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Multi-soliton interactions and the inverse problem of wave crests

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Declaration

I declare that this thesis is my original, unaided work. It is being submitted for the degree of Doctor of Philosophy in Natural Sciences at Tallinn Technical University, Tallinn, Estonia. It has not been submitted before for any degree or examination at any other university.

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Sisukokkuvõte

Antud töö eesmärgiks on leida detailne kirjeldus paljude solitonide üheaegsele vastastikmõjule. Selleks defineeriti, konstrueeriti ja analüüsiti Kortewegi-de Vriesi (KdV) tüüpi võrrandite klassi mitmiksolitonilisi lahendeid. Analüüsi täpsustuseks esitati interaktsioonisolitoni mõiste ja leiti uudne mitmiksolitoniliste lahendite lahutus. Selle lahutuse kohaselt on mitmiksolitoniline lahend solitonide ja interaktsioonisolitonide lineaarne superpositsioon. Interaktsioonisolitoni mõistet on rakendatud paljude solitonide vastastikmõju interpreteerimisel. Vastastikmõju tulemusel tekkivate lainemustrite analüüs vajab selget geomeetrilist interpretatsiooni. Suvalise arvu solitonide vastastikmõju mustrite geomeetriliseks kirjelduseks on konstrueeritud uudne algoritmiline meetod. Kõiki uusi mõisteid on demonstreeritud kahe solitoni vastastikmõju juhu jaoks. On toodud ka vastavaid näiteid kolme ja viie solitoni vastastikmõju juhtudest. Solitonide vastastikmõju illustreerivate mudelitena on kasutatud KdV, KdV-Sawada-Kotera ja Kadomtsevi-Petviashvili (KP) võrrandeid. Antud töö praktilise rakenduse demonstreerimiseks on vaadeldud laineharjade pöördülesannet. On tõestatud, et kahe KP solitoni vastastikmõju korral on sellel pöördülesandel ühene lahend. On uuritud selle lahendi tundlikkust võimalikele mõõtmisvigadele.

Abstract

The aim of this thesis is to find a detailed description of multi-soliton interactions. For that multi-soliton solutions of KdV type equations are defined, constructed, and analyzed in phase variables. As a result of the analysis, the concept of an interaction soliton is introduced and a novel decomposition of multi-soliton solutions is proposed. According to the decomposition, a multi-soliton solution is a linear superposition of solitons and interaction solitons. The concept of the interaction soliton is exploited to interpret multi-soliton interactions. A geometric representation of interaction patterns is introduced and an algorithmic way to construct the interaction patterns for an arbitrary number of solitons is proposed and implemented. All new concepts are illustrated for two-soliton interactions, examples are given also for three- and five-soliton interactions. For exemplifying models of soliton interactions, the KdV, KdV-Sawada-Kotera, and KP equations are used. As a practical application of these findings, an inverse problem of wave crests is introduced. The uniqueness of a solution to the inverse problem is proved for the KP two-soliton interactions. Sensitivity of this solution is analyzed against possible measurement errors.

Contents

Foreword

In many fields of physics the studies of finite amplitude wave phenomena in dispersive medium often lead to simplified mathematical models that support *soliton* solutions. Various aspects of soliton phenomena have been studied intensively by many authors over more than recent three decades with profound details. The aim of this thesis is to complement this knowledge by *a detailed description of multi-soliton interactions*. This description is needed for many practical applications, for instance, for solving both the direct and inverse problems in multi-directional wave phenomenon (e.g. surface waves).

List of Publications

Publication I P. Peterson and E. van Groesen. A direct and inverse problem for wave crests modelled by interactions of two solitons. *Phys. D*, 141:316–332, 2000.

Publication II P. Peterson and E. van Groesen. Sensitivity of the inverse wave crest problem. *Wave Motion*, 34:391–399, 2001.

Publication III P. Peterson. Construction and decomposition of multi-soliton solutions of KdV type equations (submitted).

Publication IV P. Peterson. Construction of multi-soliton interaction patterns of KdV type equations (submitted).

In this thesis various items from the above publications are referred, for example, as Figure **II**.3, Eq. (**III**.10), Theorem **IV**.3.1, etc.

My research has been an essential part of all these **Publications**. I have performed all the analytical work, computations of plots, and writing of the **Publications**. In the following survey on the subject of my thesis, I claim to express my original thoughts and results from the **Publications** that are furnished with several well-known facts, established by others, with references to relevant sources, to make this thesis self-contained and as easy to follow as possible.

Definitions

KdV type equation	$\equiv$	PDE for $u = u(x, y, t)$ that is bi-linearizable into Hirota bilinear form $P(D_x, D_y, D_t) = 0$
$P = P(\mu, \nu, \omega)$	$\equiv$	Hirota polynomial of the corresponding KdV type equation
$\boldsymbol{x}$	$\equiv$	vector of real variables $(x, y)^T$
t	$\equiv$	time parameter
u_g	$\equiv$	g-soliton solution $u_g(\boldsymbol{x}, t) = U_g(\mathbf{K}\boldsymbol{x} + \boldsymbol{\omega} t)$
U_g	$\equiv$	g-soliton solution in phase variables
S_{index}	$\equiv$	soliton term with index $= \kappa(\boldsymbol{\alpha})$ in the decomposition $U_g = \sum_{\boldsymbol{\alpha}\in\{0,1\}^g} S_{\kappa(\boldsymbol{\alpha})}$
$\boldsymbol{k}$	$\equiv$	wave vector $(\mu, \nu)^T$
$\mathbf{K}$	$\equiv$	wave matrix with transposed wave vectors as row vectors
$\boldsymbol{\mu}, \boldsymbol{\nu}$	$\equiv$	columns of the wave matrix $\mathbf{K}$
$\boldsymbol{\varphi}$	$\equiv$	vector of phase variables $\boldsymbol{\varphi} = \mathbf{K}\boldsymbol{x} + \boldsymbol{\omega} t$
Δ_{ij}	$\equiv$	phase shift parameter $\Delta_{ij} = -\ln A_{ij}$
A_{ij}	$\equiv$	phase shift coefficient $A_{ij} = -\frac{P(\mu_i-\mu_j,\nu_i-\nu_j,\omega_i-\omega_j)}{P(\mu_i+\mu_j,\nu_i+\nu_j,\omega_i+\omega_j)}$

1 Introduction

Soliton theory — a theory of nonlinear partial differential equations (nonlinear PDEs) — started to evolve more than thirty years ago. Today this theory is deep and mathematically beautiful, but still developing fast, driven by many applications in various fields of physical science and engineering.

One of the self-evident aims of soliton theory is to provide methods for solving nonlinear PDEs by analytical means. Indeed, different methods have been developed during these years. These include, for example, the inverse scattering theory (IST) [Gardner, Green, Kruskal, and Miura, 1967] that allows to solve the initial-value problems of many (integrable) nonlinear PDEs; the Hirota n-linear ($n > 1$) formalism [Hirota, 1980, Grammaticos, Ramani, and Hietarinta, 1994] according to which whole families of special solutions can be directly constructed, even for non-integrable equations; etc.

In this thesis the Hirota formalism is exploited. In the framework of Hirota n-linear formalism, the basic idea of solving nonlinear PDEs is the following: to find the change of independent variables such that the constitutive equation for new variables is n-linear (for some positive n). If such a change of variables exists, constructing various solutions is rather straightforward because finding solutions to n-linear equations is relatively easy. For simplicity, in the following $n = 2$ is assumed. Grammaticos, Ramani, and Hietarinta [1990] classified such, bi-linearizable, equations into five groups: KdV, mKdV, NLS, SG, and BO type equations (this list may not be complete). This classification is based on the different ways how the corresponding equations are bi-linearized. Also the properties of solutions are expected to be different for members from different classes of PDEs (see Hietarinta [1987a,b,c, 1988]). In this thesis only KdV type equations are considered and the notion "KdV type equations" is used *strictly* in the sense of Hirota bilinear formalism.

For practical applications as well as for theoretical understanding of soliton phenomenon, a complete analysis of soliton solutions is needed. The key question in such an analysis is: *"How do solitons interact?"*. The aim of this thesis is to clarify this problem for the cases where an *arbitrary* number of solitons is in simultaneous interaction. Such a problem has not been addressed or solved before. Although several attempts have been made to describe interactions between up to three solitons [Anker and Freeman, 1978, Moloney and Hodnett, 1991], no conclusive results have been reported for the general case.

This thesis consists of two parts. The first part (Section 2, except 2.1.3) is devoted to well-known results on soliton interactions in order to prepare ground for explaining novel results of this thesis. The second part consists of an overview of **Publications** that are listed in Page 6 and appended to this thesis. The overview contains:

- Two-soliton interactions in one and two space dimensions in Sections 2.1.3 and 3.1, respectively (**Publication I**).

- Multi-soliton interactions in two space dimensions in Section 3.2 (**Publication III** and **Publication IV**).
- The inverse problem of wave crests as an application of the above results in Section 4 (**Publication I** and **Publication II**).

In Section 5 possible extensions to this study are discussed. Summary of the results is given in Section 6.

2 Solitons and their interactions

Soliton phenomena exist on two conditions. First, there must exist stable solitary waves that travel with constant configurations (shape, speed, etc.) as long as they do not meet any external obstacles. Second, if a solitary wave meets another of its kind, they interact, but without destroying each other's identities (*elastic interaction*). Such solitary waves are called *solitons*.

The soliton phenomenon is essentially a nonlinear phenomenon. First, solitons can exist due to a delicate equilibrium between (linear) scattering (the dispersion phenomenon) and (nonlinear) convective (the shock wave phenomenon) actions. And second, the world path (the trajectory in space-time space) of a soliton suffers phase shifts caused by interactions with other solitons. The values of the phase shifts depend on the amplitudes of interacting solitons.

Soliton phenomena can certainly be observed in Nature. In order to take advantage of the existence of solitons for practical applications, it is important to learn more about the interaction process between solitons. The question "What actually happens during the interaction process of solitons?" is fundamental also from a theoretical point of view. For example, one of the most robust tests for a mathematical model to be (non-)integrable, is to study interactions of its solitary waves. If the interaction is inelastic, then the model is unlikely to be integrable.

2.1 Soliton interactions in $(1+1)$-dimensional models

2.1.1 Soliton interactions in the Korteweg-de Vries model

The simplest model for the soliton phenomenon is the Korteweg-de Vries (KdV) equation. Originally it was derived in 1895 by Korteweg and de Vries to model unidirectional propagation of small but finite amplitude long surface waves in shallow water. Later on, the KdV equation proved to be a rather universal model for long wave phenomena in dispersive media in many fields of physics. In the following the basic results about KdV soliton interactions are reviewed.

The KdV equation (in normalized variables) reads

$$u_t + 6uu_x + u_{xxx} = 0, \tag{1}$$

where $u = u(x,t)$ and subscripts denote partial derivatives. The KdV equation has a solitary wave solution (one-soliton solution) of the form

$$u_1(x,t;\mu) = \frac{\mu^2}{2}\operatorname{sech}^2\frac{\mu}{2}(x-\mu^2 t), \tag{2}$$

where μ is a free parameter. Note that the amplitude of this solitary wave (soliton) is proportional to its speed. Its world path is a straight line $x-\mu^2 t = 0$ where function u_1 obtains its maximum for fixed t. The proof for the stability of (2) to shape disturbances is given by Benjamin [1972].

For the KdV equation one can find an exact solution for the situation where two solitary waves, with different amplitudes, meet each other. To be specific, such a two-soliton solution can be given by the following formula (see any text-book on solitons):

$$u_2(x,t;\mu_1,\mu_2) = 2\frac{\partial^2}{\partial x^2}\ln\theta, \tag{3}$$

where

$$\theta = 1 + e^{\varphi_1} + e^{\varphi_2} + A_{12}e^{\varphi_1+\varphi_2}, \tag{4}$$

$$\varphi_i = \mu_i(x-\mu_i^2 t), \qquad i = 1,2, \tag{5}$$

$$A_{12} = (\mu_1-\mu_2)^2/(\mu_1+\mu_2)^2, \tag{6}$$

and $\mu_{1,2}$ are free parameters. It is easy to find that for $\mu_1 > \mu_2 > 0$ (that is assumed throughout of this Section) the following approximations hold:

$$u_2(x,t;\mu_1,\mu_2) \approx u_1(x,t;\mu_1) + u_1(x-\delta_2,t;\mu_2) \qquad \text{as } t\to-\infty, \tag{7}$$

$$u_2(x,t;\mu_1,\mu_2) \approx u_1(x-\delta_1,t;\mu_1) + u_1(x,t;\mu_2) \qquad \text{as } t\to+\infty, \tag{8}$$

where $\delta_{1,2} = \Delta_{12}/\mu_{1,2}$ are phase shift constants and $\Delta_{12} = -\ln A_{12}$ is a phase shift parameter. In the KdV case, $\delta_{1,2}$ are positive because $0 < A_{12} < 1$ holds. The interpretation of the above approximation is that the interaction between two solitons causes the larger soliton (with μ_1) to shift forwards in x-space by δ_1, while the smaller soliton (with μ_2) gets shifted backwards by $\delta_2 > \delta_1$.

The above analysis can be extended for situations where an arbitrary number of different solitons interact with each other (this is possible for the KdV equation and a number of other equations that are integrable (in Hirota sense). A result of this analysis is that the total phase shift of one soliton equals to the sum of phase shifts obtained by pair-wise interactions with other solitons. In fact, for the g number of solitons, the i-th soliton experiences a total phase shift

$$\frac{1}{\mu_i}\sum_{j=1}^{g}([\mu_j<\mu_i]-[\mu_j>\mu_i])\Delta_{ij} \tag{9}$$

as t varies from $-\infty$ to ∞. Here symbol $[B]$ denotes function that has value 1 if the Boolean expression B is true, and 0 otherwise.

Visualization (of numerical experiments) gives some idea on what happens during the interaction of two solitons. When two KdV solitons meet, the resulting wave profile (as seen in the middle of interaction process) can have two different shapes. If the ratio μ_2/μ_1 is smaller than $1/3$ (one soliton is considerably smaller than the other), then the profile has only one maximum. But if $1 > \mu_2/\mu_1 > 1/3$ (the amplitudes of solitons are close) then the resulting wave profile has two maxima. See e.g. Yoneyama [1984] for relevant figures. Note that if $\mu_2 \to \mu_1$ (nearly equal solitons), then the visualization may give a misleading illusion of "bouncing off" solitons (see LeVeque [1987]) — it should be treated as a simple limiting case of the general behavior (see Hodnett and Moloney [1989] for more extensive discussion about this matter).

More detailed information on the soliton interaction process is obtained from the analysis of a two-soliton solution (3). Universal to many such studies is that the two-soliton solution is decomposed into a sum of terms which are attached with some relevant meaning. However, different but equally reasonable approaches may give sometimes contradictory results. Below some well-known approaches are described.

Historically, the inverse scattering theory (IST) provided the first systematic method to derive soliton solutions for various evolution equations (see e.g. Miura [1976] for a review about IST). Based on the IST, Gardner, Greene, Kruskal, and Miura [1974] (GGKM) found that a g-soliton solution can be represented as a sum of g terms, each representing a soliton and being given in terms of discrete spectrum of the associated Sturm-Liouville equation. This "eigenvalue and eigenfunction decomposition" have been exploited by Caenepeel and Malfliet [1985] to explore the internal structure of the two-soliton solution. By writing

$$u_2 = u^{(1)} + u^{(2)} \tag{10}$$

for all time moments, where $u^{(1,2)}$ represent the GGKM soliton terms, Caenepeel and Malfliet visualize the graphs of $u^{(1,2)}$ for the case $\mu_2/\mu_1 < 1/3$ (see above). They conclude that at the beginning of interaction process the larger soliton diminishes its amplitude and accelerates at the same time (results positive phase shift; my note: the acceleration contradicts the amplitude-speed relation of solitons). And the smaller soliton splits into two parts, the right-hand part "leaking" to the left-hand part (results negative phase shift; my note: the splitting contradicts the solitary character of solitons). Moloney and Hodnett [1986], Hodnett and Moloney [1989] derive equivalent representation (10) from Hirota method but they write soliton terms $u^{(1,2)}$ in a more elegant form (similar to one-soliton solution (2)):

$$u^{(1)} = \frac{\mu_1^2}{2} g_x(\varphi_1, \varphi_2) \operatorname{sech}^2 \frac{1}{2} g(\varphi_1, \varphi_2), \tag{11}$$

$$u^{(2)} = \frac{\mu_2^2}{2} g_x(\varphi_2, \varphi_1) \operatorname{sech}^2 \frac{1}{2} g(\varphi_2, \varphi_1), \tag{12}$$

where $g(\varphi_1, \varphi_2) = \varphi_1 + \ln((1 + A_{12}e^{\varphi_2})/(1 + e^{\varphi_2}))$. Yoneyama [1984] shows that $u^{(1,2)}$ satisfy the so-called Interacting KdV equations:

$$u^{(i)}{}_t + 6(u^{(1)} + u^{(2)})u^{(i)}{}_x + u^{(i)}{}_{xxx} = 0.$$

In order to track the world paths of solitons during the interaction, two different definitions of soliton positions have been proposed. The first one, proposed by Caenepeel and Malfliet [1985], defines the position of the i-th soliton as the center of its mass:

$$x_G^{(i)}(t) = \frac{\int x u^{(i)}\,dx}{\int u^{(i)}\,dx},$$

where integration is performed over the real axis. The second one, given by Moloney and Hodnett [1986], defines the path of the i-th soliton as the function $x_P^{(i)} = x_P^{(i)}(t)$ such that

$$\int_{-\infty}^{x_P^{(i)}} u^{(i)}\,dx = \int_{x_P^{(i)}}^{+\infty} u^{(i)}\,dx$$

holds. For $t \to \pm\infty$, soliton paths $x_G^{(i)}$ and $x_P^{(i)}$ coincide which, however, is not true for finite t. Following the path of the smaller soliton according to $x_G^{(2)}$, Caenepeel and Malfliet [1985] find that this soliton has negative velocity (authors note: the negative velocity contradicts the assumption of unidirectional wave propagation). The same observation is made in Moloney and Hodnett [1986] where the evolution of $x_P^{(2)}$ is followed. More drastically, the latter authors find that the speed of the smaller soliton may become infinite.

In conclusion, various contradictions, a few of them have been pointed out above (see also Bryan and Stuart [1992]), indicate that the GGKM representation (10) *is not suitable* for describing the interaction process of solitons in a detailed manner.

A different decomposition from the GGKM one, is proposed by Bryan and Stuart [1992]. They introduce the decomposition of a g-soliton solution as "a linear superposition of accelerating solitary waves and interaction terms". The total number of terms in this superposition is $2g$. The authors show that shortcomings in the interpretation of the GGKM decomposition can be avoided with their decomposition. However, they do not give any meaning to the interaction terms.

In **Publication I** a novel decomposition of a two-soliton solution is introduced (this result is generalized in **Publication III** for g-soliton solutions). In this decomposition, a soliton solution is written as linear superposition of solitons and interaction solitons (in general, the total number of soliton terms being $2^g - 1$). The main advantage of this representation over previously described ones is that, first of all, it supports a clear description of what happens during the interaction of solitons. Conclusions for interactions between unidirectional solitons are given in Section 2.1.3 below.

2.1.2 Soliton interactions in the KdV-Sawada-Kotera model

The majority of studies on soliton interactions concentrate on one equation — the KdV equation. However, there is a number of equations that support soliton solutions with exactly the same analytical form as given in (3), and therefore, all these equations should have similar descriptions about soliton interactions. Obviously, the KdV equation is preferred as a representative for demonstrating soliton phenomena because the KdV equation is the simplest possible equation that has nonlinear and dispersion terms which seem to be necessary for the soliton phenomenon. However, it turns out that *the KdV equation is too simple* to demonstrate all aspects of soliton phenomena.

To illustrate this, consider the KdV-Sawada-Kotera equation studied in Hirota and Ito [1983]

$$u_t + a(3u^2 + u_{xx})_x + b(15u^3 + 15uu_{xx} + u_{xxxx})_x = 0 \tag{13}$$

that, clearly, for $b = 0$ becomes the KdV equation and for $a = 0$ the Sawada-Kotera equation. For all values of a and b, the KdV-Sawada-Kotera equation supports multi-soliton solutions. A two-soliton solution of the KdV-Sawada-Kotera equation is in the form as given in (3) but with the following phase variables and the phase shift coefficient:

$$\varphi_i = \mu_i(x - (a\mu_i^2 + b\mu_i^4)t), \qquad i = 1, 2, \tag{14}$$

$$A_{12} = \frac{(\mu_1 - \mu_2)^2(3a + 5b(\mu_1^2 - \mu_1\mu_2 + \mu_2^2))}{(\mu_1 + \mu_2)^2(3a + 5b(\mu_1^2 + \mu_1\mu_2 + \mu_2^2))} \tag{15}$$

where $\mu_{1,2}$ are free parameters. Hirota and Ito [1983] reported that for different amplitude ratios the interaction of two solitons can be one of the following scenarios: (i) two solitons interact through emitting and absorbing the third soliton, (ii) two solitons fuse to one soliton and then this splits into two solitons, (iii) two solitons become singular after colliding with each other. It turns out that the value of phase shift coefficient A_{12} determines which one of these scenarios is realized. The corresponding intervals for the three interaction scenarios are: (i) $0 < A_{12} < 1$, (ii) $1 < A_{12}$, (iii) $A_{12} < 0$, respectively.

Note that for the KdV equation only the scenario (i) is possible. And even then, it is impossible to see the third soliton. Namely, this third soliton can become visible (i.e. separated from the initial two solitons) only if the phase shift parameter $\Delta_{12} = -\ln A_{12}$ is large enough (i.e. the phase shift constants are larger than the widths of solitons). In the KdV case, this is possible only if two solitons are nearly equal. In **Publication I** it is shown that the amplitude of the third soliton decreases to zero as $\mu_2 \to \mu_1$. Therefore, we can observe only the bouncing effect of the KdV solitons where the third soliton is too small to be visible.

The two-soliton solution of the KdV-Sawada-Kotera equation shows also that unidirectional solitons can have resonances [Hirota and Ito, 1983]. The condition for the resonance is $|\Delta_{12}| = \infty$, that is, the phase shifts of solitons are infinitely large. For the interaction scenario (i) this means the situation where one soliton spontaneously splits into two solitons. And for the scenario (ii) the resonance condition means that two solitons fuse into one — the so-called resonance soliton.

2.1.3 Soliton interactions in KdV type equations

Interactions of two solitons can be treated in a unified manner. This is described in **Publication I** for multi-directional wave models of KdV type. Below these results are given for unidirectional wave models for complementing the studies described above.

Unified treatment of soliton interactions is possible because all $(1+1)$-dimensional KdV type models have two-soliton solutions with the same analytical form $u_2(x,t) = U_2(\varphi_1, \varphi_2)$, where

$$U_2 = L[1 + e^{\varphi_1} + e^{\varphi_2} + A_{12}e^{\varphi_1+\varphi_2}], \tag{16}$$

$L[\theta] = 2(\mu_1\partial_1 + \mu_2\partial_2)^2 \ln\theta$ (in **Publication III** $L[\theta]$ is written in terms of Hirota derivatives allowing a more general analysis), $\varphi_{1,2} = \mu_{1,2}x + \omega_{1,2}t$ are phase variables, and $\mu_{1,2}$ are free parameters. From the definition of KdV type equations (see **Publication I** or **Publication III** for details) each KdV type equation can be written in the Hirota bilinear form which is defined by a related even polynomial $P = P(\mu, \omega)$. Often this polynomial is in the form of the dispersion relation. For example, for the KdV-Sawada-Kotera equations the corresponding polynomial is $P = \mu\omega + a\mu^4 + b\mu^6$. This polynomial also determines the relations for the phase shift coefficient A_{12} and the parameters $\omega_{1,2}$ in terms of free parameters $\mu_{1,2}$:

$$\begin{aligned} P(\mu_1, \omega_1) &= 0, & (17)\\ P(\mu_2, \omega_2) &= 0, & (18)\\ P(\mu_1 - \mu_2, \omega_1 - \omega_2) + A_{12}P(\mu_1 + \mu_2, \omega_1 + \omega_2) &= 0. & (19) \end{aligned}$$

The unified analysis of two-soliton interactions is based on the following approach. First, function U_2 in (16) is studied in terms of (canonical) phase variables in the $(\mu_{1,2}, \omega_{1,2}, \Delta_{12})$-parameter space. Note that here no relations between these parameters are assumed. Finally, the description of how two solitons interact for a given KdV type equation is determined by the behavior of U_2 on the manifold given by equations (18)– (19) in the parameter space.

The general behavior of the function U_2 is illustrated in Figure **I**.2 for positive and negative values of the phase shift parameter Δ_{12}. Its structure is best seen for large phase shift values as shown in Figure **I**.4. The graph of U_2 consists of three parts. Two parts out of three correspond to two soliton identities, each part consisting of two long-crested humps being parallel to one of the phase shift variable axis and being shifted relative to each other near to the origin of phase variables (the interaction region). The third part of the graph is localized to the interaction region and is a diagonal soliton-like hump connecting the former two parts. In **Publication I** this third part of the graph is called the *interaction soliton*. It turns out that the function U_2 can be analytically decomposed into a sum of three terms that correspond to the parts above (see **Publication I** for details and **Publication III** for the generalization of this analysis to an arbitrary number of solitons, see Section 3.2.1). This novel decomposition is given as

$$U_2 = S_1 + S_2 + S_{12}, \tag{20}$$

where $S_{1,2}$ are called soliton terms and S_{12} is the interaction soliton term. Plots of these terms are shown in Figure **I**.5 and Figure **I**.6.

Publication I introduces a geometrical representation of two-soliton interactions. The representation consists of line slices representing the positions of supporting regions of soliton terms $S_{1,2,12}$. A collection of these lines is called the *phase pattern set* and denoted by P_U. Phase pattern sets for U_2 are illustrated in Figure **I**.8. **Publication IV** generalizes this geometrical representation for an arbitrary number of solitons (see Section 3.2.2).

In order to describe soliton interactions in the real space using the concepts above, the relation between phase and real-time variables must be used:

$$\begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} x + \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} t \tag{21}$$

or $\boldsymbol{\varphi} = \boldsymbol{\mu} x + \boldsymbol{\omega} t$ for short. It shows that the image of the real space $\{x \in \mathbb{R}\}$ in the space of phase variables is a line $\boldsymbol{\mu}\mathbb{R}$ that has direction vector $\boldsymbol{\mu}$. As the time parameter t is increased, the image of the real space shifts in the direction of vector $\boldsymbol{\omega}$. Since the phase pattern set P_U represents supporting regions of the function U_2, the positions of solitons in the real space are defined by the intersection points of the phase pattern set P_U and the image of the real space. In **Publication I** the set of soliton positions for a given time moment t is denoted by $P_u(t)$ and called the *real pattern set*. The real pattern set $P_u(t)$ represents supporting regions of a two-soliton solution u_2. For one space dimension, the set $P_u(t)$ reads

$$P_u(t) = \frac{1}{\boldsymbol{\mu}^T \boldsymbol{\mu}} \boldsymbol{\mu}^T (P_U \cap (\boldsymbol{\mu}\mathbb{R} + \boldsymbol{\omega} t) - \boldsymbol{\omega} t). \tag{22}$$

This result is a special case of Proposition **IV**.4.1.

Figure 1 illustrates the above description of two-soliton interactions for the KdV-Sawada-Kotera equation. This case corresponds to scenario (i) described in Section 2.1.2 where two solitons interact through emitting and absorbing the interaction soliton.

In conclusion, two-soliton interactions for a wide class of KdV type equations can be described in a unified manner using the concept of an interaction

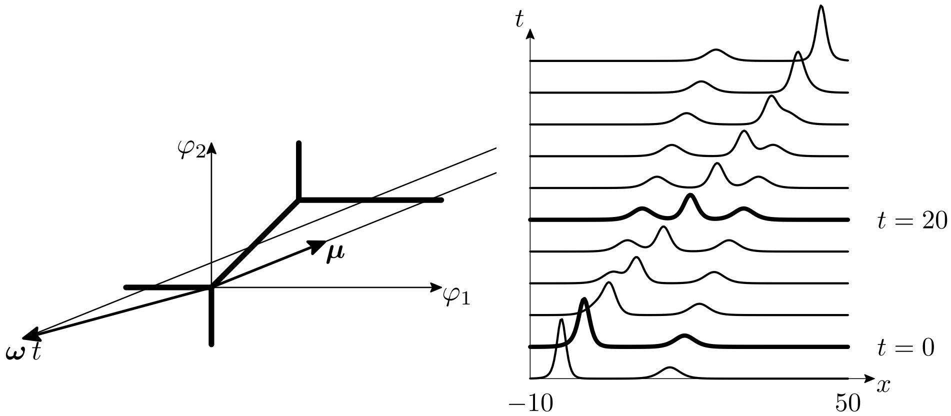

Figure 1: Interaction of two KdV-Sawada-Kotera ($a = 1, b = -1/5$) solitons with $\mu_1 = 1.987452$, $\mu_2 = 0.8$ (parameters taken from Hirota and Ito [1983]). *Left plot:* The positions of solitons are found as the intersection points of the phase pattern set (bold lines) and the image of the real space (thin lines, with direction vector $\boldsymbol{\mu}$) for two time moments $t = 0$ and $t = 20$. *Right plot:* At a certain distance before the larger soliton meets the smaller one, it splits into the smaller soliton and the interaction soliton. Interaction ends when the interaction soliton merges with the smaller soliton resulting the larger soliton.

soliton that is defined through decomposition (20). According to this description, solitons interact through the interaction soliton that can exist only limited time and that is either created by a splitting process of one of the solitons (scenario (i)) or by a collision process of two solitons (scenario (ii)).

A complete analysis of two-soliton interaction for a given KdV type equation is now reduced (a) to studying the corresponding $\Delta_{12} - \mu_{1,2}$ diagram in order to determine possible interaction scenarios and (b) to constructing real pattern set $P_u(t)$ in order to visualize approximate world paths of solitons. No extensive numerical evaluation of soliton solutions (neither from the analytical form nor by integrating the equation) is needed for the complete analysis.

2.2 Soliton interactions in $(2 + 1)$-dimensional models

Soliton phenomena are observed also for multi-directional wave propagation. Figure 2 illustrates the oblique interaction of two solitary waves in shallow water. This photo demonstrates vividly all the features of the soliton phenomenon: existence of stable solitary waves, wave crests in different sides of the interaction region are parallel (indicating elastic interaction) and shifted (indicating nonlinear interaction).

The oblique interaction of two solitary waves is analyzed by Miles [1977a,b] where the author derives two-soliton solution directly from the boundary-value problem for inviscid irrotational motion using the Boussinesq approximation.

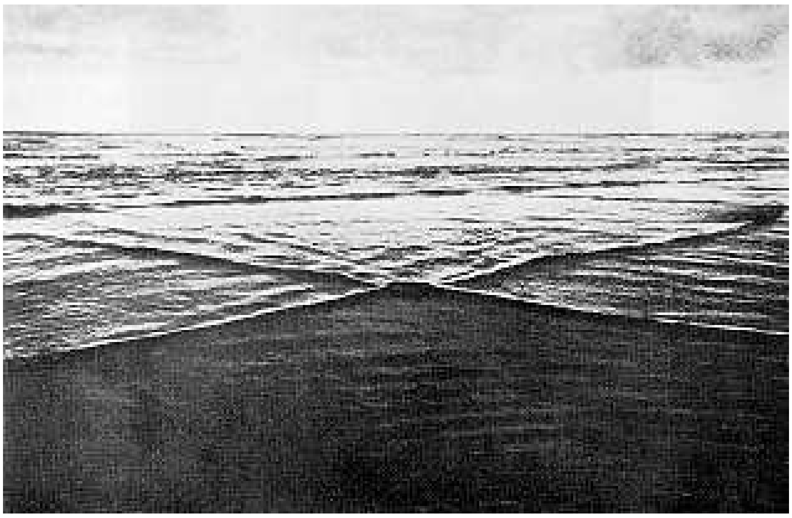

Figure 2: Fig. 12 in Hammack et al. [1995]: "Oblique interaction of two nearly solitary waves in shallow water, off the coast of Oregon. The interaction occurred in water about 1 m deep. (Photograph courtesy of T. Toedtemeier.)"

The corresponding model equation for weakly oblique interactions was originally derived by Kadomtsev and Petviashvili [1970] to study the transversal stability of KdV solitons (in fact, KdV solitons turn out to be stable to transverse perturbations, see also Matsukawa, Watanabe, and Tanaca [1988]). The Kadomtsev-Petviashvili (KP) equation models weakly multi-directional propagation of small but finite amplitude long surface waves in shallow water. The KP equation (in normalized variables) reads

$$(u_t + 6uu_x + u_{xxx})_x + 3u_{yy} = 0, \tag{23}$$

where it is assumed that waves travel mainly in the direction of x-axis. Hammack, Scheffner, and Segur [1989], Hammack et al. [1995] demonstrated in laboratory experiments the oblique interaction of two wave trains and compared them to the corresponding analytical solutions of the KP equation. They found that the KP equation is a remarkably good model for the phenomenon of oblique interaction of waves also for relatively large interaction angles.

A related problem of oblique interaction of waves but with "head on" collisions is studied by Johnson [1996]. To model this phenomenon, the author introduced the two-dimensional Boussinesq equation

$$u_{tt} + (-u_x + 6uu_x - u_{xxx})_x - u_{yy} = 0. \tag{24}$$

This equation extends the classical Boussinesq equation to two spatial dimensions analogously like the KP equation extends the KdV equation, that is, the angles of propagation directions of waves relative to the x-axis are assumed to be small. All these equations belong to the class of KdV type equations, and therefore have two-soliton solutions with the same analytical form (3). Only expressions for the phase shift coefficient A_{12} differ — their concrete expressions are defined by the dispersion relations of the corresponding linearized equations (see e.g. **Publication I** for details). Note that different from the KP equation, the two-dimensional Boussinesq equation does not support multi-soliton solutions [Hietarinta, 1987a].

Publication I analyses two-soliton solutions of $(2+1)$-dimensional KdV type equations. In particular, the KP equation is considered as a model equation. The analysis shows that for different amplitude ratios three types of interactions are possible: (i) with appearance of positive phase shifts, (ii) with appearance of negative phase shifts, and (iii) interaction becomes singular. All these interaction types may be realized within the KP model showing that it is representative for demonstrating the soliton phenomenon in two space dimensions. The interaction types are analogous to the three interaction scenarios described in the previous Section for $(1+1)$-dimensional models. This analogy is very much expected because in both cases the analytical forms of soliton solutions are identical. However, interaction of solitons in two space dimensions requires different interpretation from that in one space dimension. This is covered in the following Sections of this thesis.

Anker and Freeman [1978], Freeman [1980] describe three-soliton interactions in terms of the motion of KP two-soliton resonant interactions (resonant triads). However, the authors recognize among other limitations that this description is very much an idealization because at each interaction point the condition for a resonant interaction is not met.

3 Soliton interaction patterns

A soliton in two space dimensions is localized in all directions except in the one that is perpendicular to its propagation direction. The positions of such solitons are determined by their crests lines. Interacting solitons in two space dimensions form patterns that consist of soliton crests (or crest lines). Such interaction patterns evolve in time according to the evolution of solitons.

3.1 Two-soliton interaction patterns

For two unidirectional solitons in two space dimensions the interaction pattern consists of two parallel crest lines (of solitons) that approach each other as the faster soliton catches the smaller soliton. As a result of the interaction process (that is assumed to be nonsingular in what follows), the crest lines of the faster and slower solitons have exchanged their order and, as the time elapses, recede relative to each other. During the interaction two scenarios are possible. According to the first scenario, the crest line of the faster soliton splits into two parallel lines corresponding to the smaller and the interaction soliton, respectively. The crest line of the interaction soliton approaches to the crest line of the initial slower soliton. At the end of the interaction, the latter crest lines merge into one line that corresponds to the shifted crest line of the faster soliton. In the second scenario, the crest lines of the two solitons merge into one line corresponding to the interaction soliton. At the end of interaction this crest line splits into two lines corresponding to crest lines of shifted solitons.

Consider now oblique interaction of two solitons in two space dimensions. In **Publication I** the case where two solitons propagate in different directions is analyzed. The analysis is based on the decomposition of a two-soliton solution into a superposition of two soliton terms and the interaction soliton term (see (20)). First, a geometrical representation of the two-soliton solution is introduced in phase variables. In this representation the so-called phase pattern set P_U consists of line slices that represent the supporting regions of the terms in the decomposition. Two qualitatively different phase pattern sets are possible for different signs of the phase shift parameter Δ_{12}. These are illustrated in Figure **I**.8.

Phase variables are related to real-time variables by

$$\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix} = \begin{bmatrix}\mu_1 & \nu_1\\ \mu_2 & \nu_2\end{bmatrix}\begin{pmatrix}x\\ y\end{pmatrix} + \begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix} t, \tag{25}$$

or $\boldsymbol{\varphi} = \mathbf{K}\boldsymbol{x} + \boldsymbol{\omega} t$ for short. The unidirectional case, where the rows of $\mathbf{K}$ are collinear, is considered in the paragraph above. Here, assuming non-collinear wave vectors, the wave matrix $\mathbf{K}$ defines bijective mapping between the spaces of phase and real variables, respectively. This means that the whole set P_U is an image of the two-soliton interaction pattern in the real space: $P_u(t) = \mathbf{K}^{-1}P_U - \boldsymbol{w}t$ where $\boldsymbol{w} = \mathbf{K}^{-1}\boldsymbol{\omega}$. In addition, the real pattern set $P_u(t)$ is stationary in the reference frame moving in the direction of $-\boldsymbol{w}$. This is illustrated in Figure **I**.9 and Figure **I**.10 for the negative and positive phase shift cases, respectively. Actual interaction patterns for two-soliton solutions are shown in Figure **I**.11.

In conclusion, the two-soliton interaction patterns are stationary and consist of the crest lines of two pairs of half-line solitons that are connected by the crest line of an interaction soliton in two possible ways defined by the sign of the phase shift parameter.

3.2 Multi-soliton interaction patterns

Simultaneous interactions between more than two solitons form nonstationary and complicated patterns. The problem of predicting such patterns from a predefined set of wave parameters (wave amplitudes, traveling directions, initial positions, etc.) is called the *direct problem of wave crests*.

In **Publication IV** the direct problem is solved for an arbitrary number of interacting solitons. Below the most important steps in this solution are outlined.

3.2.1 Multi-soliton solution and its decomposition

Publication III analyses multi-soliton solutions of KdV type equations in phase variables. A strict definition of such solutions is given (Definition **III**.4). From that and from the general bi-linearization transformation for KdV type equations, multi-soliton solutions in phase variables, $U_g(\boldsymbol{\varphi})$, are constructed (Theorem **III**.3.1). The analytical form of multi-soliton solutions is well-known (it originates from Hirota [1971]) but here a strict proof is given that it is applicable for all KdV type equations, and that is a novel result in **Publication III**.

A multi-soliton solution in real space-time variables, $u_g(\boldsymbol{x}, t)$, is given, for a fixed time moment, as a restriction of $U_g(\boldsymbol{\varphi})$ on the image of the real space in the space of phase variables:

$$u_g(\boldsymbol{x}, t) = U_g(\mathbf{K}\boldsymbol{x} + \boldsymbol{\omega} t). \tag{26}$$

Publication III introduces a decomposition of multi-soliton solutions into a superposition of soliton and interaction soliton terms (Theorem **III**.4.1). This generalizes the decomposition described in **Publication I** for an arbitrary number of solitons. The decomposition is given as

$$U_g = \sum_{\boldsymbol{\alpha} \in \{0,1\}^g} S_{\kappa(\boldsymbol{\alpha})}, \tag{27}$$

where $\kappa(\boldsymbol{\alpha})$ denotes the index of the corresponding soliton term (see **Publication III** for details).

The interpretation of this decomposition is based on the following statement: interaction of two solitons (of any kind) involves an interaction soliton that connects the two solitons and their shifted counterparts. Now, interaction solitons (initiated by, say, two solitons) interact with a third soliton in the same way, generating a new, higher order interaction soliton. This description can be easily prolonged. The decomposition (27) contains g solitons, all possible interaction solitons (the number is $2^g - g - 1$), and a so-called vacuum soliton $S_\emptyset$ which is identically zero.

The properties of soliton terms $S_{\kappa(\boldsymbol{\alpha})}$ are given in **Publication III**, these include the asymptotic behavior (Proposition **III**.4.3) and the amplitudes (Proposition **III**.4.4) of the terms $S_{\kappa(\boldsymbol{\alpha})}$. It turns out that the interaction soliton terms $S_{\kappa(\boldsymbol{\alpha})}$ are localized in certain directions in the space of phase variables. The number of such directions is larger for higher order interaction soliton terms. This property is closely related to the appearance of interaction solitons in the real space where these have a limited life time. This is clarified further in the following Section 3.2.3 where actual interaction patterns are constructed.

3.2.2 Phase pattern set of multi-soliton solution

A phase pattern set represents the positions of supporting regions of soliton solutions in phase variables, and therefore, through the relation (26), completely determines interaction patterns in the real space. The phase pattern set, P_U, of a g-soliton solution is defined in the g-dimensional space of phase variables which, in general, would make the analysis rather difficult. It turns out that due to special "recursive" properties of multi-soliton solutions (see Definition **III**.4), the analysis is still feasible in spite of possible high dimensions of the space of phase variables. **Publication IV** gives a definition of a phase pattern set that is specific for multi-soliton solutions (Definition **IV**.1).

The crucial part in **Publication IV**, as well in solving the direct problem, is the *recognition* how to construct phase pattern sets for multi-soliton solutions with an arbitrary number of solitons. This recognition is formulated in Theorem **IV**.3.1 that claims that the phase pattern set is a projected set (to the space of phase variables) of the ridges of a special $(g+1)$-dimensional polyhedron. This polyhedron is completely defined by the phase shift matrix of the corresponding multi-soliton solution. For the two soliton case the construction of a phase pattern set is illustrated in Figure **IV**.2. Figure **IV**.4 shows the phase pattern set for three-soliton solution.

Recall that there are two types of (nonsingular) two-soliton interactions which are distinguished by the sign of the corresponding phase shift parameter. For g-soliton interactions the number of different interaction types is equal to $1 + g(g-1)/2$. For example, for the KP three-soliton solution the regions of nonsingular solutions in the $\boldsymbol{\mu}$-space are shown in Figure **IV**.3. And a particular interaction type is determined by the signs of three phase shift parameters $\Delta_{12,13,23}$ making the number of different interaction types equal to 4.

Different parts of a phase pattern set represent the supporting regions of the corresponding soliton terms in the decomposition (27). This is illustrated for the three soliton case in Figure **IV**.6.

The fact that the phase pattern set is now defined as a geometrical figure, allows its algorithmic construction using methods from Computational Geometry. In this thesis, for manipulating multi-dimensional polyhedra the software library `cddlib` by Fukuda [2000] is used.

3.2.3 Interaction patterns of multi-soliton solutions

In this work, interaction patterns, that are formed as a result of simultaneous interactions of many solitons, are defined by the real pattern set of the corresponding multi-soliton solution.

Publication IV defines the real pattern set, $P_u(t)$, for two space dimensions as follows

$$P_u(t) = \{\boldsymbol{x} \in \mathbb{R}^2 | \mathbf{K}\boldsymbol{x} + \boldsymbol{\omega} t \in P_U\}, \tag{28}$$

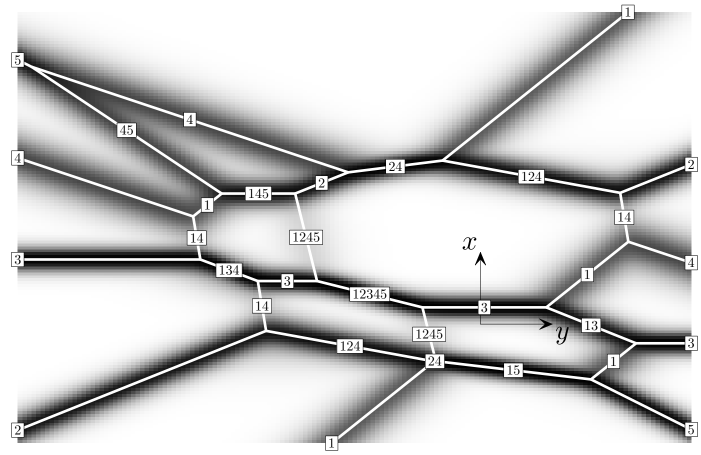

Figure 3: The real pattern set (white lines) of a five-soliton solution (abstract case, corresponds to Figure **I**.1). Labels are the indices of the soliton terms of the corresponding decomposition. (Fig. 10 in **Publication IV**.)

where P_U is a phase pattern set of the corresponding multi-soliton solution. The real pattern set represents the supporting regions of the multi-soliton solution, $u_g(\boldsymbol{x}, t)$, in real variables.

For interpretation and also for practical applications, the real pattern set can be written as follows (Proposition **IV**.4.1):

$$P_u(t) = (\mathbf{K}^T\mathbf{K})^{-1}\mathbf{K}^T(P_U \cap (\mathbf{K}\mathbb{R}^2 + \boldsymbol{\omega}_\perp t)) - \boldsymbol{w}t, \qquad (29)$$

where $\boldsymbol{w} = (\mathbf{K}^T\mathbf{K})^{-1}\mathbf{K}^T\boldsymbol{\omega}$ and $\boldsymbol{\omega}_\perp$ is a g-vector perpendicular to the two-dimensional hyperplane $\mathbf{K}\mathbb{R}^2$: $\boldsymbol{\omega}_\perp = \boldsymbol{\omega} - \mathbf{K}\boldsymbol{w}$. For a fixed time moment, the two-dimensional real space, represented as $\mathbb{R}^2$, has an image, $\mathbf{K}\mathbb{R}^2$, in the g-dimensional space of phase variables. This image is a two-dimensional hyperplane. The supporting regions of $u_g(\boldsymbol{x}, t)$ are defined by the intersection between the hyperplane $\mathbf{K}\mathbb{R}^2 + \boldsymbol{\omega}t$ and the phase pattern set P_U. The operator given by $(\mathbf{K}^T\mathbf{K})^{-1}\mathbf{K}^T$ maps this intersection into the real space. The result, shifted by $\boldsymbol{w}t$, is the real pattern set $P_u(t)$. As the time parameter t is increased, the hyperplane, representing the image of the real space, shifts in the direction of vector $\boldsymbol{\omega}_\perp$. As a result, the intersection between the hyperplane and the phase pattern set varies. These variations are reflected in the non-stationarity of $P_u(t)$. The above description is illustrated for three-soliton interactions in Figure **IV**.8 and Figure **IV**.9.

Note that multi-soliton interaction patterns can be also stationary. The stationarity condition is given by $\boldsymbol{\omega}_\perp = \mathbf{0}$.

Finally, Figure 3 illustrates a snapshot of a five-soliton interaction pattern. This exemplifies the complexity of possible soliton interaction patterns, as well as the usefulness of the interaction soliton concept, introduced in this work,

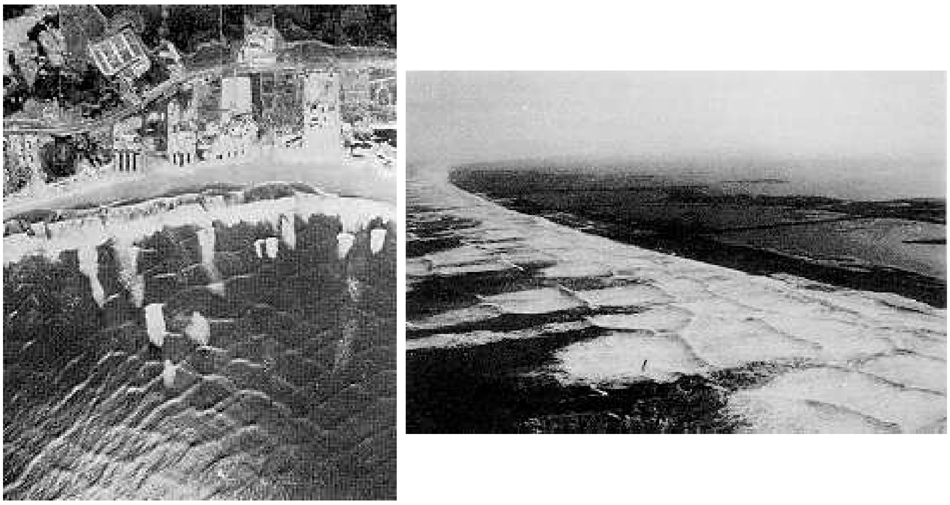

Figure 4: Left (Fig.11 in Hammack et al. [1995]): "Aerial photograph of waves off the southern coast of Long Island.... Beyond the surf zone, the wave patterns are two-dimensional, and approximately periodic. They have flat troughs, sharp crests, and approximately hexagonal shape." Right (Fig.13 in Hammack et al. [1995]): "Aerial photograph of waves in shallow water, south of the Oregon Inlet on Pea Island, off the coast of North Carolina. (Photograph courtesy of C. Miller.)"

to interpret the multi-soliton interaction patterns.

4 Application: The inverse problem of wave crests

Consider wave patterns that arise from wave interactions of soliton type. For example, the aerial photos in Figure 4 show hexagonal shape interaction patterns that are typical for interaction of two multi-directional soliton trains.

The aim of the *inverse problem of wave crests*, first introduced in **Publication I**, is to predict wave parameters (most important wave amplitudes) from the geometry of interaction patterns of waves. Such a photographic measurement technique has several advantages over the more widely used point measurement technique. To name a few, first, it gives an instant overview of the overall surface condition. Second, no physical contact is required with the water surface that could otherwise lead to artificial perturbations, or that would be actually impossible due to some external conditions (e.g. extreme weather conditions on sea).

To tackle the inverse problem, a solid mathematical model of these waves is needed so that its solutions (surface elevation) could be related to abstract notions such as traveling directions and amplitudes of waves. Since soliton interactions seem to be rather robust phenomena in (the theory of) water waves, in the following waves are associated with solitons and for a model equation a member from the family of KdV type equations is assumed.

In **Publication I**, the inverse problem of wave crests is solved for two wave interactions modeled by two-soliton solutions. Two typical interaction patterns are shown in Figure **I**.11. These plots show important quantities, defining the geometry of two-wave interaction patterns, to be the interaction angle α_{12} and the phase shifts $\delta_{1,2}$ of the solitons. The inverse problem is solved by relating the interaction patterns with the phase pattern sets of two-soliton solutions as illustrated in Figure **I**.12. A phase pattern set is completely determined by a phase shift parameter Δ_{12} that depends on amplitudes. In particular, for the KP case we have

$$\Delta_{12} = -\ln \frac{\rho^2 - (\mu_1 - \mu_2)^2}{\rho^2 - (\mu_1 + \mu_2)^2}, \tag{30}$$

where $\mu_{1,2}^2/2$ are soliton amplitudes and $\rho = 2\tan\alpha_{12}/2$. In fact, mainly because the phase shifts depend on the amplitudes of waves, it is possible to solve this inverse problem. The final equation, given by Eq. (**I**.26), has two distinct solutions, as shown in Figure **I**.13, that correspond to the two interaction types with negative and positive phase shifts, respectively. As a result, **Publication I** concludes that, *for a specified phase shift, the solution of the inverse problem is unique.*

Publication II studies the sensitivity of the "photographic" measurement method, introduced in **Publication I**, to conclude about the practical applicability of the method. For that specific solutions are constructed and analyzed against possible errors in phase shift and interaction angle measurements. In particular, solutions for the cases with small interaction angle, moderate phase shifts, and/or symmetric interactions are found. The main conclusions from **Publication II** are the following. First, the method is most sensitive to errors in measuring the interaction angle, especially, if the angle is small. And finally, the method of calculating the heights of the waves from a "photographic" record is applicable if the positions of the wave crests can be well defined from interaction patterns (e.g. if the crests are long enough) and the interaction angle and the phase shifts can be determined accurately.

5 Discussion

Below some open problems and extensions to the subject of this thesis are discussed.

- In Section 2.1.3, the interaction of unidirectional solitons was considered only with the example of the two-soliton solution. The corresponding multi-soliton interactions are, technically, the degenerated cases of the multi-directional soliton interactions that are explained in Section 3 in full, and therefore not repeated earlier. Although, multi-soliton interactions have a different interpretation in the unidirectional case. Basically, the notion of interaction patterns needs to be replaced by the notion of world paths of solitons.

- In this thesis only solitons of "solitary type" were considered. It is a known fact that KdV type equations have also solutions of "cnoidal type". The corresponding multi-soliton solutions can be expressed in terms of the Riemann theta function. These can be derived using exactly the same approach as presented in this thesis for solitary type multi-soliton solutions. The results of Hammack et al. [1989] indicate that also for cnoidal type multi-soliton solutions the concept of an interaction soliton could be introduced and exploited for constructing the corresponding interaction patterns. However, it is currently not clear how to construct the corresponding phase pattern sets that must be now a g-periodic lattice in the space of phase variables. Clarification of this problem is left for future research.

- In this thesis soliton solutions of only KdV type equations were considered. Similar analysis could be developed also for other classes of bi-linearizable equations because the analysis here used only the very basic properties of soliton interactions that are common also for solutions of those other classes of equations. Adjustments to the current analysis are required because the functional forms of soliton solutions are different for different classes of equations.

- In this thesis the existence of a unique solution to the inverse problem of wave crests was proved, but only for two-soliton interactions of the KP equation. However, initial considerations show that this result could be extended also for multi-soliton interactions, in spite of the fact that then the interaction patterns are nonstationary.

  These considerations are based on the fact that the KP g-soliton solution is defined by $2g$ number of independent parameters, and to determine these from the single interaction picture, there must exist the same number of independent relations between the analytically given g-soliton solution and the corresponding interaction pattern. Indeed, this set of equations exists and is composed as follows. The first g number of relations are given by g number of traveling directions that are measured as relative angles between the soliton wave crests far from the interaction region and the x-axis. The rest g relations are defined as g number of total phase shifts that each soliton experiences if to follow its crest line from one end to another. Note that the listed quantities, the interaction angles and the total phase shifts, can be measured from interaction pictures and, most importantly, these are invariant in time although the interaction patterns are nonstationary.

However, practical applicability of such a posed solution of the inverse problem for multi-soliton solutions is questionable because in order to measure all those invariants, a relatively large observation area is needed.

- An interesting application of the analysis of soliton interactions would be the modeling of freak wave phenomena in a deterministic way. The freak wave phenomena have been observed in open sea where very high amplitude waves, more than two times the significant wave height, occur by some, still unsolved, reasons. These phenomena have received a great interest nowadays due to devastating consequences if ships meet such extreme waves.

  The idea of modeling deterministic freak waves with soliton interaction is based on the fact that the amplitudes of interaction solitons can be larger than the linear superposition of single waves would predict. So, imagine, if one generates (in laboratory environment) a group of relatively small amplitude solitary waves, traveling possibly in different directions, and in such a way that at some space-time point an interaction soliton occurs that has remarkably high amplitude, then the interactions between a ship model and such a high amplitude wave could be tested. Using the technique presented in this thesis, it would be possible to predict the space-time points of such "freak interaction solitons", and moreover, to provide initial conditions for wave generators to generate wave fields with the freak wave phenomena.

  Since freak wave phenomena are observed in a deep sea, then possibly a suitable model equation, that also supports soliton solutions, is the Benjamin-Ono equation. Therefore, it is important to extend the present analysis also for other classes of equations to describe the corresponding multi-soliton interactions in a deterministic way.

6 Summary

In this thesis, a novel approach for analyzing and describing multi-soliton interaction is presented. This approach consists of the following steps and implications:

- Multi-soliton solutions are constructed in the most general form for the KdV type equations using Hirota bilinear formalism. This construction is possible because such solutions have the same functional form.

- Multi-soliton solutions are analyzed in canonical phase variables. In this way the most important properties of such solutions are revealed. Actual multi-soliton solutions are the restrictions of multi-soliton solutions in phase variables to the image of the real space-time space in the space of phase variables. After finding the restriction, the appearance of multi-soliton solutions becomes seemingly very complex, but the building blocks of this complexity are now explicitly known.

- All possible interaction scenarios are found by analyzing the multi-soliton solutions without assuming any particular governing equation. After the analysis, taking some KdV type equation to be a model, it is concluded that only those interaction scenarios are realized that are defined on a certain manifold in the parameter space (of wave matrix and phase shift matrix). This manifold is defined by the Hirota conditions of the given KdV type equation. These Hirota conditions include also the dispersion relations and the relations for phase shift coefficients.

As a result of the chosen approach, the analysis of multi-soliton solutions resulted in a novel decomposition. According to this decomposition, a multi-soliton solution is a linear superposition of solitons and all possible interaction solitons. An interaction soliton is defined to be the connecting wave form between pairwisely interacting solitons (of any kind) and their shifted counterparts.

For visualization of soliton interactions, a geometrical representation of interaction patterns is introduced. The process consists of constructing the phase and real pattern sets that represent supporting regions of multi-soliton solutions in phase and real variables, respectively. The real patterns sets, representing actual interaction patterns, are, in general, nonstationary and seemingly complex. The visualization process is useful and efficient only due to the recognition that the phase pattern sets can be constructed algorithmically for an arbitrary number solitons. According to this novel idea, the phase pattern set is a set of ridges of a certain $(g+1)$-dimensional polyhedron, defined by phase shift matrix, that is projected to the g-dimensional space of phase variables. Then the real pattern set is obtained by finding an intersection between the phase pattern set and a G-dimensional hyperplane ($G = 1, 2$ is the number of the real space dimensions) and by mapping the result to the real space. As the time parameter is increased, this G-dimensional hyperplane is shifted in the space of phase variables and that results in nonstationary interaction patterns in the real space.

All new concepts and results described above are illustrated for two-soliton interactions. In particular, the interaction of two unidirectional solitons of the KdV-Sawada-Kotera equation was demonstrated to illustrate the nature of an interaction soliton in one space dimension. Multi-directional soliton interactions are demonstrated using the KP equation as a practical example model. The corresponding two- and three-soliton interactions are described in detail. Also an example of five-soliton interactions is considered to illustrate the usefulness of the interaction soliton concept to describe general multi-soliton interactions. It is important to remember that the description of soliton interaction, introduced in this thesis, is simply applicable for all KdV type equations. Nevertheless, a particular application may need different interpretations depending (i) on the dimension of the real space, (ii) on the assumptions made in deriving the governing equation, and (iii) on the physical background of the wave phenomenon, in general.

As an application of the introduced soliton interaction analysis, the inverse problem of wave crests is introduced. This problem is about predicting wave parameters (most importantly, the amplitudes) from the single snapshot of interaction patterns of solitons. It is proved that for two oblique interacting KP solitons the inverse problem has a unique solution. In addition, the sensitivity of the inverse problem to measurement errors is analyzed and conclusions for practical applicability of the method are drawn.

In conclusion, the main goal of this thesis, to find an answer and a detailed description about what happens during the simultaneous interaction of an arbitrary number of solitons, is achieved with the crucial aid of the concept of an interaction soliton.

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Publication I

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PEARU PETERSON AND E. VAN GROESEN

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Publication II

Sensitivity of the inverse wave crest problem

Pearu Peterson and E. van Groesen

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Publication III

Construction and decomposition of multi-soliton solutions of KdV type equations

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Publication IV

Construction of multi-soliton interaction patterns of KdV type equations

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