R"] \\
 & \mathcal{G}_R &
\end{tikzcd}$$

which extends to a diagram

$$\begin{tikzcd}
 & V(\mathcal{G}_S) \arrow[dd] & \overline{n'} \arrow[l] \arrow[r] \arrow[dd, "f"] & \mathrm{D}(\mathcal{G}_S) \arrow[dd, "\mathrm{D}(\phi)"] & \overline{m'} \arrow[l] \arrow[r] \arrow[dd, "g"] & V(\mathcal{G}_S) \arrow[dd] & \\
n \arrow[ur] \arrow[dr] & & & & & & m \arrow[ul] \arrow[dl] \\
 & V(\mathcal{G}_R) & \overline{n} \arrow[l] \arrow[r] & \mathrm{D}(\mathcal{G}_R) & \overline{m} \arrow[l] \arrow[r] & V(\mathcal{G}_R) &
\end{tikzcd}$$

Let $\mathcal{F}\colon \mathsf{Cospan}^{\sim}\,\mathsf{FinSet} \to \mathbb{CB}_{\Sigma}$ be the morphism from Theorem 4.1.2 and let $\mathcal{F}(n \xrightarrow{f} n' \xleftarrow{\mathrm{id}} n') = -\boxed{\mathcal{F}(f)}-$ and $\mathcal{F}(m \xrightarrow{g} m' \xleftarrow{\mathrm{id}} m') = -\boxed{\mathcal{F}(g)}-$. If we take the above diagram apart and rearrange the pieces we get the following:

1.

$$\begin{tikzcd}
 & V(\mathcal{G}_S) \arrow[dd] & & \\
n \arrow[ur] \arrow[dr] & & & \overline{n'} \arrow[ull] \arrow[dl, "f"] \\
 & V(\mathcal{G}_R) & \overline{n} \arrow[l] &
\end{tikzcd}$$

which is a diagram of two cospans, so in $\mathsf{Cospan}^{\sim}\,\mathsf{FinSet}$ the bottom cospan is less than the top one. Therefore, by applying $\mathcal{F}$, which preserves the ordering, we get

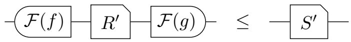
$$-\boxed{\alpha}-\boxed{\mathcal{F}(f)}- \quad \leq \quad -\boxed{\delta}-$$

Note that $-\boxed{\mathcal{F}(f)}-$ seems to point in the wrong direction, which is because $\mathcal{F}(f)$ is a comap rather than a map.

2.

$$\begin{tikzcd}
\overline{n'} \arrow[r] \arrow[d, "f"] & \mathrm{D}(\mathcal{G}_S) \arrow[d, "\mathrm{D}(\phi)"] & \overline{m'} \arrow[l] \arrow[d, "g"] \\
\overline{n} \arrow[r] & \mathrm{D}(\mathcal{G}_R) & \overline{m} \arrow[l]
\end{tikzcd}$$

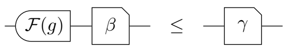
to which we can apply Lemma 4.2.20 and obtain



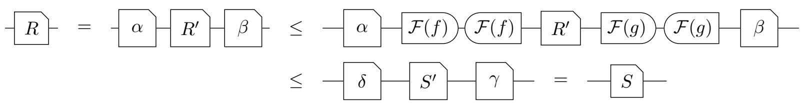
3.

$$\begin{tikzcd}
 & & V(\mathcal{G}_S) \arrow[dd] & \\
\overline{m'} \arrow[urr] \arrow[dr, "g"'] & & & m \arrow[ul] \arrow[dl] \\
 & \overline{m} \arrow[r] & V(\mathcal{G}_R) &
\end{tikzcd}$$

which again gives an inequality in $\mathsf{Cospan}^{\sim}\,\mathsf{FinSet}$ and by applying $\mathcal{F}$ we get



Putting together all the pieces of the puzzle, we finally get

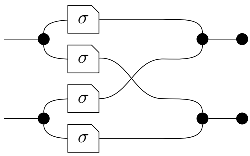


□

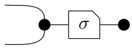
Example 4.2.22. If $R, S\colon n \to m$ are morphisms in $\mathbb{CB}_{\Sigma}$, then Theorem 4.2.21 allows us to decide if $R \leq S$ holds by checking for the existence of a morphism $\alpha\colon \mathcal{G}_S \to \mathcal{G}_R$ that makes the diagram

$$\begin{tikzcd}
 & \mathcal{G}_S \arrow[dd, "\alpha"] & \\
n \arrow[ur, "\iota_S"] \arrow[dr, "\iota_R"'] & & m \arrow[ul, "\omega_S"'] \arrow[dl, "\omega_R"] \\
 & \mathcal{G}_R &
\end{tikzcd}$$

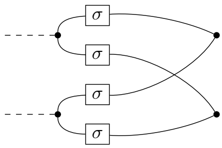
commute. To better let the reader appreciate the use of this, let us consider an example. Let $\Sigma = \Sigma_{1,1} = \{\sigma\}$. Consider the morphism $R\colon 2 \to 0$ in $\mathbb{CB}_\Sigma$ defined as



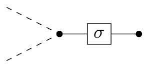
and $S\colon 2 \to 0$ defined as



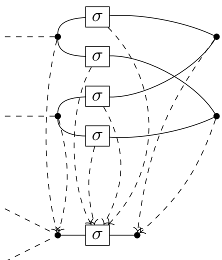
These have corresponding universal graphs, given by $\mathcal{G}_R$ which can be illustrated as



and $\mathcal{G}_S$ illustrated as



To prove that $R \leq S$, it suffices to find an interface-preserving homomorphism $\mathcal{G}_S \to \mathcal{G}_R$. Such a morphism is visualised as follows:



which shows that indeed $R \leq S$.

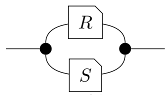
4.3 A combinatorial characterisation for $\mathbb{CB}_{\Sigma/E}$

In $\mathbb{CB}_\Sigma$, every inequality is mediated by a morphism of hypergraphs via Theorem 4.2.21. In a general Frobenius theory, this is often not true. We can postulate an axiom $R \leq S$ without there being a morphism between $\mathcal{G}_R$ and $\mathcal{G}_S$ in either direction. That is a categorically strange situation, because category theory is all about morphisms. But there is a trick we can use here, which is that thanks to Proposition 1.1.8 we have $R \leq S$ if and only if $R \cap S = R$. But it turns out that there is always a morphism $\mathcal{G}_R \to \mathcal{G}_{R\cap S}$, which means we are in business.

Proposition 4.3.1. *If $R, S\colon n \to m$ are morphisms in $\mathbb{CB}_\Sigma$ and $\mathcal{G}_{R\cap S}$ a universal graph for $R \cap S$, then $\mathcal{G}_{R\cap S}$ is a weak pushout*

$$\begin{tikzcd} n+m \arrow[r] \arrow[d] & \mathcal{G}_R \arrow[d] \\ \mathcal{G}_S \arrow[r] & \mathcal{G}_{R\cap S} \end{tikzcd}$$

in FHyp_Σ.

Proof. In Proposition 1.1.8 we proved that $R \cap S =$  .

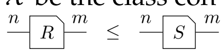
Sending this through the isomorphism $\pi\colon \mathbb{CB}_\Sigma \to \mathsf{DiscCospan}^{\sim}\,\mathsf{FHyp}_\Sigma$ gives the composite

$$\begin{tikzcd} & & & \mathcal{G}_{R\cap S} & & & \\ & n \arrow[urr] & & \mathcal{G}_R + \mathcal{G}_S \arrow[u] & & m \arrow[ull] & \\ n \arrow[ur] & & n+n \arrow[ul] \arrow[ur] & & m+m \arrow[ul] \arrow[ur] & & m \arrow[ul] \end{tikzcd}$$

which is a weak colimit diagram. It is easy to check that this also makes the above diagram a weak pushout. ☐

We will therefore associate an inequality $R \leq S$ with the induced morphism $\mathcal{G}_R \to \mathcal{G}_{R\cap S}$.

This allows us, starting from a Frobenius theory $\mathcal{T}$, to define a class of morphisms which corresponds to all the inequalities that hold in $\mathcal{T}$.

Definition 4.3.2. Given a Frobenius theory $\mathcal{T}$, let $\mathcal{X}$ be the class consisting of the following morphisms: Whenever 

holds in $\mathcal{T}$ and

$$\begin{array}{ccc} n+m & \longrightarrow & \mathcal{G}_R \\ \downarrow & & \downarrow p \\ \mathcal{G}_S & \longrightarrow & \mathcal{G}_{R\cap S} \end{array}$$

is a weak pushout in FHyp_Σ, then $p \in \mathcal{X}$. Let $\mathcal{W} = \mathcal{W}(\mathcal{X})$ be the witnesses (see Definition 2.3.1) generated from $\mathcal{X}$. We refer to $\mathcal{W}$ as the system of witnesses associated with the theory $\mathcal{T}$.

When we are given a presentation of a Frobenius theory (Σ, E), we have two choices of how to assign a class of witnesses to it. The first is to ignore that we know a set of axioms E, use the class $\mathcal{X}$ that corresponds to all inequalities that hold in $\mathbb{CB}_{\Sigma/E}$ and generate a class of witnesses from it as $\mathcal{W}(\mathcal{X})$. The second and certainly more economic one is to identify the axioms in E with a set of morphisms of hypergraphs and consider the class $\mathcal{W}(E)$, the closure of E under the properties of witnesses. Our goal is to prove that these two constructions coincide. For that it will be useful to start with the less economic one, because $\mathcal{X}$ already has some nice properties that make $\mathcal{W}(\mathcal{X})$ convenient to work with. It turns out that $\mathcal{X}$ satisfies all the preconditions of Lemma 2.3.4:

Lemma 4.3.3. *$\mathcal{X}$ contains identities and is furthermore closed under coproducts and weak pushouts.*

Proof. This simply follows from the fact that inequalities in $\mathcal{T}$ are compatible with monoidal product and composition. $\square$

Lemma 4.3.4 (Workhorse lemma)**.** *Let $\mathbb{CB}_{\Sigma/E}$ be a Frobenius theory and let $\mathcal{W}$ be the corresponding system of witnesses. Let $R, S\colon n \to m$ be morphisms in $\mathbb{CB}_\Sigma$ with corresponding universal graphs $\mathcal{G}_R$, $\mathcal{G}_S$ and let*

$$\begin{tikzcd} & \mathcal{G}_R \arrow[dd, "w"] & \\ n \arrow[ur, "\iota_R"] \arrow[dr, "\iota_S"'] & & m \arrow[ul, "\omega_R"'] \arrow[dl, "\omega_S"] \\ & \mathcal{G}_S & \end{tikzcd}$$

be commutative with $w \in \mathcal{W}$. Then $R = S$ in $\mathbb{CB}_{\Sigma/E}$.

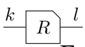
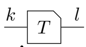
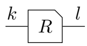
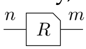
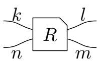
Proof. Let L be the class of all the morphisms that satisfy the property of the Lemma, that is all those morphisms w such that whenever

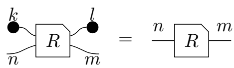
$$\begin{tikzcd} & \mathcal{G}_R \arrow[dd, "w"] & \\ n \arrow[ur, "\iota_R"] \arrow[dr, "\iota_S"'] & & m \arrow[ul, "\omega_R"'] \arrow[dl, "\omega_S"] \\ & \mathcal{G}_S & \end{tikzcd}$$

then $R = S$ in $\mathbb{CB}_{\Sigma/E}$. We want to prove that $\mathcal{W} \subseteq L$. We can use Lemma 2.3.4 to argue inductively.

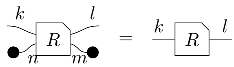
If the property holds for a morphism $w \; ; \; g$, then it holds for w. The property clearly holds for identities and is stable under composition. Therefore it remains to check the property for $\omega = d \in \mathcal{X}$. Hence, d arises from a weak pushout

$$\begin{tikzcd} k + l \arrow[r] \arrow[d] & \mathcal{G}_R \arrow[d, "d"] \\ \mathcal{G}_T \arrow[r] & \mathcal{G}_S \end{tikzcd}$$

such that  $\leq$ . Here, the same $\mathcal{G}_R$ appears with different interfaces. For clarity, we will slightly abuse notation and denote the corresponding string diagrams with the same letter, but labelling the interface explicitly, so $k \to \mathcal{G}_R \leftarrow l$ will be denoted as  and $n \to \mathcal{G}_R \leftarrow m$ as . Combining the two interfaces gives a string diagram  corresponding to $k + n \to \mathcal{G}_R \leftarrow l + m$ such that

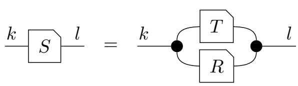


and

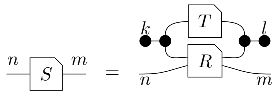


From what we already know for $\mathbb{CB}_\Sigma$, we have $S \leq R$ in $\mathbb{CB}_\Sigma$, so also in $\mathbb{CB}_{\Sigma/E}$. It therefore suffices to show $R \leq S$ in $\mathbb{CB}_{\Sigma/E}$.

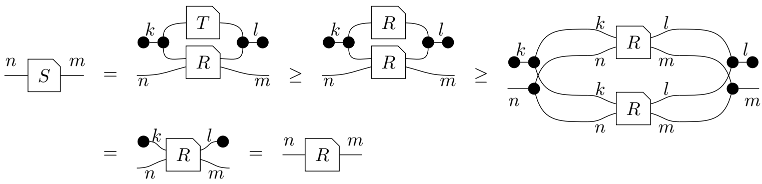
Since $\mathcal{G}_S$ is a weak pushout, by Proposition 4.3.1 we have



and since the interface $n \to \mathcal{G}_S \leftarrow m$ factors through $\mathcal{G}_R$, we have



From this we get



$\square$

This allows us to give a combinatorial characterisation of $\mathbb{CB}_{\Sigma/E}$, similar to the one for $\mathbb{CB}_\Sigma$ in Theorem 4.2.21.

Definition 4.3.5. Let $\mathcal{W}$ be a class of witnesses. Let $\mathsf{DiscCospan}^{\mathcal{W}}\,\mathsf{FHyp}_\Sigma$ be the full subcategory of $\mathsf{Cospan}^{\mathcal{W}}\,\mathsf{FHyp}_\Sigma$ (see Definition 2.3.2) on the natural numbers seen as discrete hypergraphs (compare Definition 4.2.6).

Theorem 4.3.6. *We have* $\mathbb{CB}_{\Sigma/E} \cong \mathsf{DiscCospan}^{\mathcal{W}}\,\mathsf{FHyp}_\Sigma$

Proof. For any inequality $A \leq B$ in E we have a commutative diagram

$$\begin{tikzcd} & \mathcal{G}_A \arrow[d, "e"] & \\ X \arrow[ur, bend left] \arrow[r] \arrow[dr, bend right] & \mathcal{G}_{A\cap B} & Z \arrow[ul, bend right] \arrow[l] \arrow[dl, bend left] \\ & \mathcal{G}_B \arrow[u] & \end{tikzcd}$$

in FHyp_Σ where $e \in \mathcal{W}$. This means that

$$\mathbb{CB}_\Sigma \to \mathsf{DiscCospan}^{\sim}\,\mathsf{FHyp}_\Sigma \to \mathsf{DiscCospan}^{\mathcal{W}}\,\mathsf{FHyp}_\Sigma$$

satisfies E, so by Proposition 3.2.3, there is an induced morphism $\mathbb{CB}_{\Sigma/E} \to \mathsf{DiscCospan}^{\mathcal{W}}\,\mathsf{FHyp}_\Sigma$. This morphism is identity on objects and full since every discrete cospan of hypergraphs comes from a string diagram. It suffices to see that this morphism is faithful, for which it suffices to show that it reflects the ordering. Assuming we have a commutative diagram

$$\begin{tikzcd}
 & \mathcal{G}_A \arrow[d, "w"] & \\
X \arrow[ur, bend left] \arrow[r] \arrow[dr, bend right] & \mathcal{G}_{A'} & Z \arrow[ul, bend right] \arrow[l] \arrow[dl, bend left] \\
 & \mathcal{G}_B \arrow[u] &
\end{tikzcd}$$

in FHyp_Σ with $w \in \mathcal{W}$. Then by Lemma 4.3.4 we have $-\boxed{A}- = -\boxed{A'}-$ and therefore by Theorem 4.2.21 we have

$$-\boxed{A}- \quad = \quad -\boxed{A'}- \quad \leq \quad -\boxed{B}-$$

$\square$

Starting from a presentation of a Frobenius theory (Σ, E), we could also consider the class of witnesses directly generated from the morphisms corresponding to E. In fact, this gives the same class of morphisms.

Theorem 4.3.7. *Let (Σ, E) be a Frobenius theory and let $\mathcal{W}$ be the corresponding system of witnesses. Identify E with the set of morphisms $\mathcal{G}_A \to \mathcal{G}_{A\cap B}$ for $A \leq B \in E$. Then*

$$\mathcal{W}(E) = \mathcal{W}$$

Proof. Since $E \subseteq \mathcal{W}$ and since $\mathcal{W}$ is a class of witnesses, we have $\mathcal{W}(E) \subseteq \mathcal{W}$. We therefore show the converse now. For any inequality $A \leq B$ in E we have a commutative diagram of hypergraphs

$$\begin{tikzcd}
 & \mathcal{G}_A \arrow[d, "e"] & \\
n \arrow[ur, bend left] \arrow[r] \arrow[dr, bend right] & \mathcal{G}_{A\cap B} & m \arrow[ul, bend right] \arrow[l] \arrow[dl, bend left] \\
 & \mathcal{G}_B \arrow[u] &
\end{tikzcd}$$

By the universal property of $\mathbb{CB}_{\Sigma/E}$ in Proposition 3.2.3, this induces a morphism of Cartesian bicategories

$$F\colon \mathsf{DiscCospan}^{\mathcal{W}}\,\mathsf{FHyp}_\Sigma \cong \mathbb{CB}_{\Sigma/E} \to \mathsf{DiscCospan}^{\mathcal{W}(E)}\,\mathsf{FHyp}_\Sigma$$

Since $\mathcal{W} = \mathcal{W}(\mathcal{X})$, it suffices to show $\mathcal{X} \subseteq \mathcal{W}(E)$. Let now $A \leq B$ be any inequality (not necessarily an axiom) that holds in $\mathbb{CB}_{\Sigma/E}$ and

$$\begin{tikzcd} n+m \arrow[r] \arrow[d] & \mathcal{G}_A \arrow[d, "x"] \\ \mathcal{G}_B \arrow[r] & \mathcal{G}_{A\cap B} \end{tikzcd}$$

a weak pushout. Since F respects the ordering, we get $F(A) \leq F(B)$, which by definition means that there is a commutative diagram

$$\begin{tikzcd} & \mathcal{G}_A \arrow[d, "w"] & \\ n \arrow[ur, bend left] \arrow[r] \arrow[dr, bend right] & Z & m \arrow[ul, bend right] \arrow[l] \arrow[dl, bend left] \\ & \mathcal{G}_B \arrow[u] & \end{tikzcd}$$

in FHyp_Σ with $w \in \mathcal{W}(E)$. By the universal property of the weak pushout, we get a morphism $\mathcal{G}_{A\cap B} \to Z$ such that

$$\begin{tikzcd} n+m \arrow[r] \arrow[d] & \mathcal{G}_A \arrow[d, "x"] \arrow[ddr, bend left, "w"] & \\ \mathcal{G}_B \arrow[r] \arrow[drr, bend right] & \mathcal{G}_{A\cap B} \arrow[dr] & \\ & & Z \end{tikzcd}$$

and therefore $x \in \mathcal{W}(E)$. $\square$

Therefore $\mathbb{CB}_{\Sigma/E}$ can be described using witnesses generated in a simpler way:

Corollary 4.3.8. *We have* $\mathbb{CB}_{\Sigma/E} \cong \mathsf{DiscCospan}^{\mathcal{W}(E)}\,\mathsf{FHyp}_\Sigma$.

This has the important consequence that inequalities that hold in $\mathbb{CB}_{\Sigma/E}$ correspond to morphisms in $\mathcal{W}(E)$. This will be fundamental in 5, so it deserves special attention:

Corollary 4.3.9. *For* $A, B\colon n \to m$ *morphisms in* $\mathbb{CB}_\Sigma$*, let* $\alpha\colon \mathcal{G}_A \to \mathcal{G}_{A\cap B}$ *be the canonical morphism. Then* α *is a witness generated from* E *if and only if* $A \leq B$ *in* $\mathbb{CB}_{\Sigma/E}$.

Chapter 5

Completeness

Introduction

This chapter is devoted to finally proving the completeness theorem, as already hinted at in Section 3.3. To that end, we will make heavy use of the results established in Chapter 4, where we saw that we can understand free Cartesian bicategories in terms of hypergraphs. This is already a first step in an entirely categorical perspective on the problem of completeness. It will prove convenient to refer explicitly to a presentation of a Frobenius theory. Since this chapter's result are constructive, for practical applications this presentation can and should be chosen carefully.

In order to show completeness we need to prove that every property that is shared by all models of $\mathbb{CB}_{\Sigma/E}$ already has a proof in $\mathbb{CB}_{\Sigma/E}$. As mentioned in the introduction, our aim is to transfer this problem along the love triangle into the world of hypergraphs. We can translate the set of axioms E into a set of morphisms of hypergraphs, by identifying $R \leq S$ with the morphism $\mathcal{G}_R \to \mathcal{G}_{R \cap S}$ as we did in Section 4.3. It will turn out that models of $\mathbb{CB}_{\Sigma/E}$ correspond to those hypergraphs that satisfy a lifting property, called E-injectivity. Furthermore, we will see that the inequalities that hold in all models correspond to morphisms that satisfy an analogous lifting property with respect to those E-injective objects. We call those morphisms cofibrations.

On the other hand, Chapter 4 has taught us that inequalities in $\mathbb{CB}_{\Sigma/E}$ correspond to witnesses generated by E. This allows us to formulate the question of completeness in entirely categorical terms: Is every cofibration with respect to E-injective objects already a witness?

Seen this way, this question is of general categorical interest. We therefore detach the study from the category of hypergraphs and study it in full generality. The context for that general treatment is the theory of locally finitely presentable categories, categories that come with a well-behaved notion of finite object, called compact objects. Since all string diagrams are finite, their corresponding hypergraphs are finite too, which is why compact objects play an important role for us.

The categorical counterpart of the completeness theorem comes in the form of a mild sufficient condition on the set E which implies that every cofibration between compact objects is already a witness. In order to prove this, we will, starting from an object X, construct an injective object X' with a morphism $X \to X'$ in a way that is inspired by the orthogonal reflection defined in [2].

This construction will be iterative and each step will be done for us by a functor $\mathcal{F}_E$. Intuitively, applying the functor to an object X does the following: For any morphism $f \in E$, $\mathcal{F}_E$ looks for all matchings of the codomain of f in X, and glues the domain of f into X. Logically speaking, for every axiom of our theory, $\mathcal{F}_E$ matches all possible left-hand sides of axioms and glues in the corresponding right-hand sides.

However, it could be that in the process of gluing we introduce new matchings that are not accounted for – which is why the construction needs to be iterated. In the colimit, this iterative application will turn out to yield an injective object. But we will also prove that each step of the construction gives rise to a witness. We can then use a compactness argument to show that for any given cofibration, finitely many applications (and a finite subset of E) suffice. But since witnesses are closed under composition, this will conclude the proof.

Outline of the Chapter In Section 5.1 we will establish a correspondence between Σ-hypergraphs and models of $\mathbb{CB}_\Sigma$, that is Σ-structures. This correspondence is categorically interesting, because for any hypergraph G, its Hom-functor $\mathrm{Hom}(\_, G)$ maps colimits of hypergraphs to limits in Set, which means that it induces a morphism of Cartesian bicategories

$$\mathcal{M}_G \colon \mathbb{CB}_\Sigma \cong \mathsf{DiscCospan}^\sim \mathsf{FHyp}_\Sigma \to \mathsf{Span}^\sim \mathsf{Set} \cong \mathsf{Rel}$$

The Lifting Lemma 5.1.4 will then establish an important connection between logical properties of $\mathcal{M}_G$ and categorical lifting properties of G. Its proof is similar in spirit to the Yoneda Lemma. We will use this to give the categorical translation of the completeness problem that we tackle in the remainder of this chapter.

Section 5.2 gives a short overview over the theory of locally finitely presentable categories based on [2], where the interested reader can find further details.

Section 5.3 is a brief intermezzo that shows how to prove a special case of the completeness theorem, the case for Frobenius theories with no axioms, i.e. with $E = \varnothing$. It is not strictly necessary to do that since it would follow from the general completeness theorem, but it is an instructive example to see what the categorical formulation of completeness boils down to in this special case.

Section 5.4 is the technical meat of the thesis. It develops the categorical machinery necessary to prove the completeness theorem, most importantly the functor $\mathcal{F}_E$, where $\mathcal{F}_E(X)$ is one step towards an "injectivisation" of X. More generally, denoting by $\mathcal{F}_{E^n}$ the n-fold application of $\mathcal{F}_E$, we can construct a sequence

$$X \longrightarrow \mathcal{F}_E(X) \longrightarrow \mathcal{F}_{E^2}(X) \longrightarrow \mathcal{F}_{E^3}(X) \longrightarrow \cdots$$

the colimit of which we call $\mathcal{F}_{E^\omega}(X)$. This object will turn out to be E-injective. But importantly, the morphisms $X \to \mathcal{F}_{E^n}(X)$ furthermore have the property of being witnesses with respect to E. From this we can prove that any cofibration is a witness using a compactness argument.

5.1 Hypergraphs and models

The isomorphism $\pi\colon \mathbb{CB}_\Sigma \to \mathsf{DiscCospan}^\sim\,\mathsf{FHyp}_\Sigma$ that we have constructed in Chapter 4 gives a useful combinatorial perspective on string diagrams, but it has additional important logical consequences. This section is devoted to those.

The first comes from the observation that for any Σ-hypergraph G, the functor $\mathrm{Hom}(\_, G)$ maps finite colimits in FHyp_Σ to limits in Set. This means that it sends coproducts to products and pushouts to pullbacks, so we get

$$\mathcal{M}_G\colon \mathbb{CB}_\Sigma \cong \mathsf{DiscCospan}^\sim\,\mathsf{FHyp}_\Sigma \to \mathsf{Span}^\sim\,\mathsf{Set} \cong \mathsf{Rel}$$

By contravariance of $\mathrm{Hom}(\_, G)$, $\mathcal{M}_G$ is furthermore order-preserving and therefore a morphism of Cartesian bicategories.

Definition 5.1.1. Let G be a Σ-hypergraph. Let $\mathcal{M}_G\colon \mathbb{CB}_\Sigma \to \mathsf{Rel}$ be the morphism of Cartesian bicategories induced by $\mathrm{Hom}(\_, G)$ as described above. We call $\mathcal{M}_G$ the Σ-structure associated with G. If $\mathcal{G}_R$ is a universal graph for $R\colon n \to m$, we write $\mathcal{M}_{\mathcal{G}_R} = \mathcal{U}_R$ and call it a universal model of R.

Intuitively, Σ-hypergraphs and Σ-structures are very similar. A Σ-structure S comes with a relation $S(\sigma) \subseteq S(1)^n \times S(1)^m$ for every $\sigma \in \Sigma_{n,m}$. We can think of this as a hypergraph that has at most one σ-hyperedge between any tuple of vertices. Conversely, given a hypergraph G, the Σ-structure $\mathcal{M}_G$ is obtained in a relatively simple manner. The set $\mathcal{M}_G(\sigma)$ consists of all those tuples of vertices that are joined by some σ-hyperedge in G. The following does therefore not come as a surprise.

Proposition 5.1.2. *For any model $\mathcal{M}\colon \mathbb{CB}_\Sigma \to \mathsf{Rel}$, there is a hypergraph G such that $\mathcal{M} \cong \mathcal{M}_G$.*

Remark 5.1.3. Universal graphs and universal models are only unique up to mutual morphisms, not up to isomorphism. There is therefore no justification to speak of "the" universal model/graph. Nevertheless, we use the notation $\mathcal{G}_R$ for any universal graph of R and $\mathcal{U}_R$ for any universal model of R, because the particular choice of representative does not matter. For example, an inequality $R \leq S$ gives rise to a morphism $\mathcal{G}_S \to \mathcal{G}_R$ independently of which representative is chosen.

Universal graphs play an important role for us. As we have seen, inequalities between string diagrams translate to morphisms between their universal graphs. This translation can be further refined into a property very similar to one appearing in [4].

Lemma 5.1.4 (Lifting Lemma). *Let $A, B\colon n \to m$ in $\mathbb{CB}_\Sigma$ be string diagrams and let $\alpha\colon \mathcal{G}_A \to \mathcal{G}_{A\cap B}$. Let G be a Σ-hypergraph. We have $\mathcal{M}_G(A) \subseteq \mathcal{M}_G(B)$ if and only if for any morphism $f\colon \mathcal{G}_A \to G$, there is a morphism $g\colon \mathcal{G}_{A\cap B} \to G$ such that*

$$\begin{tikzcd}
\mathcal{G}_A \arrow[r, "\alpha"] \arrow[d, "f"] & \mathcal{G}_{A\cap B} \arrow[dl, dashed, "g"] \\
G &
\end{tikzcd}$$

Proof. We prove both implications.

- Assume $\mathcal{M}_G(A) \subseteq \mathcal{M}_G(B)$. By definition of $\mathcal{M}_G$ we have a commutative diagram

$$\begin{tikzcd}
& \mathrm{Hom}(\mathcal{G}_A, G) \arrow[dl, "\iota_A;-"'] \arrow[dr, "\omega_A;-"] \arrow[dd, "\beta"] & \\
\mathrm{Hom}(n, G) & & \mathrm{Hom}(m, G) \\
& \mathrm{Hom}(\mathcal{G}_B, G) \arrow[ul, "\iota_B;-"] \arrow[ur, "\omega_B;-"'] &
\end{tikzcd}$$

in Set. This means that the diagram

$$\begin{tikzcd}
n+m \arrow[r] \arrow[d] & \mathcal{G}_A \arrow[d] \arrow[dr, bend left, "f"] & \\
\mathcal{G}_B \arrow[r] \arrow[rr, bend right, "\beta(f)"'] & \mathcal{G}_{A\cap B} \arrow[r, dashed] & G
\end{tikzcd}$$

commutes, hence the dashed morphism exists by the property of the weak pushout $\mathcal{G}_{A\cap B}$ as required.

- If, conversely, for any morphism $f\colon \mathcal{G}_A \to G$ there is a $g\colon \mathcal{G}_{A\cap B} \to G$ as above, pick one such $g(f)$ for each f and consider $\alpha(f) = \mathcal{G}_B \to \mathcal{G}_{A\cap B} \xrightarrow{g(f)} G$. This makes the diagram

$$\begin{tikzcd}
 & \mathrm{Hom}(\mathcal{G}_A, G) \arrow[dl] \arrow[dr] \arrow[dd, "\alpha"] & \\
\mathrm{Hom}(n, G) & & \mathrm{Hom}(m, G) \\
 & \mathrm{Hom}(\mathcal{G}_B, G) \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

commute, and therefore we have $\mathcal{M}_G(A) \leq \mathcal{M}_G(B)$.

□

5.1.1 Injectivity and cofibrations

The importance of the Lifting Lemma 5.1.4 can hardly be overstated. It translates the logical property of $\mathcal{M}_G$ satisfying an inequality into a categorical lifting property of G. This motivates us to take a closer look at lifting properties and the logical properties they correspond to.

Definition 5.1.5. Let $b\colon X \to Y$ and $f\colon X \to Z$ be a pair of morphisms. A morphism $g\colon Y \to Z$, that makes the diagram

$$\begin{tikzcd}
X \arrow[r, "b"] \arrow[d, "f"] & Y \arrow[dl, dashed, "g"] \\
Z &
\end{tikzcd}$$

commute, is called a lift of f along b.

In this language, the Lifting Lemma 5.1.4 makes a connection between inequalities and lifts. Since a model of $\mathbb{CB}_{\Sigma/E}$ is a model of $\mathbb{CB}_{\Sigma}$ that simultaneously satisfies several inequalities, the corresponding categorical notion will be the notion of an injective object, an object that satisfies several lifting properties.

Definition 5.1.6. Let E be a class of morphisms. An object I is called E-injective if any morphism $f\colon X \to I$ has a lift along any morphism in E. We denote the class of E-injective objects by $E\,\text{-}\mathsf{Inj}$.

Example 5.1.7. In the category of left R-modules over a ring R the class of injective modules is given by $E\,\text{-}\mathsf{Inj}$ for E the class of monomorphisms.

Remark 5.1.8. Another way to state Definition 5.1.6 is as follows: An object I is E-injective if the contravariant Hom-functor $\mathrm{Hom}(\_, I)$ maps morphisms in E to surjective functions.

For a presentation (Σ, E) of a Frobenius theory, where E is a set of inequalities between morphisms in $\mathbb{CB}_{\Sigma}$, we can identify E with a set of morphisms in FHyp_{Σ}, where the inequality $A \leq B$ is identified with $\mathcal{G}_A \to \mathcal{G}_{A \cap B}$, as we did in Chapter 4. In the following, we will see E as such a set of morphisms.

Proposition 5.1.9. *If (Σ, E) is a presentation of a Frobenius theory, then a hypergraph G is E-injective if and only if $\mathcal{M}_G$ is a model of (Σ, E).*

But the Lifting Lemma 5.1.4 can be read in two ways. Instead of fixing the inequality and looking for Σ-structures that satisfy it, we can do the converse. Fixing a structure, the Lifting Lemma 5.1.4 characterises categorically which inequalities that structure satisfies, or more generally, given a class of structures, we can categorically characterise inequalities that they satisfy simultaneously. This will then correspond to a categorical property that is “dual” to that of injectives, starting from a class of objects rather than a class of morphisms.

Definition 5.1.10. Let J be a class of objects. A morphism $c\colon X \to Y$ is called a J-cofibration if any morphism $f\colon X \to I$ with $I \in J$ has a lift with respect to c.

Remark 5.1.11. The name cofibration is in analogy to how the term is used in the theory of model categories, see [27]. The notable difference is that we focus on injective objects rather than injective morphisms and hence our cofibrations are defined with respect to a class of objects.

Interestingly, one can build a factorisation system with transfinite composites of witnesses as left class and injective morphisms as right class, which is an application of the small object argument. Since injective morphisms are of less interest here, we leave the investigation of that factorisation system to possible future work.

Remark 5.1.12. A morphism $c\colon X \to Y$ is a J-cofibration if and only if $\mathrm{Hom}(c, I)\colon \mathrm{Hom}(Y, I) \to \mathrm{Hom}(X, I)$ is a surjective function for every $I \in J$. In other words, a J-cofibration is a morphism that is mapped to a surjective function by every $\mathrm{Hom}(\_, I)$ for $I \in J$.

The relevance of the notion of J-cofibration is as follows:

Proposition 5.1.13. *Let (Σ, E) be a presentation of a Frobenius theory, $R, S\colon n \to m$ morphisms in $\mathbb{CB}_\Sigma$ and $\alpha\colon \mathcal{G}_R \to \mathcal{G}_{R\cap S}$ the canonical morphism. Let J be the class of E-injective objects. The following are equivalent:*

- *α is a J-cofibration*
- *every model $\mathcal{M}\colon \mathbb{CB}_{\Sigma/E} \to \mathsf{Rel}$ satisfies $\mathcal{M}(R) \subseteq \mathcal{M}(S)$*

Proof. Since any models of $\mathbb{CB}_\Sigma$ is of the form $\mathcal{M}_G$ for a Σ-hypergraph G by Proposition 5.1.2 and since $\mathcal{M}_G$ defines a model of $\mathbb{CB}_{\Sigma/E}$ if and only if G is E-injective by Proposition 5.1.9, the claim follows from the Lifting Lemma 5.1.4. □

5.1.2 Categorical outlook on Completeness

There is an interplay between classes of injectives and classes of cofibrations.

Proposition 5.1.14. *Cofibrations and injective objects form an (antitone) Galois connection between classes of objects and classes of morphisms, that is:*

1. *If $E \subseteq E'$, then $E'\text{-}\mathsf{Inj} \subseteq E\text{-}\mathsf{Inj}$.*
2. *If $I \subseteq I'$, then $I'\text{-}\mathsf{Cofib} \subseteq I\text{-}\mathsf{Cofib}$.*
3. *Let E be a class of morphisms and I a class of objects. Then $I \subseteq E\text{-}\mathsf{Inj}$ if and only if $E \subseteq I\text{-}\mathsf{Cofib}$.*

In general, a Galois connection induces closure operators, one of which comes in handy for our particular case.

Lemma 5.1.15. *For any class of objects J, $J\text{-}\mathsf{Cofib}$ is a class of witnesses as defined in Definition 2.3.1.*

Proof. We have to show that cofibrations are closed under composition, coproducts and pushouts. By Remark 5.1.12, cofibrations are those morphisms that are mapped to surjective functions in Set under all $\mathrm{Hom}(\_, I)$ for $I \in J$. These functors are contravariant, so it suffices to show that surjective functions in Set are closed under the duals of those operations, so that surjective functions form a class of covers in Set. That is easily seen to be the case. □

Corollary 5.1.16. *Let E be a set of morphisms, $\mathcal{W}(E)$ be the class of witnesses generated by E, then*

$$\mathcal{W}(E) \subseteq (E\text{-}\mathsf{Inj})\text{-}\mathsf{Cofib}$$

Taking all of this together gives us a dictionary between logical and categorical notions in Figure 2. With this dictionary, completeness becomes a kind of dual of Corollary 5.1.16 in the following sense: Proposition 5.1.13 gives a neat categorical description of completeness: We need to prove that every property that is shared by all models is a logical consequence of the axioms, in other words we need to prove that every J-cofibration, where J is the class of E-injective objects, is "provable" from E. But we have seen in Corollary 4.3.9 that this notion of provable is exactly a witness generated from E. We therefore have the following categorical incarnation of the completeness theorem. The remainder of this chapter is dedicated to developing the proof.

logical	categorical
$A, B \colon n \to m$ in $\mathbb{CB}_\Sigma$	$\mathcal{G}_A \to \mathcal{G}_{A\cap B}$
Set E of axioms $A \leq B$	Set E of morphisms $\mathcal{G}_A \to \mathcal{G}_{A\cap B}$
$R \leq S$ in $\mathbb{CB}_{\Sigma/E}$	$\mathcal{G}_R \to \mathcal{G}_{R\cap S} \in \mathcal{W}(E)$
$\mathcal{M}_G$ is a model of $\mathbb{CB}_{\Sigma/E}$	G is E-injective
$\mathcal{M}(R) \leq \mathcal{M}(S)$ $\forall \mathcal{M} \colon \mathbb{CB}_{\Sigma/E} \to \mathsf{Rel}$	$\mathcal{G}_R \to \mathcal{G}_{R\cap S}$ is a cofibration with respect to E-Inj

Figure 2: Dictionary between logical and categorical properties

Sketch of Theorem (Completeness theorem)**.** *Let $\mathcal{C}$ be a "sufficiently nice" category. Let E be a "sufficiently nice" set of morphisms in $\mathcal{C}$. Let f be a "sufficiently nice" morphism. The following are equivalent:*

- *f is a cofibration with respect to E-injective objects.*
- *f is a witness generated by E.*

In the following we will have to see what sufficiently nice should mean. Spoiler: The concept of a locally finitely presentable category will be tremendously helpful.

5.2 Locally finitely presentable categories

Considering hypergraphs rather than models has convenient benefits for us, as the former category has the rich structure of a locally finitely presentable category. This structure turns out to be essential for our proof of the completeness theorem, so we review it here. The standard textbook introduction to the topic is [2]. Intuitively, a locally finitely presentable category has a set of objects that are "finite" in a suitable sense, such that every other object is constructed from the finite ones. Since the word finite is overloaded with meaning all throughout mathematics, in this context one typically speaks of compact objects. Since category theory is all about relationships between objects, essentially treating objects as black boxes, it is remarkable that there is a categorical notion that captures the intuition of "finiteness" fairly well. For example, for the category of vector spaces, compact objects are precisely finite dimensional vector spaces.

Definition 5.2.1. A directed set is a preordered set where each finite subset has an upper bound.

Directed sets are used in topology and analysis to define a notion of limit that is more general than the usual notion of sequential limit induced by sequences. We can see directed sets as categories in the usual way every preordered set can be seen as a category – the objects of the category are elements of the set with exactly one arrow $a \to b$ if and only if $a \leq b$. But there is also a more straightforward way of categorifying the notion of directedness, which is the concept of a filtered category.

Definition 5.2.2. A filtered category is one where every finite subdiagram admits a cocone.

Just like colimits over finite diagrams can be constructed from initial objects, coproducts and coequalisers, this is also true for cocones. This gives an equivalent characterisation of filtered categories.

Lemma 5.2.3. *A category $\mathcal{C}$ is filtered if and only if it has the following three properties:*

- *There is an object in $\mathcal{C}$*
- *For any two objects $X, Y \in \mathcal{C}$ there is an object Z and morphisms $X \to Z$ and $Y \to Z$.*
- *For any parallel pair of arrows $f, g \colon X \to Y$, there is $h \colon Y \to Z$ such that*
$$f \,;\, h = g \,;\, h$$

Example 5.2.4. The most important example of a filtered category for us is the category of natural numbers with a morphism $n \to m$ if and only if $n \leq m$. Diagrams over this category are sequences

$$X_0 \longrightarrow X_1 \longrightarrow X_2 \longrightarrow X_3 \longrightarrow \cdots$$

It turns out that filtered diagrams, diagrams indexed over a filtered category, and colimits of such diagrams define a useful notion of finiteness.

Definition 5.2.5. A functor is called *finitary* if it preserves filtered colimits. An object C in a category is called *compact* if the Hom-functor $\mathrm{Hom}(C, \_)$ is finitary.

Example 5.2.6.

- In the category Set, the compact objects are exactly finite sets.
- In the category of vector spaces over a field, the compact objects are exactly finite dimensional vector spaces.
- In the category of groups, the compact objects are precisely finitely presented groups.

Since compact objects are supposed to capture a notion of finiteness, the following is something one should expect.

Lemma 5.2.7. *Compact objects are closed under finite colimits.*

Proof. Filtered colimits commute with finite limits. The closure of compact objects under finite colimits follows by the Yoneda Lemma. □

Finite sets in Set have an additional interesting property. Every set can be constructed as the union over its finite subsets. Categorifying this property is what motivates the notion of a locally finitely presentable category.

Definition 5.2.8. A locally finitely presentable category is a cocomplete category with a set of compact objects, where every object is the colimit over the (filtered) category of compact objects mapping into it.

Example 5.2.9.

- The category Set of sets and functions is locally finitely presentable. The compact objects are exactly the finite sets.

- Every presheaf category $[\mathcal{C}^{\mathrm{op}}, \mathsf{Set}]$ is locally finitely presentable. The compact objects are finite colimits of representables. In particular, the category Hyp_Σ of Σ-hypergraphs, introduced in Chapter 4, is locally finitely presentable with the finite hypergraphs as compact objects.

We will need a stronger notion of finiteness, that is not standard in the literature. In the category of sets, there is only a finite number of morphisms between finite sets. An equivalent, more categorical, way of saying this is that for a finite set X, the Hom-functor $\mathrm{Hom}(X, \_)$ preserves compactness – given a finite set Y, also the set of morphisms $\mathrm{Hom}(X, Y)$ is finite.

Definition 5.2.10. We call a functor *humble* if it preserves filtered colimits and sends compact objects to compact objects. Similarly, an object $X \in \mathcal{C}$ is called humble if $\mathrm{Hom}(X, \_) \colon \mathcal{C} \to \mathsf{Set}$ is a humble functor.

For such a humble object X and any compact object Y, $\mathrm{Hom}(X, Y)$ is therefore a compact object in Set, which is to say it is finite.

Remark 5.2.11. Since there are only finitely many functions between finite sets, humble sets and compact sets agree. The same is true for presheaf categories, humble presheaves and compact presheaves agree. Therefore, for Σ-hypergraphs, humble hypergraphs are exactly the finite ones.

5.3 Completeness for free theories

The special case of $E = \varnothing$ is instructive, so even though it will later follow from the general completeness Theorem, we will investigate it closer. For $E = \varnothing$, any object is E-injective because that becomes a vacuous condition. That means that cofibrations have to have the lifting property with respect to identities, which forces them to be exactly split monos. But that is also the smallest class of witnesses.

Theorem 5.3.1. *The class of cofibrations with respect to $\varnothing$-injective objects coincides with the witnesses generated by $\varnothing$.*

Corollary 5.3.2 (Completeness for free theories)**.** *If $R, S\colon n \to m$ in $\mathbb{CB}_\Sigma$ satisfy $\mathcal{M}(R) \leq \mathcal{M}(S)$ for all models $\mathcal{M}\colon \mathbb{CB}_\Sigma \to \mathsf{Rel}$, then $R \leq S$ in $\mathbb{CB}_\Sigma$.*

5.4 Proof of the completeness theorem

We shall now develop the tools to prove Theorem 5.4.27. The general idea is the following: Given an object X, we will develop a morphism $X \to X'$ that is a suitable filtered colimit of witnesses such that X' is E-injective. Then for $f\colon X \to Y$ a cofibration, we get a diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & X' \\ Y \arrow[ur, dashed, "g"'] & \end{tikzcd}$$

where g exists because f is a cofibration and X' is E-injective. Now, if Y is compact, since X' is a filtered colimit of witnesses, there is a witness $e\colon X \to X''$ and

$$\begin{tikzcd} X \arrow[r, "e"] \arrow[d, "f"'] & X'' \\ Y \arrow[ur, "g'"'] & \end{tikzcd}$$

Therefore also f is a witness.

5.4.1 Expansions

We will now construct this "injectivisation". The idea for the construction is to construct from X a new object X' with a morphism $\zeta\colon X \to X'$ such that for any $e\colon A \to B \in E$ and $f\colon A \to X$, there is $g\colon B \to X'$ such that

$$\begin{tikzcd} X \arrow[r, "\zeta"] & X' \\ A \arrow[u, "f"] \arrow[r, "e"] & B \arrow[u, dashed, "g"'] \end{tikzcd}$$

It will be convenient to construct such an object indirectly via a universal property and then hope that it actually exists (Spoiler alert: it does).

Definition 5.4.1. Let $e\colon A \to B$ be a morphism. An *expansion* along e is a morphism $\phi\colon X \to Y$ together with a function $\beta\colon \operatorname{Hom}(A, X) \to \operatorname{Hom}(B, Y)$ such that for every $g\colon A \to X$, the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "\phi"] & Y \\ A \arrow[u, "g"] \arrow[r, "e"] & B \arrow[u, "\beta(g)"'] \end{tikzcd}$$

More generally for a set E of morphisms, an expansion along E is a morphism $\phi\colon X \to Y$ together with a family of functions $(\beta_e)_{e\in E}$, where for $e\colon A \to B \in E$, $\beta_e\colon \operatorname{Hom}(A, X) \to \operatorname{Hom}(B, Y)$ makes the above diagram commute.

Proposition 5.4.2. *For a set of morphisms E, denote the set of expansions from X to Y by $\operatorname{Exp}_E(X, Y)$. This is functorial in both variables, depending contravariantly on X and covariantly on Y, so it defines a functor $\operatorname{Exp}_E\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathsf{Set}$, $(X, Y) \to \operatorname{Exp}_E(X, Y)$.*

The point of Exp_E is that $\operatorname{Exp}_E(X, \_)$ will be a representable functor and a representation $\operatorname{Hom}(X', \_) \cong \operatorname{Exp}_E(X, \_)$ of it will be the first step in making X injective.

Proposition 5.4.3. $\operatorname{Exp}_E(X, Y)$ *is the pullback*

$$\begin{tikzcd} \operatorname{Exp}_E(X,Y) \arrow[r] \arrow[d] & \prod_{e\colon A\to B\in E} \operatorname{Hom}(A,X) \to \operatorname{Hom}(B,Y) \arrow[d, "q"] \\ \operatorname{Hom}(X,Y) \arrow[r, "p"] & \prod_{e\colon A\to B\in E} \operatorname{Hom}(A,X) \to \operatorname{Hom}(A,Y) \end{tikzcd}$$

where

$$p(\phi)_e(f) = f \mathbin{;} \phi$$

and

$$q(\beta)_e(f) = e \mathbin{;} \beta_e(f)$$

The idea is now to look for the initial expansion of X along E. Formally that corresponds to considering $\mathrm{Exp}_E(X, \_)$ as a functor in its second variable and hoping that it is a representable functor (Spoiler alert: It is.).

Lemma 5.4.4. *Let X be an object. There is a diagram $\Delta\colon \mathcal{D} \to \mathcal{C}$ such that*

$$\mathrm{Exp}_E(X, \_) \cong \mathrm{Cocone}(\Delta, \_)$$

Proof. Let $\mathcal{D}$ be the category consisting of an object $*$ together with one span $* \leftarrow P_{e,f} \to Q_{e,f}$ for every $e\colon A \to B \in E$ and every $f\colon A \to X$. Note that there are no non-trivial compositions in $\mathcal{D}$. Now set $\Delta(*) = X$ and for $e\colon A \to B \in E$ and $f\colon A \to X$,

$$\Delta(P_{e,f} \to *) = A \xrightarrow{f} X,$$

$$\Delta(P_{e,f} \to Q_{e,f}) = A \xrightarrow{e} B$$

Now we need to prove that a cocone over Δ is the same thing as an expansion along E. Note that every $\alpha \in \mathrm{Cocone}(\Delta, Y)$ in particular defines a morphism $\alpha_*\colon X \to Y$ just like for expansions. The difference between the view as expansions and the view as cocones is how the remaining data is organised: For each $e\colon A \to B$ and each $f\colon A \to B$ the cocone α gives rise to a morphism $\alpha_{Q_{e,f}}\colon B \to Y$, while for expansions, the data is "curried" as $\beta_e(f)$. The condition of being a cocone makes the same diagrams commute as an expansion needs to satisfy and every expansion defines a unique cocone because the components $\alpha_{P_{e,f}}$ can be reconstructed from the commutative diagrams. □

We are particularly interested in the case where this Δ is a finite diagram. For that, E needs to be finite and every codomain in E needs to only allow finitely many morphisms into the object X being expanded.

Proposition 5.4.5. *In the case that E is finite and X is compact and the morphisms in E have humble domains, the diagram Δ from Lemma 5.4.4 is finite.*

Corollary 5.4.6. *In a cocomplete category $\mathcal{C}$, $\mathrm{Exp}_E(X, \_)$ is representable for each X resulting in a functor $\mathcal{F}_E \colon \mathcal{C} \to \mathcal{C}$ such that*

$$\mathrm{Hom}(\mathcal{F}_E\,(X)\,, Y) \cong \mathrm{Exp}_E(X, Y)$$

Proof. The details are spelled out in the Appendix in Lemma A.1.23. $\square$

Example 5.4.7. Consider some examples for the construction of $\mathcal{F}_E$ at work.

- Let $\Sigma = \emptyset$ and $E = \{ \ \leq \ \}$, where the left-hand side is the empty diagram representing $\mathrm{id}_0 \colon 0 \to 0$ in $\mathbb{CB}_\Sigma$. Now models of $\mathbb{CB}_{\Sigma/E}$ (in Rel) are precisely the non-empty sets. $\mathcal{G} = \varnothing$ is the empty hypergraph, the initial object in FHyp_Σ, so there is a unique morphism $\mathcal{G} \to G$, so Δ will consist of only a single span, namely

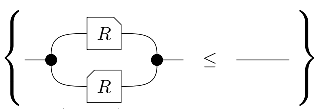
$$\begin{tikzcd} \mathcal{G} & & G \\ & \varnothing \arrow[ul] \arrow[ur] & \end{tikzcd}$$

  The colimit of this diagram is the coproduct $G + \mathcal{G}$ and we have $\mathcal{G} = 1$, so $\mathcal{F}_E\,(G) = G + 1$.

- Let $\Sigma = \emptyset$ and $E = \{ \ \leq \ \}$. There are two models of this theory, namely the set with only one element and the empty model. Now, $\mathcal{G} = 2$ the set with two elements, hence the diagram Δ consists of one span

$$\begin{tikzcd} \mathcal{G} & & G \\ & 2 \arrow[ul] \arrow[ur, "{(x,y)}"'] & \end{tikzcd}$$

  for every $(x, y) \in G(1)^2$. Now we have $\mathcal{G} = 1$, so the colimit over Δ will glue together every pair of elements of $G(1)$. If G is non-empty, that will yield $\mathcal{F}_E\,(G) = 1$, if G is empty, Δ will be the diagram containing only G, so $\mathcal{F}_E\,(G)$ will be the empty set as well.

- Let $\Sigma = \{R \colon 1 \to 1\}$ and $E =$ .

  Models of this theory are sets equipped with an antisymmetric relation R. Let G be a Σ-hypergraph, so a directed graph.

Now $\mathcal{G}_{-\bullet\begin{smallmatrix}\boxed{R}\\ \boxed{R}\end{smallmatrix}\bullet-}$ has two vertices that are mutually related via R, so a morphism $\alpha\colon \mathcal{G}_{-\bullet\begin{smallmatrix}\boxed{R}\\ \boxed{R}\end{smallmatrix}\bullet-} \to G$ picks out precisely two such vertices. Therefore, Δ contains a span

$$\begin{tikzcd} \mathcal{G}_{-} & & G \\ & 2 \arrow[ul] \arrow[ur, "{(x,y)}"'] & \end{tikzcd}$$

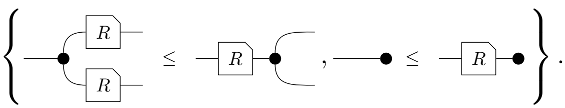
for every such pair of vertices (x, y). Now, $\mathcal{G}_{-} = 1$ as before, so the colimit over Δ will glue together every pair of elements of $G(1)$ that are mutually related via $G(R)$, in other words we quotient G in order to enforce antisymmetry of R.

- Let $\Sigma = \{R\colon 1 \to 1\}$ and $E = \{-\boxed{R}-\boxed{R}- \;\leq\; -\boxed{R}-\}$ Models of this theory are sets equipped with a transitive relation R. Let G be a Σ-hypergraph, so a directed graph. Then $\mathcal{G}_{-\boxed{R}-\boxed{R}-}$ consists of three vertices a, b, c with $(a, b), (b, c) \in \mathcal{G}_{-\boxed{R}-\boxed{R}-}(R)$. A morphism $\mathcal{G}_{-\boxed{R}-\boxed{R}-} \to G$ corresponds to a configuration of three vertices like that in G. Now the diagram Δ contains one span

$$\begin{tikzcd} \mathcal{G}_R & & G \\ & 2 \arrow[ul, "\iota_R+\omega_R"] \arrow[ur, "{(x,z)}"'] & \end{tikzcd}$$

for every x, y, z with $(x, y), (y, z) \in G(R)$. We have $\mathcal{G}_R(1) = \{\iota_R, \omega_R\}$ a two-element set and we have $\mathcal{G}_R(R) = \{(\iota_R, \omega_R)\}$. Since $\iota_R+\omega_R\colon 2 \to \mathcal{G}_R$ induces an isomorphism on the underlying sets, the colimit of Δ has the same underlying set as G. The effect of the colimit is to make sure that there is an edge $x \to z$ for every pair of edges $x \to y$, $y \to z$. In other words $\mathcal{F}_E(G)$ is the first step of a transitive closure of G. This example also shows that $\mathcal{F}_E(G)$ is in general not E-injective. To obtain an E-injective object, one needs to iterate this construction, which we will investigate in the following.

- Let $\Sigma = \{R\colon 1 \to 1\}$ and

$E =$ 

A model of this theory is a set equipped with an endofunction. Let G be any Σ-hypergraph, so a directed graph. We have $\mathcal{G}_{-\bullet} = 1$, so Δ contains for every $x \in G(1)$, a span $\mathcal{G}_{-\boxed{R}-\bullet} \xleftarrow{\iota_R} 1 \xrightarrow{x} G$, where $\mathcal{G}_{-\boxed{R}-\bullet} = \mathcal{G}_R$ as before, but note how we glue only along the left interface this time. The effect of this part of the colimit is to freely adjoin an image under R for every $x \in G(1)$. The other class of spans in Δ are those coming from the first inequality in E, where we need to consider $\mathcal{G}_{-\bullet\genfrac{}{}{0pt}{}{\boxed{R}-}{\boxed{R}-}}$, consisting of three vertices a, b, c with $(a, b), (a, c) \in \mathcal{G}_{-\bullet\genfrac{}{}{0pt}{}{\boxed{R}-}{\boxed{R}-}}(R)$. So the other class of spans in Δ are $\mathcal{G}_{-\boxed{R}-\bullet\genfrac{}{}{0pt}{}{-}{-}} \xleftarrow{\iota+\omega} 3 \xrightarrow{(x,y,z)} G$, where $(x, y), (x, z) \in G(R)$. Here $\mathcal{G}_{-\boxed{R}-\bullet\genfrac{}{}{0pt}{}{-}{-}} = \mathcal{G}_R$, with interface $\iota\colon 1 \to \mathcal{G}_R$ the usual left interface but the right interface, $\omega\colon 2 \to \mathcal{G}_R$, collapses both points. The effect of this part of the colimit is to glue y and z together whenever both $(x, y) \in G(R)$ and $(x, z) \in G(R)$.

Proposition 5.4.8. *Since $\mathcal{F}_E(X)$ is the universal expansion of X along E, in particular there is a morphism $\zeta_X\colon X \to \mathcal{F}_E(X)$. This morphism is natural in X, so we get a natural transformation $\zeta\colon \mathrm{id}_{\mathcal{C}} \to \mathcal{F}_E$.*

The important aspect about the colimit description of $\mathcal{F}_E$ is that the diagram Δ has a peculiar shape that can be best described as a star of spans. There is a central point (called $*$ in the proof of 5.4.4), attached to which are a number of spans. Using abstract nonsense, we can prove handy tools about those diagrams and their colimits. But let us first fix some nomenclature.

Definition 5.4.9. A marked span is a graph of the shape $* \xleftarrow{A} \xrightarrow{B}$, with one of the two external points marked $*$ so as to break the symmetry. We call the edge pointing to $*$ the inner edge, and the remaining edge the outer one. A starspan is a graph that consists of a set of marked spans, glued together along their $*$-points.

Colimits over starspan diagrams are interesting, because they can be broken down into pushouts of simpler colimits.

Lemma 5.4.10 (Iteration). *Let G be a starspan and S_1, S_2 a partition of its marked spans into two classes. Let G_i be the starspan having S_i as its* *set of spans. Given a diagram $\Delta\colon G \to \mathcal{C}$, we can naturally restrict it to diagrams $\Delta_i\colon G_i \to \mathcal{C}$. Let $X = \Delta(*)$ then*

$$\begin{tikzcd}
\operatorname{colim} \Delta_1 \arrow[r] & \operatorname{colim} \Delta \\
X \arrow[u] \arrow[r] & \operatorname{colim} \Delta_2 \arrow[u]
\end{tikzcd}$$

is a pushout diagram.

Proof. This follows from Corollary A.3.3 and the observation that

$$\begin{tikzcd}
\operatorname{colim} G_1 \arrow[r] & \operatorname{colim} G \\
* \arrow[u] \arrow[r] & \operatorname{colim} G_2 \arrow[u]
\end{tikzcd}$$

is a pushout. $\square$

Lemma 5.4.10 has the consequence that for a finite starspan, we can compute the colimit one pushout at a time. Since witnesses are closed under pushouts and composition, that has the following important consequence:

Corollary 5.4.11. *Let G be a finite starspan and $\mathcal{C}$ be a category, E a set of morphisms. Let $\Delta\colon G \to \mathcal{C}$ such that every outer edge is a witness with respect to E. Then the induced morphism $f\colon \Delta(*) \to \operatorname{colim} \Delta$ is a witness as well.*

Proof. This is done by induction on the number of spans in G. If G has no spans, the statement becomes vacuous. Consider the case where G can be written as a pushout

$$\begin{tikzcd}
s \arrow[r] & G \\
* \arrow[u] \arrow[r] & G' \arrow[u]
\end{tikzcd}$$

where s is a single span. Let $\Delta\colon G \to \mathcal{C}$ be a diagram. Restricting Δ to G' and to s gives diagrams Δ' and Δ_s respectively. By induction hypothesis, the canonical morphism $X \to \operatorname{colim} \Delta'$ is a witness. The colimit of Δ_s is the following pushout

$$\begin{array}{ccc} X & \longrightarrow & \operatorname{colim} \Delta_s \\ \uparrow & & \uparrow \\ A & \longrightarrow & B \end{array}$$

where by assumption on Δ, the morphism $A \to B$ is a witness. This makes the induced morphism $X \to \operatorname{colim} \Delta_s$ a witness as well. By Lemma 5.4.10, the diagram

$$\begin{array}{ccc} \operatorname{colim} \Delta' & \longrightarrow & \operatorname{colim} \Delta \\ \uparrow & & \uparrow \\ X & \longrightarrow & \operatorname{colim} \Delta_s \end{array}$$

is a pushout, so the induced morphism $X \to \operatorname{colim} \Delta$ is a witness as well. $\square$

Most importantly, this observation applies to $\mathcal{F}_E$ and shows that the morphism $\zeta_X : X \to \mathcal{F}_E(X)$ is a witness generated from E.

Lemma 5.4.12. *If E is finite and morphisms in E have humble domains and X is compact, then $\zeta_X : X \to \mathcal{F}_E(X)$ is a witness generated from E.*

Proof. Follows from Corollary 5.4.11 and Lemma 5.4.4. $\square$

Proposition 5.4.13. *The set $\operatorname{Exp}_E(X, Y)$ depends contravariantly on the set of morphisms E in the sense that for any inclusion $E \subseteq E'$, there is a canonical forgetful function $\operatorname{Exp}_{E'}(X, Y) \to \operatorname{Exp}_E(X, Y)$, forgetting part of the data.*

Corollary 5.4.14. *$\mathcal{F}_E$ depends covariantly on E in the sense that whenever $E \subseteq E'$, there is a natural transformation $\mathcal{F}_E \to \mathcal{F}_{E'}$.*

Proof. This is an application of Lemma A.1.23. $\square$

Lemma 5.4.15. *Let $\mathcal{D}$ be a directed system of sets of morphisms. Then*

$$\operatorname{Exp}_{\bigcup \mathcal{D}}(X, Y) \cong \lim_{D \in \mathcal{D}} \operatorname{Exp}_D(X, Y)$$

Proof. An element of $\lim_{D\in\mathcal{D}} \operatorname{Exp}_D(X,Y)$ is a family of expansions x_D, one for each $D \in \mathcal{D}$ such that whenever $D \subseteq D'$, forgetting the additional data in $x_{D'}$ yields x_D. It is clear that an expansion along $\bigcup \mathcal{D}$ gives rise to such a compatible family by restriction. So it suffices to prove the converse, that every compatible family of expansions gives rise to an expansion along the whole of $\bigcup \mathcal{D}$.

Assume that we have a compatible family of expansions, where x_D is an expansion along D. Since the family is compatible, the morphism part of the x_D has to be always the same. Because let $D_1, D_2 \in \mathcal{D}$. Since $\mathcal{D}$ is directed, there is $D_1, D_2 \subseteq D \in \mathcal{D}$. Now let $\phi_D \colon X \to Y$ be the morphism associated with x_D and likewise for D_1 and D_2. Since forgetting along $D_1 \subseteq D$ and $D_2 \subseteq D$ leaves the morphism unchanged, we get $\phi_{D_1} = \phi_D = \phi_{D_2}$. The whole family therefore agrees on a morphism, call it from now on $\phi \colon X \to Y$. We choose that as the morphism part of the expansion along $\bigcup \mathcal{D}$. We now need to define a family of morphisms $\beta_e \colon \operatorname{Hom}(A,X) \to \operatorname{Hom}(B,Y)$ for each $e \colon A \to B \in E$. We are given such families $(\beta_D)_e$ for each $D \in \mathcal{D}$. Fix $e \in \bigcup \mathcal{D}$, then there is $D \in \mathcal{D}$ with $e \in D$. Unfortunately, this D will in general not be unique. Nevertheless, we have a canonical choice for β_e because it turns out that if $e \in D_1$ and $e \in D_2$, then $(\beta_{D_1})_e = (\beta_{D_2})_e$. Since $\mathcal{D}$ is directed, there is a $D \in \mathcal{D}$ such that $D_1, D_2 \subseteq D$. Since the family is compatible, β_{D_1} and β_{D_2} are the result of forgetting from β_D. But since $e \in D_1, D_2, D$, it is not affected by forgetting and therefore

$$(\beta_{D_1})_e = (\beta_D)_e = (\beta_{D_2})_e$$

It is easy to see that (ϕ, β) makes the required diagrams commute and is therefore an expansion. Forgetting the additional structure of β along any inclusion $D \subseteq \bigcup \mathcal{D}$ yields β_D. $\square$

Corollary 5.4.16. *For a directed system $\mathcal{D}$ of sets of morphisms and $E = \bigcup \mathcal{D}$, we have*

$$\mathcal{F}_E \cong \operatorname*{colim}_{D\in\mathcal{D}} \mathcal{F}_D$$

Proof.

$$\begin{aligned}\operatorname{Hom}(\mathcal{F}_E(X), Y) &\cong \operatorname{Exp}_E(X,Y) \cong \lim_{D\in\mathcal{D}} \operatorname{Exp}_D(X,Y)\\ &\cong \lim_{D\in\mathcal{D}} \operatorname{Hom}(\mathcal{F}_D(X), Y) \cong \operatorname{Hom}(\operatorname*{colim}_{D\in\mathcal{D}} \mathcal{F}_D(X), Y)\end{aligned}$$

This isomorphism is natural in X and Y, therefore the claim follows by the Yoneda Lemma A.1.19. □

Since every set is the union of all its finite subsets (which is a directed system), we can use Corollary 5.4.16 to reduce the case of infinite E to the case of finite E. It will also be useful to establish sufficient conditions for when $\mathrm{Exp}_E(X, Y)$ preserves filtered colimits in X and respectively in Y.

Lemma 5.4.17.

1. *If E is finite and all morphisms in E have compact domains, then $\mathrm{Exp}_E(\_, Y)$ preserves filtered colimits for all Y.*
2. *If E is finite, all morphisms in E have humble domains and compact codomains and X is compact, then $\mathrm{Exp}_E(X, \_)$ preserves filtered colimits.*

Proof. Both follow from the characterisation of Exp in Proposition 5.4.3 and the fact that filtered colimits commute with finite limits. □

Corollary 5.4.18. *Let E be a finite set of morphisms.*

1. *If each morphism in E has compact domain, then $\mathcal{F}_E$ preserves filtered colimits, that is it is a finitary functor.*
2. *If each morphism in E has humble domain and compact codomain, $\mathcal{F}_E$ is a humble functor. (see Definition 5.2.10)*

Proof. This follows from Lemma 5.4.17 by the Yoneda Lemma A.1.19. □

5.4.2 Iterated expansions

Definition 5.4.19. For a finite sequence $\mathcal{E} = (E_1, E_2, \dots, E_n)$ of morphisms, let

$$\mathcal{F}_{\mathcal{E}} = \mathcal{F}_{E_1} \,;\, \mathcal{F}_{E_2} \,;\, \dots \,;\, \mathcal{F}_{E_n}$$

Note that in the case $n = 0$, this reduces to $\mathcal{F}_{()} = \mathrm{id}_{\mathcal{C}}$. For an infinite sequence $\mathcal{E} = (E_1, E_2, \dots)$, let $\mathcal{E}_n$ be the finite subsequence consisting of the first n sets. There is a natural transformation $\mathcal{F}_{\mathcal{E}_n} \to \mathcal{F}_{\mathcal{E}_{n+1}}$ induced from the natural transformation $\mathrm{id}_{\mathcal{C}} \to \mathcal{F}_{E_{n+1}}$. This arranges the $\mathcal{F}_{\mathcal{E}_n}$ into a diagram

$$\mathrm{id}_{\mathcal{C}} = \mathcal{F}_{\mathcal{E}_0} \longrightarrow \mathcal{F}_{\mathcal{E}_1} \longrightarrow \mathcal{F}_{\mathcal{E}_2} \longrightarrow \cdots$$

and we define $\mathcal{F}_{\mathcal{E}} = \operatorname*{colim}_n \mathcal{F}_{\mathcal{E}_n}$

Definition 5.4.20. As a notational shortcut, let E^n be the sequence of length n that is constantly E, so $E^n = (E, E, \ldots, E)$. Let E^ω be the infinite sequence that is constantly E.

Lemma 5.4.21. *For a (finite or infinite) sequence of sets $\mathcal{E}$, there is a natural transformation $\zeta\colon \mathrm{id}_{\mathcal{C}} \to \mathcal{F}_{\mathcal{E}}$. If $\mathcal{E}$ is a finite sequence of finite sets, each of which consisting of morphisms with humble domains and if X is compact, then also $\mathcal{F}_{\mathcal{E}}X$ is compact and $\zeta_X\colon X \to \mathcal{F}_{\mathcal{E}}(X)$ is a witness generated from E.*

Proof. We do an induction on the length n of the sequence $\mathcal{E}$. If $n = 0$, the statement is trivial, because ζ is the identity. If $\mathcal{E}$ has length $n+1$, write $\mathcal{E} = \mathcal{E}' * (E)$ where $*$ means concatenation, E is the last element in the sequence $\mathcal{E}$ and $\mathcal{E}'$ the remaining sequence of length n. Now $\zeta_X^{\mathcal{E}}$ factors as

$$X \xrightarrow{\zeta^{\mathcal{E}'}} \mathcal{F}_{\mathcal{E}'}(X) \xrightarrow{\zeta^{E}} \mathcal{F}_E(\mathcal{F}_{\mathcal{E}'}(X)) = \mathcal{F}_{\mathcal{E}}(X)$$

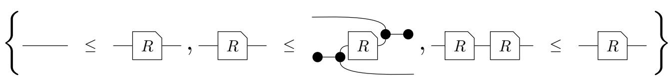
By induction hypothesis, $\mathcal{F}_{\mathcal{E}'}(X)$ is compact and therefore, because all morphisms in E have humble domain, so is $\mathcal{F}_E(\mathcal{F}_{\mathcal{E}'}(X)) = \mathcal{F}_{\mathcal{E}}(X)$. Now ζ_X is a witness as a composite of witnesses. $\square$

Proposition 5.4.22. *If all morphisms in E have compact domain then $\mathcal{F}_{E^\omega}X$ is E-injective, regardless of X.*

Example 5.4.23. For these examples let $\Sigma = \{R\colon 1 \to 1\}$ and G a Σ-hypergraph.

- Consider the example of a transitive relation again, that is $E = \{ -\boxed{R}-\boxed{R}- \;\le\; -\boxed{R}- \}$. Now $\mathcal{F}_{E^\omega}(G)(*) = G(*)$ so the underlying vertex set didn't change. But $\mathcal{M}_{\mathcal{F}_{E^\omega}(G)}(R)$ is the transitive closure of $\mathcal{M}_G(R)$.

- Consider

  $E =$ 

  the theory of equivalence relations. Again, $\mathcal{F}_{E^\omega}(G)$ has the same underlying set as G but $\mathcal{M}_{\mathcal{F}_{E^\omega}(G)}$ is equipped with the equivalence relation generated by $\mathcal{M}_G(R)$.

The functor $\mathcal{F}_{\mathcal{E}}$ depends on $\mathcal{E}$ in a similar way to how $\mathcal{F}_E$ depends on E.

Proposition 5.4.24. *For sequences $\mathcal{E}$ and $\mathcal{E}'$ of the same length, whenever $\mathcal{E} \subseteq \mathcal{E}'$ in the sense of componentwise inclusion, there is a natural transformation $\mathcal{F}_{\mathcal{E}} \to \mathcal{F}_{\mathcal{E}'}$.*

Proposition 5.4.25. *Let $\mathcal{D}$ be a directed system of sets of morphisms, with each $D \in \mathcal{D}$ finite and each morphism in D with compact domain. Let $E = \bigcup \mathcal{D}$, then*

$$\mathcal{F}_{E^n} \cong \operatorname*{colim}_{D \in \mathcal{D}^n} \mathcal{F}_D$$

and therefore

$$\mathcal{F}_{E^\omega} \cong \operatorname*{colim}_{n \in \mathbb{N}} \operatorname*{colim}_{D \in \mathcal{D}^n} \mathcal{F}_D$$

Since every set is a directed colimit over its finite subsets, this has the following important consequence:

Lemma 5.4.26. *Let E be a set of morphisms with compact domain, and let $\mathcal{P}_f(E)$ be the set of finite subsets of E. Then*

$$\mathcal{F}_{E^\omega} \cong \operatorname*{colim}_{n} \operatorname*{colim}_{\mathcal{E} \in (\mathcal{P}_f(E))^n} \mathcal{F}_{\mathcal{E}}$$

This is the final piece of the puzzle. We are now able to prove the main theorem of the thesis:

Theorem 5.4.27. *Let $\mathcal{C}$ be a locally finitely presentable category. Let E be a set of morphisms in $\mathcal{C}$ with humble domain and compact codomain (see Definition 5.2.10). Then a morphism between compact objects that is a cofibration with respect to E-injective objects, is a witness generated by E.*

Proof. Let $f\colon X \to Y$ be an E-cofibration. Consider the canonical morphism $\zeta\colon X \to \mathcal{F}_{E^\omega}(X)$ and the diagram

$$\begin{tikzcd} X \arrow[r, "\zeta"] \arrow[d, "f"] & \mathcal{F}_{E^\omega}(X) \\ Y & \end{tikzcd}$$

Since by Proposition 5.4.22 $\mathcal{F}_{E^\omega}(X)$ is E-injective and since f is a cofibration, there is an induced morphism

$$\begin{tikzcd} X \arrow[r, "\zeta"] \arrow[d, "f"] & \mathcal{F}_{E^\omega}(X) \\ Y \arrow[ur, dashed, "g"] & \end{tikzcd}$$

Since, by Lemma 5.4.26 $\mathcal{F}_{E^\omega}(X)$ can be written as a filtered colimit and since Y is compact, there is an $n \in \mathbb{N}$ and a sequence $\mathcal{E} = (E_1, \ldots, E_n)$ of finite subsets of E such that $g \colon Y \to \mathcal{F}_{E^\omega}(X)$ factors through $\mathcal{F}_{\mathcal{E}}(X)$. Hence we get a commutative diagram

$$\begin{tikzcd} X \arrow[r, "\zeta'"] \arrow[d, "f"] & \mathcal{F}_{\mathcal{E}}(X) \arrow[r] & \mathcal{F}_{E^\omega}(X) \\ Y \arrow[ur, dashed, "h"] \arrow[urr, bend right, "g"'] & & \end{tikzcd}$$

Since $\zeta' \colon X \to \mathcal{F}_{\mathcal{E}}(X)$ is a witness generated from E, this makes f a witness as desired. $\square$

From this we can immediately deduce the logical incarnation of the completeness theorem

Theorem 5.4.28 (Completeness)**.** *If $A, B \colon n \to m$ are morphisms in $\mathbb{CB}_{\Sigma/E}$ such that $\mathcal{M}(A) \subseteq \mathcal{M}(B)$ for all models $\mathcal{M}$. Then $A \leq B$ in $\mathbb{CB}_{\Sigma/E}$.*

Proof. By Prop 5.1.13, $\alpha \colon \mathcal{G}_A \to \mathcal{G}_{A \cap B}$ is a cofibration with respect to E-injective objects. Since the set E of axioms satisfies all the niceness properties and $\mathcal{G}_A$ and $\mathcal{G}_{A\cap B}$ are compact, Theorem 5.4.27 applies and therefore α is a witness generated from E. By Corollary 4.3.9, that means $A \leq B$ in $\mathbb{CB}_{\Sigma/E}$. $\square$

Appendix A

Category Theory

Introduction

It is very important to have a wide array of different mathematical theories and techniques at one's disposal to solve problems. Often times it is possible to translate a problem in one theory into a problem in another and there is always the hope that someone solved it in the different framework. A historical example of this kind is the translation of topological problems about "holes" in spaces into algebraic problems, known as algebraic topology. The extensive use and usefulness of mappings between different mathematical theories called for a rigorous treatment and for a formalisation of "mathematical theory". Such a formalisation emerged in the form of categories, introduced by Saunders Mac Lane and Samuel Eilenberg. A great exposition of the topic can be found in [33].

The idea is very simple and beautiful. While the objects of study differ vastly across theories, each theory comes with its own flavour of structure-preserving morphism. For set theory those are simply functions, for group theory those are group homomorphisms, for topology those are homeomorphisms, for geometry those are isometries, and so on. If we treat the actual structures as opaque, looking at only the relevant functions between them, a mathematical theory looks like a web of objects connected by arrows. This then is the general framework of working with several interconnected theories. But it requires a paradigm shift. Different theories only look alike for as long as you do not look inside the objects. The question then becomes: How far do you get in such a way? How much of a theory can be captured using its arrows alone? The answer turns out to be even better than "everything". Referring to arrows alone, you can do exactly what matters. You cannot, for example, find out the variable name in a polynomial ring. But that is great news, because you shouldn't. That is an implementation detail at best and should not (and does not) matter to the theory. Colloquially, properties like names that are not invariant under isomorphism, are considered "evil".

This appendix is intended as a reference to categorical properties used in the thesis. For an introduction to category theory see [33].

A.1 Categories

A.1.1 Definitions

Definition A.1.1. A category $\mathcal{C}$ consists of a collection of objects $\mathrm{Ob}(\mathcal{C})$ and for each pair of objects $X, Y \in \mathrm{Ob}(\mathcal{C})$ a set $\mathrm{Hom}(X, Y)$ of homomorphisms equipped with an associative composition $\mathrm{Hom}(X, Y) \times \mathrm{Hom}(Y, Z) \to \mathrm{Hom}(X, Z)$, and identities $\mathrm{id}_X \in \mathrm{Hom}(X, X)$ that are both-sided units for composition. We denote the composite of $f\colon X \to Y$ and $g\colon Y \to Z$ as $f \mathbin{;} g$.

Remark A.1.2. There are two ways in which to denote composition of morphisms in a category. Traditionally, the composition of two functions is denoted as $g \circ f$, which is also read as g after f. In this notation, f is applied first, then g. The convention we use here is the converse, where $f \mathbin{;} g$ means f is to be applied first, followed by g.

Example A.1.3.

- One of the most important examples of a category is the category Set whose objects are sets and whose morphisms are functions.

- Similarly, there is the category Rel of relations, where objects are sets and a morphism $R\colon X \to Y$ is a relation $R \subseteq X \times Y$.

- For any algebraic structure there is a category where morphisms are the structure-preserving functions. So there is a category of groups, a category of monoids, a category of rings, etc.

- For any ring R, there is a category $R-\mathsf{mod}$ whose objects are R-modules and whose morphisms are module homomorphisms.

- Every category $\mathcal{C}$ has an *opposite category* $\mathcal{C}^{\mathrm{op}}$, which has the same objects as $\mathcal{C}$, but all arrows reversed.

Remark A.1.4. The word "collection" in Definition A.1.1 is deliberately vague. It is necessary to be vague about it, because of what is known in the literature as size issues. Due to Russell's paradox, the collection of all sets cannot be a set itself, otherwise it would contain itself and that opens a whole can of worms. One way of solving this problem is to enrich the underlying set-theory to allow for different kinds of collections. The simplest way is to distinguish sets and classes, such that the collection of sets is itself a class, rather than a set. Another approach is to use Grothendieck universes, which makes "setness" a notion relative to a universe. Whenever one wants to form a collection like that of all sets in a given universe, it becomes an entity in a different, larger, universe.

Many introductions to category theory are written using one or the other approach, so for example a category is often defined as having a class of objects and classes of morphisms, but that is not the only way. As far as I can tell, none of the possibilities is considered canonical. Since the differences in underlying set-theories don't matter to the applications in this thesis, I take an approach of the "bring your own collection" kind and leave the choice of set-theory to the reader.

Furthermore, note the restriction to sets for homomorphisms. For each pair of objects, we ask for a set of morphisms. In the literature one can also find the broader definition that allows for a collection of such morphisms (with the same caveats as above). What is defined here is then called a locally small category. Since all categories appearing in this thesis are locally small and since it will be an important property for us, we make it part of the definition.

Typically, morphisms in a category are visualised in diagrams such as

$$\begin{tikzcd}
& T \arrow[d, "h"] \arrow[ddl, bend right, "f"'] \arrow[ddr, bend left, "g"] & \\
& P \arrow[dl, "k"] \arrow[dr, "l"'] & \\
X & & Y
\end{tikzcd}$$

Such a diagram is called commutative if any of the possible paths from one object in the diagram to another composes to the same morphism. Typically in category theory, properties on objects are defined through commutative diagrams. Such properties are called universal properties. It is easiest to understand this idea through examples, which we will look at now, before going into generality. The easiest examples of universal properties are those of being initial or final.

Definition A.1.5. An object 0 in a category is called initial if for any object X, there is a unique morphism $0 \to X$. An object 1 in a category is called terminal if for any object X, there is a unique morphism $X \to 1$.

Example A.1.6.

- In the category Set, the empty set is the only initial object. Final objects are precisely those with only one element.
- In the category of abelian groups, initial and final objects agree and are given (up-to isomorphism) by the trivial group.
- In the category of rings, initial objects are isomorphic to the ring $\mathbb{Z}$ of integers, final objects are isomorphic to the trivial ring where $0 = 1$.

Universal properties can also be more complicated and involve several morphisms at once.

Definition A.1.7. Given objects X, Y in a category $\mathcal{C}$, an object $X \times Y$ is called a product of X and Y if it has the following properties:

- There are morphisms $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$.

- For any object T and morphisms $f\colon T \to X$, $g\colon T \to Y$, there is a unique morphisms $(f,g)\colon T \to X \times Y$ such that

$$\begin{tikzcd}
& T \arrow[ddl, bend right, "f"'] \arrow[d, dashed, "{(f,g)}"] \arrow[ddr, bend left, "g"] & \\
& X \times Y \arrow[dl, "\pi_X"] \arrow[dr, "\pi_Y"'] & \\
X & & Y
\end{tikzcd}$$

commutes.

Example A.1.8.

- In the category of sets, the Cartesian product is an example of a product in the sense of A.1.7.
- In the category of groups, the direct product satisfies A.1.7.

Every definition and theorem in category theory gives rise to a dual, which is obtained by turning around all the arrows in the definition. As a convention, a name is often prefixed with "co-" to indicate its dual.

Definition A.1.9. Given objects X, Y in a category $\mathcal{C}$, an object $X+Y$ is called a coproduct of X and Y if it has the following properties:

- There are morphisms $\iota_X : X \to X+Y$ and $\iota_Y : Y \to X+Y$.
- For any object T and morphisms $f\colon X \to T$, $g\colon Y \to T$, there is a unique morphisms $f+g\colon X+Y \to T$ such that

$$\begin{tikzcd}
& T & \\
& X + Y \arrow[u, dashed, "f+g"] & \\
X \arrow[uur, bend left, "f"] \arrow[ur, "\iota_X"'] & & Y \arrow[uul, bend right, "g"'] \arrow[ul, "\iota_Y"]
\end{tikzcd}$$

commutes.

Category theory is all about focusing on structure-preserving maps. In a rather paradoxical way, we have now introduced the mathematical structure of categories, but not said what the morphisms should be. This needs to be fixed.

Definition A.1.10. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A *functor* $F\colon \mathcal{C} \to \mathcal{D}$ maps every object X of $\mathcal{C}$ to an object $F(X)$ of $\mathcal{D}$ and every morphism $f\colon X \to Y$ in $\mathcal{C}$ to a morphism $F(f)\colon F(X) \to F(Y)$ in $\mathcal{D}$ in such a way that

1. identities are preserved, i.e. $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$
2. composition is preserved, i.e. $F(f \mathbin{;} g) = F(f) \mathbin{;} F(g)$

What is defined above is also called a covariant functor, because it preserves the direction of arrows. There is a related notion of contravariant functor that reverses all arrows. A contravariant functor $\mathcal{C} \to \mathcal{D}$ is nothing but a covariant functor $\mathcal{C}^{\mathrm{op}} \to \mathcal{D}$, so one notion of functor suffices.

Interestingly, the structure of a category is so rich, that we do not only have morphisms between categories (the functors) but even morphisms between those morphisms. This is the notion of natural transformation, historically introduced to formalise the use of the word "natural" in mathematical practice.

Definition A.1.11. Let $F, G\colon \mathcal{C} \to \mathcal{D}$ be functors. A *natural transformation* $\eta\colon F \to G$ is a family of morphisms $\eta_X\colon F(X) \to G(X)$ in $\mathcal{D}$, one for each object X in $\mathcal{C}$. This family has to satisfy the naturality condition, that is the diagram

$$\begin{tikzcd} F(X) \arrow[r, "F(f)"] \arrow[d, "\eta_X"] & F(Y) \arrow[d, "\eta_Y"] \\ G(X) \arrow[r, "G(f)"] & G(Y) \end{tikzcd}$$

has to commute for every morphism $f\colon X \to Y$ in $\mathcal{C}$.

Remark A.1.12. Natural transformations are related to homotopies. Let $2 = 0 \to 1$ be the category with two objects and one non-identity morphism. A natural transformation $\eta\colon F \to G$ between functors

$F, G\colon \mathcal{C} \to \mathcal{D}$ is the same thing as a functor $E\colon 2 \times \mathcal{C} \to \mathcal{D}$ such that $E(0, \_) = F$ and $E(1, \_) = G$. It turns out that functoriality of E translates into the naturality condition of η and vice versa. Thus a natural transformation is the categorical analogue of passing "continuously" from one functor to another.

Definition A.1.13. For two categories $\mathcal{C}$ and $\mathcal{D}$, their product is defined as the category $\mathcal{C} \times \mathcal{D}$ where

- objects are pairs (X, Y) with X an object in $\mathcal{C}$ and Y an object in $\mathcal{D}$
- a morphism $(f, g)\colon (X, Y) \to (X', Y')$ is a pair of a morphism $f\colon X \to X'$ in $\mathcal{C}$ and $g\colon Y \to Y'$ in $\mathcal{D}$
- the identity is the pair $(\mathrm{id}_X, \mathrm{id}_Y)$
- composition is component-wise

The definitions of category and functor line up in a beautiful way to the following: For a locally small category $\mathcal{C}$, for every pair of objects $X, Y \in \mathrm{Ob}(\mathcal{C})$, there is a set $\mathrm{Hom}(X, Y)$. This set of morphisms actually depends on X and Y in a functorial way, so that Hom defines a functor, which is then, appropriately, called the Hom-functor. I like to call the existence of this Hom-functor the most important triviality in category theory.

Proposition A.1.14. *For a (locally small) category $\mathcal{C}$, the set $\mathrm{Hom}(X, Y)$ depends functorially on X and Y, defining a functor $\mathrm{Hom}\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathsf{Set}$.*

Remark A.1.15. Proposition A.1.14 is the reason why all categories in this thesis are assumed to be locally small. It is too beautiful and important of a property to be overshadowed by technicalities.

Despite being such a simple observation, the importance of Proposition A.1.14 can barely be overstated. It leads to the idea of representing a functor with an object, which in turn leads to a formalisation of universal properties and then the wonderful Yoneda Lemma.

Definition A.1.16 (Representations)**.** A functor $F\colon \mathcal{C} \to \mathsf{Set}$ is also called a copresheaf. A *representation* of such a functor is an object $X \in \mathcal{C}$ together with a natural isomorphism $\theta\colon \mathrm{Hom}(X, \_) \cong F$.

Dually, a functor $G\colon \mathcal{C}^{\mathrm{op}} \to \mathsf{Set}$ is also called a presheaf. A *representation* of such a functor is an object $Y \in \mathcal{C}$ together with a natural isomorphism $\theta\colon \operatorname{Hom}(\_, Y) \cong G$.

In either case, a functor that has a representation is called a representable functor.

Representable functors are one of the most fundamental concepts of category theory and examples are ubiquitous throughout mathematics.

Example A.1.17.

- $\operatorname{Hom}(X, \_)$ and $\operatorname{Hom}(\_, X)$ are trivially representable
- An initial object is precisely one representing $F\colon \mathcal{C} \to \mathsf{Set}$, the functor that maps every object to the one-element set (and every morphism to the identity)
- Dually, a terminal object is precisely one that represents the functor $F\colon \mathcal{C}^{\mathrm{op}} \to \mathsf{Set}$ that maps every object to the one-element set.
- A coproduct of X and Y is an object that represents the functor $T \mapsto \operatorname{Hom}(X, T) \times \operatorname{Hom}(Y, T)$.
- Dually, a product of X and Y is an object that represents the functor $T \mapsto \operatorname{Hom}(T, X) \times \operatorname{Hom}(T, Y)$.

Definition A.1.18 (Universal property)**.** An *initial property* is a functor $F\colon \mathcal{C} \to \mathsf{Set}$. Dually, a *final property* is a functor $G\colon \mathcal{C}^{\mathrm{op}} \to \mathsf{Set}$.

A universal property is either an initial property or a final property. An object is said to have a universal property if it represents the defining functor.

We have put emphasis on natural isomorphisms $\operatorname{Hom}(X, \_) \cong F$ (or the contravariant analog). It is an insight due to Nobuo Yoneda that a natural transformation $\theta\colon \operatorname{Hom}(X, \_) \to F$ is uniquely determined by $\theta(\mathrm{id}_X) \in F(X)$. This is remarkable to say the least, because the set of all such natural transformations looks very complicated, but in fact it is just $F(X)$, cleverly disguised.

Lemma A.1.19 (Yoneda Lemma)**.**

- *Let* $F\colon \mathcal{C} \to \mathsf{Set}$ *be a functor and* $X \in \mathcal{C}$ *an object. There is an isomorphism (of sets)*

$$\mathrm{Nat}(\mathrm{Hom}(X, \_), F) \cong F(X)$$

that is natural in X *and* F*.*

- *Replacing* $\mathcal{C}$ *with* $\mathcal{C}^{\mathrm{op}}$ *in the above gives the contravariant version, which we spell out explicitly: For a functor* $F\colon \mathcal{C}^{\mathrm{op}} \to \mathsf{Set}$ *and* $X \in \mathcal{C}$ *an object, There is an isomorphism (of sets)*

$$\mathrm{Nat}(\mathrm{Hom}(\_, X), F) \cong F(X)$$

that is natural in X *and* F*.*

Proof. Let $\eta\colon\ \mathrm{Hom}(X, \_) \to F$ be a natural transformation. Then we can apply η to id_X to obtain $\eta(\mathrm{id}_X) \in F(X)$. It suffices therefore to show that every choice of $x \in F(X)$ extends to a natural transformation η with $\eta(\mathrm{id}_X) = x$ and that every η is uniquely determined by $\eta(\mathrm{id}_X)$. This is a beautiful and instructive exercise left to the reader. □

Corollary A.1.20.

- *A natural transformation* $\theta\colon\ \mathrm{Hom}(\_, X) \to \mathrm{Hom}(\_, Y)$ *is equivalent to a morphism* $X \to Y$*, which is an isomorphism if and only if* θ *is.*

- *A natural transformation* $\theta\colon\ \mathrm{Hom}(X, \_) \to \mathrm{Hom}(Y, \_)$ *is equivalent to a morphism* $Y \to X$*, which is an isomorphism if and only if* θ *is.*

Corollary A.1.21. *Representations are unique up to unique isomorphism.*

Corollary A.1.22. *If two objects satisfy the same universal property, there is a unique isomorphism between them.*

Importantly for us, parameterised representations, depending on two variables, can be assembled into a functor in the following way:

Lemma A.1.23. *Given a functor $H\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{D} \to \mathsf{Set}$. If $H(X, \_)$ is representable for each $X \in \mathcal{C}$, there is a functor $F\colon \mathcal{C} \to \mathcal{D}$ such that*

$$\mathrm{Hom}(F(X), T) \cong H(X, T)$$

If two such functors H_1, H_2 are given with corresponding functors F_1, F_2 then a natural transformation $H_1 \to H_2$ induces a natural transformation $F_2 \to F_1$.

Proof. Use the axiom of choice to pick an object $F(X)$ with

$$\mathrm{Hom}(F(X), T) \cong H(X, T)$$

for each $X \in \mathcal{C}$. Functoriality of this F follows from the Yoneda Lemma: Given a morphism $f\colon X \to Y$, we get a morphism

$$\mathrm{Hom}(F(Y), T) \cong H(Y, T) \to H(X, T) \cong \mathrm{Hom}(F(X), T)$$

natural in T, from which we get a morphism $F(f)\colon F(X) \to F(Y)$ by Corollary A.1.20. That this assignment preserves composition and identities also follows from Corollary A.1.20. The second part is also a consequence of the Yoneda Lemma A.1.19. Given a natural transformation $H_1 \to H_2$, we get

$$\mathrm{Hom}(F_1(X), T) \cong H_1(X, T) \to H_2(X, T) \cong \mathrm{Hom}(F_2(X), T)$$

natural in X and T, so there is a natural transformation $F_2 \to F_1$ as required. $\square$

To finish this part, we want to briefly introduce the notion of a monoidal category. The notion of category revolves around having a sequential composition operation, a monoidal category adds on top of that an operation of parallel composition. They allow for an intuitive description through *string diagrams* which is fundamental to this thesis and will be introduced at the beginning of Chapter 1.

Definition A.1.24. A symmetric monoidal category is a category $\mathcal{C}$ together with a functor $\otimes\colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ and an object I. Furthermore, $\mathcal{C}$ is equipped with natural isomorphisms

- $\alpha\colon A \otimes (B \otimes C) \to (A \otimes B) \otimes C$
- $\lambda\colon I \otimes A \to A$

- $\rho\colon A \otimes I \to A$
- $s\colon A \otimes B \to B \otimes A$

such that the so-called coherence condition holds, that is any diagram constructed through α, λ, ρ and s commutes.

A.1.2 Limits and colimits

A very important class of universal properties comes from the following, very general, idea: Given several objects in a category, is there a way to aggregate them into one object? Suppose our objects furthermore have some morphisms between them and we would like the aggregation to respect that. How would we express that in categorical language? First, we need to formalise what we mean by "several objects with morphisms between them", but that turns out to be simple.

Definition A.1.25. Let $\mathcal{C}$ be a category. A diagram is a functor $\Delta\colon D \to \mathcal{C}$ where D is a small category.

Definition A.1.26. Given a diagram $\Delta\colon D \to \mathcal{C}$, a cone over Δ at object C is a family of morphisms $\alpha_d\colon C \to \Delta(d)$ such that for any morphism $f\colon d \to d'$ in D we have

$$\alpha_d \ ; \Delta(f) = \alpha_{d'}$$

We write $\mathrm{Cone}(C, \Delta)$ for the set of all cones over Δ at C, so the set of all such families α.

A cone is an attempt to aggregate the diagram into a single object. However, in general there can be many different cones that have nothing to do with one another.

Remark A.1.27. A cone over Δ at C is the same thing as a natural transformation $\mathrm{const}_C \to \Delta$, where $\mathrm{const}_C\colon D \to \mathcal{C}$ is the functor that is constant at C.

Corollary A.1.28. $\mathrm{Cone}(\_, \Delta)\colon \mathcal{C}^{\mathrm{op}} \to \mathsf{Set}$ *is a functor.*

Definition A.1.29. A limit of Δ is an object L representing $\mathrm{Cone}(\_, \Delta)$, so $\mathrm{Hom}(\_, L) \cong \mathrm{Cone}(\_, \Delta)$.

The dual is also very important.

Definition A.1.30. Given a diagram $\Delta\colon D \to \mathcal{C}$, a cocone over Δ at object C is a family of morphisms $\alpha_d\colon \Delta(d) \to C$ such that for any morphism $f\colon d \to d'$ in D we have

$$\Delta(f) \mathbin{;} \alpha_{d'} = \alpha_d$$

We write $\mathrm{Cocone}(\Delta, C)$ for the set of all cocones over Δ at C, so the set of all such families α.

Remark A.1.31. A cocone over Δ at C is the same thing as a natural transformation $\Delta \to \mathrm{const}_C$, where $\mathrm{const}_C\colon D \to \mathcal{C}$ is the functor that is constant at C.

Corollary A.1.32. $\mathrm{Cocone}(\Delta, C)\colon \mathcal{C} \to \mathsf{Set}$ *is a functor.*

Definition A.1.33. A colimit of Δ is an object C representing $\mathrm{Cocone}(\Delta, \_)$, i.e. $\mathrm{Hom}(C, \_) \cong \mathrm{Cocone}(\Delta, \_)$.

Example A.1.34.

- If 2 is the category with two objects and no non-identity morphisms, a limit over a diagram $\Delta\colon 2 \to \mathcal{C}$ is a product.
- Dually, a colimit over a diagram $\Delta\colon 2 \to \mathcal{C}$ is a coproduct.
- If $D = X \to Y \leftarrow Z$ is the cospan then a limit over a diagram $\Delta\colon D \to \mathcal{C}$ is called a pullback.
- Dually, if $D = X \leftarrow Y \to Z$ is the span then a colimit over a diagram $\Delta\colon D \to \mathcal{C}$ is called a pushout.

A.1.3 Adjunctions

Arguably one of the most fundamental concepts of category theory is that of adjoint functors, introduced by Daniel Kan in [30]. There is no way of doing the concept justice in an appendix, so this is intended to serve as a very rough overview.

Definition A.1.35. Given functors $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{C}$, we call F *left-adjoint* to G (conversely, we call G *right-adjoint* to F) if

$$\mathrm{Hom}_{\mathcal{D}}(F(X), Y) \cong \mathrm{Hom}_{\mathcal{C}}(X, G(Y))$$

naturally in X and Y.

The definition looks innocent enough but the concept is both remarkably powerful and ubiquitous.

Example A.1.36.

- For Vect the category of vector spaces over a field k, the forgetful functor Vect $\to$ Set, that forgets the vector space structure, has a left-adjoint Set $\to$ Vect that sends a set X to the free vector space k^X.
- In Set, the functor Set $\to$ Set, $X \mapsto A \times X$ is left adjoint to the Hom-functor $\mathrm{Hom}(A, \_)$.
- For any category $\mathcal{C}$, the unique functor $\mathcal{C} \to 1$, where 1 is the category with one object and one (identity) morphism, has a left-adjoint if and only if $\mathcal{C}$ has an initial object.

Proposition A.1.37. *Adjoints are unique up to natural isomorphism.*

Proof. Follows from Lemma A.1.23. □

Lemma A.1.38. *Right-adjoint functors preserve limits. Dually, left-adjoint functors preserve colimits.*

Proof. We prove the first statement, the second is dual. Let $G\colon \mathcal{D} \to \mathcal{C}$ be right-adjoint to $F\colon \mathcal{C} \to \mathcal{D}$. Let Δ be a diagram in $\mathcal{D}$. Let L be a limit of Δ in $\mathcal{D}$. Then we have

$$\mathrm{Hom}(\_, G(L)) \cong \mathrm{Hom}(F(\_), L) \cong \mathrm{Cone}(F(\_), \Delta) \cong \mathrm{Cone}(\_, G \circ \Delta)$$

□

A.2 Weak limits and colimits

One can define a weak notion of limits and colimits where the uniqueness requirement in the definition of the universal property is dropped. While the notion makes sense for general limits and colimits, relevant for us are weak pushouts and weak pullbacks.

Definition A.2.1 (Weak pullback)**.** A weak limit of Δ is an object L equipped with a natural epimorphism $\mathrm{Cone}(\_, \Delta) \to \mathrm{Hom}(\_, L)$, meaning that every cone over Δ factors through L but not necessarily in a unique way.

Proposition A.2.2. *Let*

$$\begin{tikzcd} P \arrow[r] \arrow[d] & A \arrow[d] \\ B \arrow[r] & C \end{tikzcd}$$

be a pullback diagram and

$$\begin{tikzcd} W \arrow[drr, bend left, "a'"] \arrow[dr, dashed, "\pi"] \arrow[ddr, bend right, "b'"'] & & \\ & P \arrow[r, "a"] \arrow[d, "b"] & A \arrow[d] \\ & B \arrow[r] & C \end{tikzcd}$$

a commutative diagram which induces the morphism $\pi\colon W \to D$ by the universal property of P. Then W is a weak pullback if and only if π is a split epi.

Proof.

- If π is a split epi, that means that the pullback diagram factors through W, which by the universal property of P implies that every cone factors through W.
- If W is a weak pullback, there is a morphism $g\colon P \to W$ by the weak universal property of W. But now $g \,;\, \pi\colon P \to P$ satisfies $g \,;\, \pi \,;\, b = g \,;\, b' = b$ and likewise $g \,;\, \pi \,;\, a = a$, so by the universal property of P we have $g \,;\, \pi = \mathrm{id}$.

□

A.3 Free categories and graphs

We will need a technical Lemma on how colimits behave under gluing of diagrams. For that it is useful to have a notion of category without composition, which is just a graph. Here, a graph consists of a set of vertices and for each pair of vertices a set of edges. A category is then nothing but a graph with extra structure, namely identities and composition.

Proposition A.3.1. *The forgetful functor from (small) categories to graphs, $F\colon \mathsf{Cat} \to \mathsf{Graph}$ has a left-adjoint L, the free category functor.*

Using the free category construction, we can talk about diagrams in a category $\mathcal{C}$ of shape a graph G, by which we mean a functor $L(G) \to \mathcal{C}$. This allows us to transfer the notions of cone, cocone, limit and colimit in the obvious way.

Lemma A.3.2 (Gluing shapes)**.** *Let $D\colon I \to \mathsf{Graph}$ be a diagram of graphs, I an index category. Let $\mathcal{D}$ be the colimit of D. Let $\Delta, \Gamma\colon \mathcal{D} \to \mathcal{C}$ be two diagrams. Let $f_i\colon D_i \to \mathcal{D}$ be the canonical morphism into the colimit. Then*

$$\mathrm{Nat}(\Delta, \Gamma) \cong \lim_i \mathrm{Nat}(f_i \mathbin{;} \Delta, f_i \mathbin{;} \Gamma)$$

Proof. The limit on the right-hand side is taken in Set. Explicitly, an element of this set is given by a family of natural transformations η_i, one for each $i \in I$, such that for a morphism $\kappa\colon i \to j$ in I, we have $D(\kappa) \mathbin{;} \eta_j = \eta_i$.

A natural transformation $\eta\colon \Delta \to \Gamma$ gives rise to a family $\eta_i = f_i \mathbin{;} \eta$, which satisfies the above condition.

The more interesting direction is the converse, given such a family η_i, can it be combined into a single natural transformation η? Note that a natural transformation $\eta\colon \Delta \to \Gamma$ can be seen as a functor $\mathcal{D} \to \mathrm{Fun}(2, \mathcal{C})$. Thus, a family η_i gives rise to functors $D_i \to \mathrm{Fun}(2, \mathcal{C})$, which combine into a functor $\eta'\colon \mathcal{D} \to \mathrm{Fun}(2, \mathcal{C})$ by the universal property of the colimit. Letting $a, b\colon \mathrm{Fun}(2, \mathcal{C}) \to \mathcal{C}$ be the evaluation at the first, respectively the second object, we get

$$f_i \mathbin{;} \eta' \mathbin{;} a = \eta_i \mathbin{;} a = f_i \mathbin{;} \Delta$$

hence, again by the universal property of the colimit, $\eta' \mathbin{;} a = \Delta$ and likewise $\eta' \mathbin{;} b = \Gamma$, so that η' defines a natural transformation $\eta\colon \Delta \to \Gamma$. $\square$

Corollary A.3.3 (Dependent colimit)**.** *Let $D\colon I \to \mathsf{Graph}$ be a diagram of graphs, $\mathcal{D}$ the colimit. Let $\Delta\colon \mathcal{D} \to \mathcal{C}$ be a diagram and $\Delta_i = f_i \mathbin{;} \Delta$ for $f_i\colon D_i \to \mathcal{D}$ the natural morphism. Then*

$$\operatorname{colim} \Delta \cong \operatorname*{colim}_i \operatorname{colim} \Delta_i$$

Proof.

$$\begin{aligned}\mathrm{Hom}(\operatorname{colim}\Delta, Y) &\cong \mathrm{Nat}(\Delta, \mathrm{const}_Y) \cong \lim_i \mathrm{Nat}(\Delta_i, \mathrm{const}_Y) \\ &\cong \lim_i \mathrm{Hom}(\operatorname{colim}\Delta_i, Y) \cong \mathrm{Hom}(\operatorname*{colim}_i \operatorname{colim}\Delta_i, Y)\end{aligned}$$

□

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