Sensitivity of the inverse wave crest problem

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Abstract. In a previous paper [1] the inverse problem for wave crests was introduced and a solution strategy for two-wave interactions was given. Here these solutions are actually constructed, in particular for the cases with small interaction angle, moderate phase shifts, and/or symmetric interactions. Two detailed examples are presented and analyzed. The sensitivity of the method is investigated, and conclusions about the practical applicability are given.

Key words: Wave pattern; Inverse problem; Sensitivity.
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1 Introduction

The so-called "inverse" problem of multi-directional waves concerns the question if from a (photographic) record of the wave *pattern* the interacting individual waves (wave heights, directions, etc) can be reconstructed. The motivation to study this problem originates from the work of Hammack et al. [2] and practical questions from hydrodynamic laboratories where waves are generated to test ships.

Since linear interactions do not disturb the interacting waves, nonlinear effects should be modelled and taken into account for well-possessedness of the problem. In [1], it was shown that taking the KP (Kadomtsev-Petviashvili) equation as a model, the inverse problem has a unique solution for two interacting waves.

In this paper we investigate how the solution can be constructed in specific cases and study the sensitivity of the result on pattern-parameters that have to be found from the measurements. A summary of the formulation and solution of the inverse problem is presented in the following section. In Section 3 we consider approximate solutions of the inverse problem that can be used as estimates when finding exact solutions with iterative methods; in several cases the approximate solutions are good enough and no iteration is needed. Then we illustrate the inverse problem with two examples. Finally, a detailed error analysis is carried out.

2 The inverse problem for two waves

Consider wave patterns that arise from two-wave interaction of soliton-type. Then the aim of the inverse problem is to find the heights of two waves using only geometrical characteristics of the wave pattern. The most important characteristics (*pattern-parameters*) are the interaction angle and the phase shifts which result from the nonlinear interaction. In fact, mainly because the phase shifts depend on the amplitudes of the waves, it is possible to solve this problem.

Occurrence of phase shift is one of the basic features in soliton phenomena and so it is natural to choose a model for the wave propagation which supports solitons. In order to tackle the inverse problem, we have used a large class of "soliton"-type of equations — *the KdV-type of equations* [3] — for which soliton interactions can be studied using Hirota's formalism [4]. For these equations the interaction of an arbitrary number of solitons is possible but in the following only two-soliton interaction will be considered. We will use the KP model, that is a specific member of this family, to demonstrate our method. The KP model is known to describe (small but finite amplitude, long) surface waves travelling in shallow water (of constant depth) "mainly" in one direction [2]. The KP equation (in normalized variables) reads

$$(u_t + 6uu_x + u_{xxx})_x + 3u_{yy} = 0, \tag{1}$$

where $u = u(x, y, t)$ denotes the surface elevation and the direction of main propagation of waves is taken the x-axis.

There are two types of two-soliton interactions: one with negative and one with positive phase shift as shown in Figure 1, respectively. A picture of the pattern shows that important quantities are the interaction angle α_{12} and the phase shift $\delta_{1,2}$ of each wave. In order to find the amplitudes of these waves uniquely, we need to know the type of the wave interaction, negative or positive. This can be seen from the pattern if, as we shall do, define the interaction angle to be acute. Then, negative shift occurs if the vertex (marked with a dot) lies on the crest line, and positive when that is not the case.

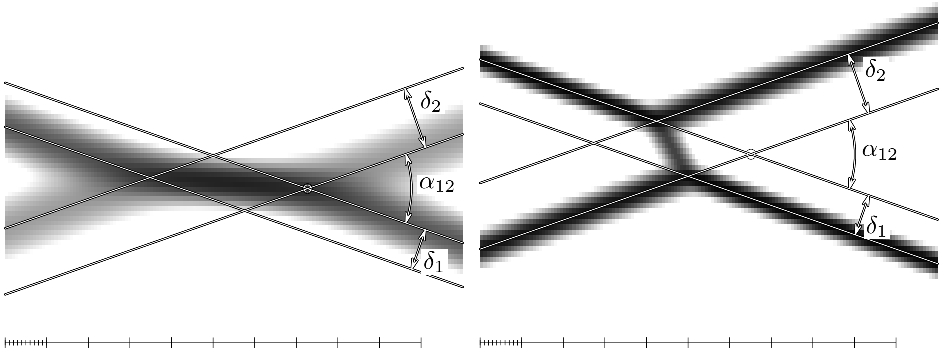

Figure 1: Interaction patterns of the two-soliton solution with negative (left) and positive (right) phase shift; indicated are the pattern-parameters $\delta_{1,2}$ (phase shifts) and α_{12} (interaction angle).

The two-soliton solution of the KP equation can be written down explicitly. It is a uniform translation of a fixed pattern in real space, that is given (at $t = 0$) by the following formula

$$u(x, y, t = 0) = 2\frac{\partial^2}{\partial x^2} \ln \left(\cosh \frac{\varphi_1 - \varphi_2}{2} + \sqrt{A_{12}} \cosh \frac{\varphi_1 + \varphi_2 - \Delta_{12}}{2} \right), \quad (2)$$

where $\varphi_1 = \mu_1(x+\frac{1}{2}\rho y)$, $\varphi_2 = \mu_2(x-\frac{1}{2}\rho y)$ are phase variables. The parameters $\mu_{1,2}$ are directly related to the wave amplitudes $a_{1,2} = \frac{1}{2}\mu_{1,2}^2$, ρ is related to the interaction angle α_{12} ($\rho = 2\tan\frac{1}{2}\alpha_{12}$), and a corresponding positive value of A_{12} is required to define an actual solution: the phase shift parameter $\Delta_{12} = -\ln A_{12}$ is negative for $A_{12} > 1$, and positive for $0 < A_{12} < 1$. For the KP two-soliton solution we have

$$A_{12} = \frac{\rho^2 - (\mu_1 - \mu_2)^2}{\rho^2 - (\mu_1 + \mu_2)^2}. \quad (3)$$

Figure 2 shows in phase variables the difference between soliton solutions with negative and positive phase shifts, respectively. Figure 1 shows the corresponding soliton solutions in real variables (propagating in the vertical direction).

A detailed analysis [1] shows that the following equations relate the parameters in the KP two-soliton solution to the pattern-parameters:

$$\mu_2 = \mu_1 \frac{\delta_1}{\delta_2}, \quad (4)$$

where μ_1 has to be found from

$$\mu_1 \delta_1 \sqrt{1 + \rho^2/4} \pm \ln \frac{\delta_2^2\rho^2 - (\delta_2 - \delta_1)^2\mu_1^2}{\delta_2^2\rho^2 - (\delta_2 + \delta_1)^2\mu_1^2} = 0, \quad (5)$$

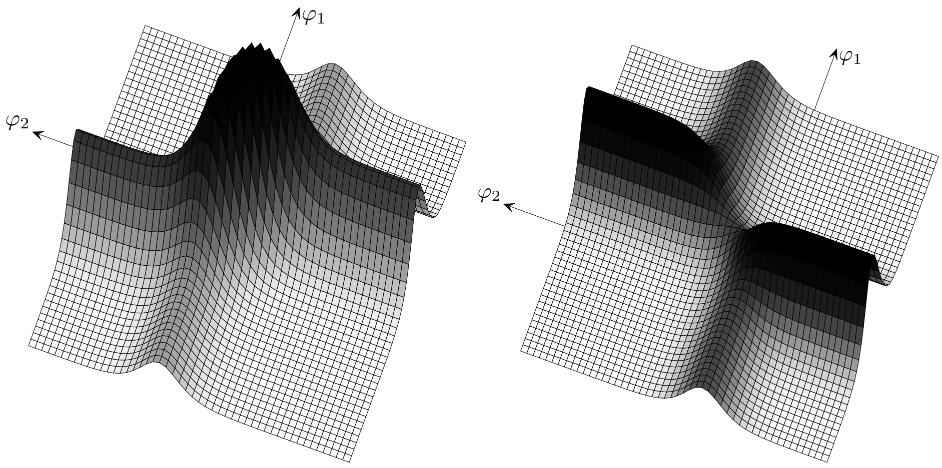

Figure 2: Two-soliton solution with negative (left) and positive (right) phase shift (in phase variables).

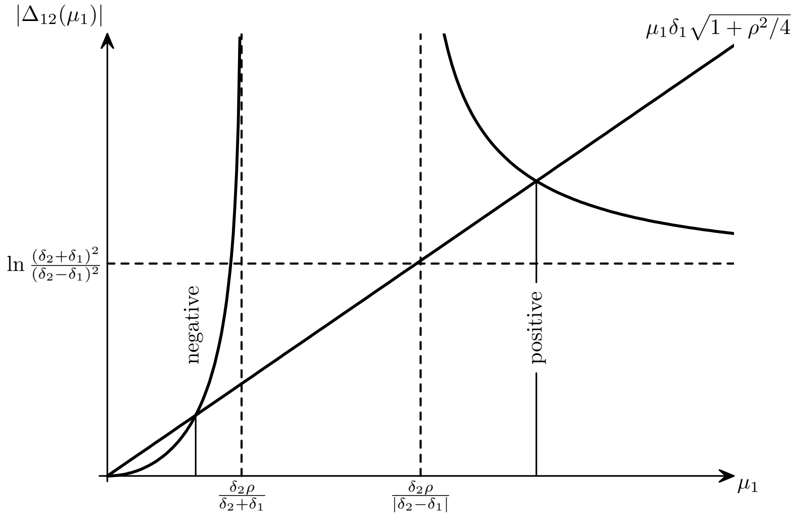

Figure 3: Graphical representation of Eq. (5) shows the existence of two nontrivial solutions. Here $|\Delta_{12}(\mu_1)|$ =the second term of (5).

where the minus-sign corresponds to negative phase shift. Figure 3 shows how the solutions of the last equation can be found from the intersection points of the relevant graphs. It shows that, in general, it has two nontrivial solutions: the solution $\mu_1 < \delta_2\rho/(\delta_2+\delta_1)$ corresponds to the negative interaction type and the other one to the positive interaction type.

These results show that the inverse problem can be solved and that the amplitudes of the waves can be found uniquely for a specific interaction type.

In the rest of this paper we will analyze the dependence of the results on the errors in the pattern-parameters in order to determine the sensitivity of the method and its applicability in practice.

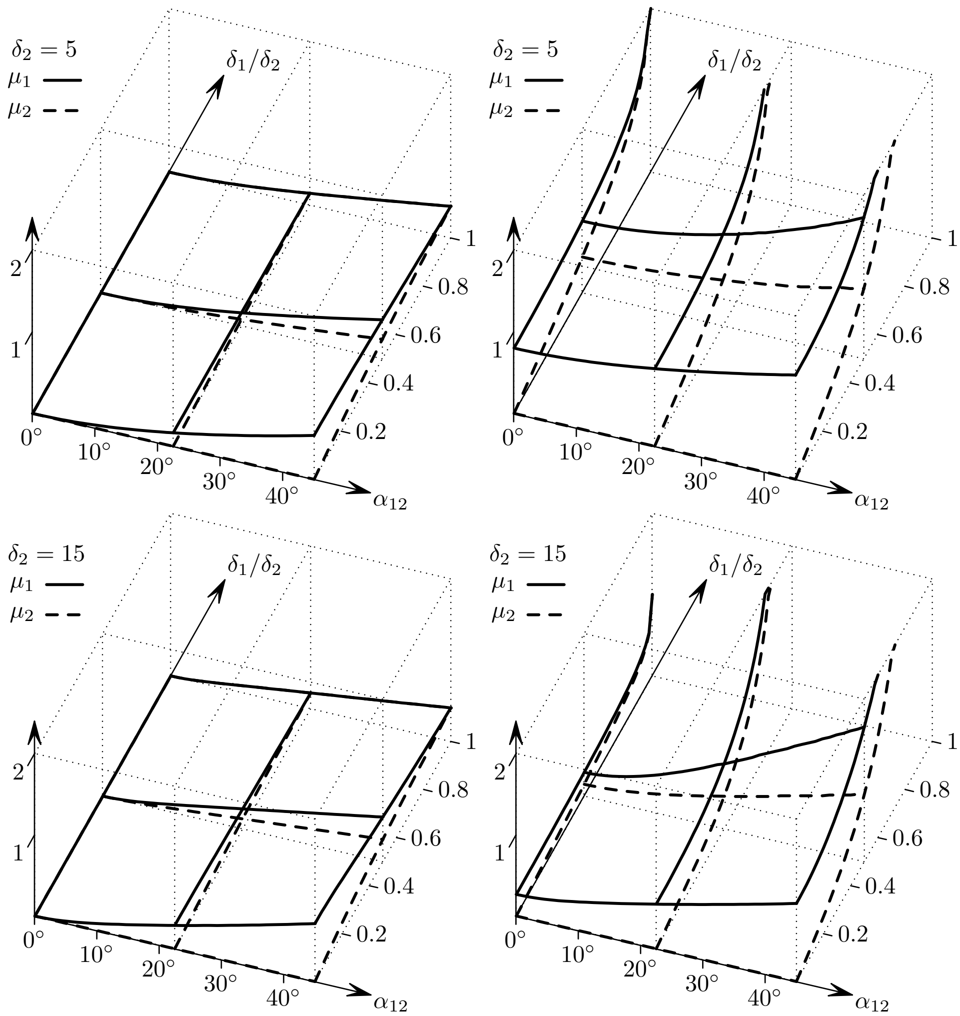

Figure 4: Solutions of Eq. (5) for interactions with negative (left) and positive (right) phase shift.

3 Special cases

Figure 4 illustrates the solutions of Eq. (5) for various sets of pattern-parameters. Vertically the parameters μ_1, μ_2 are plotted against the interaction angle α_{12} of the waves, and the value of δ_1/δ_2 for $\delta_2 = 5$ and $\delta_2 = 15$. Recall that the amplitudes of the two solitons are $\frac{1}{2}\mu_1^2$, $\frac{1}{2}\mu_2^2$, respectively.

In the following we discuss some special solutions of the inverse problem that can be used to obtain estimates for the amplitudes of the waves if special conditions hold. First, with no restriction of generality, we take $\delta_1 = r\delta_2$, $0 < r \leqslant 1$. Then Eq. (5), after taking the exponential and reordering the terms, reads

$$\left(\frac{\rho^2-(1-r)^2\mu_1^2}{\rho^2-(1+r)^2\mu_1^2}\right)^{\pm 1} = e^{-\mu_1 r\delta_2\sqrt{1+\rho^2/4}}. \tag{6}$$

Small interaction angle ($\alpha_{12} \to 0$): If the interaction angle α_{12} is decreased to zero, or equivalently, $\rho \to 0$, the solution $\mu_1 \in (0, \delta_2\rho/(\delta_2+\delta_1))$ (negative phase shift) becomes trivial: $\mu_1 \to 0$ (see the left-hand plots in Fig. 4 for $\alpha_{12} \to 0$ or Fig. 3 for $\rho \to 0$). Taking the Ansatz $\mu_1 \sim \rho^2$, the following approximate solution is found for the case with small interaction angle and negative phase shift:

$$\mu_1 = \frac{\delta_2}{4}\rho^2 + O(\rho^4). \tag{7}$$

For positive phase shift, Eq. (5) simplifies in the limit $\rho \to 0$ to

$$\mu_1\delta_1 + 2\ln\frac{|\delta_2-\delta_1|}{\delta_2+\delta_1} = 0;$$

this gives an approximate solutions for small interaction angle:

$$\mu_1 = \frac{2}{\delta_1}\ln\frac{\delta_2+\delta_1}{|\delta_2-\delta_1|} + O(\rho^2).$$

Moderate phase shifts ($\delta_1, \delta_2 \gg 1$ and $\rho > 0$): Note that the l.h.s. of Eq. (6) does not depend on δ_2. The limit process $\delta_2 \to \infty$ gives the following solutions

$$\mu_1 = \frac{\rho}{1+r}, \qquad \mu_1 = \frac{\rho}{1-r}$$

for negative and positive phase shifts, respectively. So, for large phase shifts we have the following approximate solutions:

$$\mu_1 \approx \frac{\delta_2\rho}{\delta_2+\delta_1}, \qquad \mu_1 \approx \frac{\delta_2\rho}{|\delta_2-\delta_1|}. \tag{8}$$

However, because the r.h.s. of Eq. (6) decreases exponentially with respect to the phase shift parameter, these formulas give good estimates if phase shifts are moderate (as seen in the example below), and if the interaction angle is not too small.

Graphically, these approximate solutions are the asymptotes in Fig. 3 (vertical dashed lines) and the corresponding curves are very steep if the phase shifts are large (see also Fig. 5). Formula (8) explains also the steeply rising solution curves in Fig. 4 as $r \to 1$ for interactions with positive phase shift.

Table 1: Solving the inverse problem: data corresponding to the situation shown in Fig. 1.

Phase shift:		negative	positive
Pattern-parameters	δ_1	10.0 ± 0.5	10.0 ± 0.5
(measurements):	δ_2	15.0 ± 0.5	15.0 ± 0.5
	α_{12}	$(39.0 \pm 2.0)^\circ$	$(39.0 \pm 2.0)^\circ$
Solution to the inverse	μ_1	0.42 ± 0.04	2.12 ± 0.5
problem:	μ_2	0.28 ± 0.03	1.42 ± 0.45
Wave amplitudes:	a_1	0.09 ± 0.02	2.3 ± 1.0
	a_2	0.040 ± 0.0085	1.00 ± 0.65
	a_{12}	0.20 ± 0.065	0.25 ± 0.055
Weights of relative errors	c_{δ_1}	0.38	2.00
(see Sec. 5):	c_{δ_2}	0.41	2.00
	$c_{\alpha_{12}}$	1.11	1.08

Symmetric interaction ($\delta_1 \to \delta_2$ or $r \to 1$): If the phase shifts of the two solitons are equal then from $\delta_1\mu_1 = \delta_2\mu_2$ it follows that they must have equal amplitudes: $\mu_1 = \mu_2$. But this is possible only if the phase shift is negative (because $A_{12} > 1$ in (3)). Graphically, the asymptote $\delta_2\rho/|\delta_2 - \delta_1|$ of the curve, that corresponds to positive phase shift, shifts to infinity as $\delta_1 \to \delta_2$ (see Fig. 3). Equation (5) for $\delta_1 = \delta_2$ remains transcendental:

$$\mu_1\delta_2\sqrt{1+\rho^2/4} \pm \ln\frac{\rho^2}{\rho^2 - 4\mu_1^2} = 0$$

and its solution exists only for negative phase shift (because $\rho > 2\mu_1$ must hold). For moderate phase shifts the solution is approximately $\mu_1 \approx \rho/2$ (see above) and for small interaction angle given by (7).

4 Specific examples

Table 1 illustrates two examples of the inverse problem for two wave interactions with negative and positive phase shifts. The values of the pattern-parameters δ_1, δ_2, and α_{12} correspond to the situation depicted in Fig. 1. Here a_i and a_{12} are the amplitudes of the two waves and the interaction soliton [1], respectively:

$$a_i = \tfrac{1}{2}\mu_i^2, \qquad a_{12} = \frac{(\mu_1 - \mu_2)^2 + A_{12}(\mu_1 + \mu_2)^2}{2(1+\sqrt{A_{12}})^2}.$$

To show the effect of errors in pattern-parameters when the data is obtained from a photographic recording, a choice for the relative errors of approximately

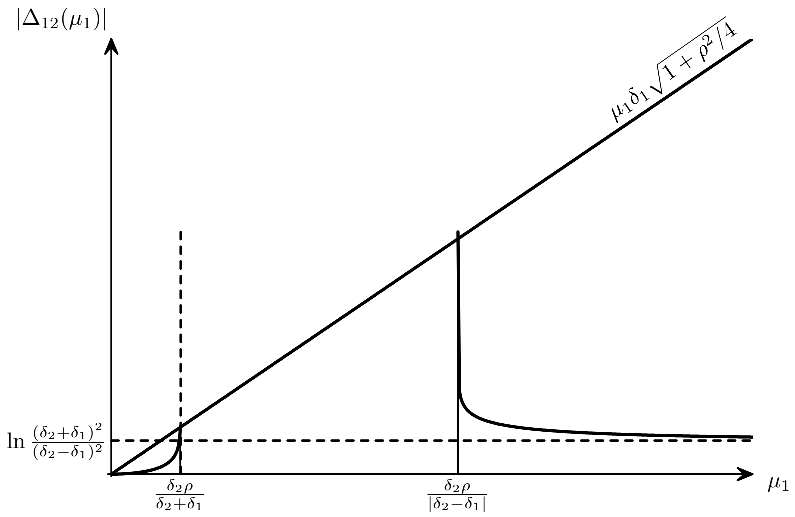

Figure 5: "Graph" of Eq. (5) for $\delta_1 = 10$, $\delta_2 = 15$, and $\alpha_{12} = 39°$.

5% is taken (the accuracy of experimental measurements in Hammack et al. [2] are between 3% and 10%, depending on the measured quantity).

Table 1 shows that the interaction with positive phase shift is more sensitive to errors than when the phase shift is negative. For example, for negative phase shift the relative error of the wave amplitudes is approximately 22%, but when phase shift is positive, the error reaches 65%! (Hammack et al. [2] found the corresponding errors to be from 26% to 33%). A detailed analysis of these errors is carried out in the next section.

Figure 5 illustrates graphically the solutions of Eq. (5) for the examples specified in the table. The figure shows that the solutions are close to the positions of the asymptotes. So, the given examples correspond to the interactions with rather large phase shifts. The large relative errors in the positive phase shift case follows from $\mu_1 \sim 1/|\delta_2 - \delta_1|$. This subtraction in the denominator is responsible for the amplification of errors if the difference in phase shifts $\delta_1 - \delta_2$ becomes small. This is also expressed in weights of relative errors (see error analysis below): for positive phase shift these are much larger than for negative phase shift (as seen in Table 1).

5 Error analysis

In this section we study the relative errors of results of the inverse problem in more detail. Note that an error estimate for μ_1 is sufficient, since then the errors for the other quantities (that are determined by μ_1) will follow from this.

Let $\Delta\delta_1$, $\Delta\delta_2$, $\Delta\alpha_{12}$ denote the absolute errors of the measured data δ_1,

δ_2, α_{12}, respectively. Then the error estimate for μ_1 is given by

$$|\Delta\mu_1| = \left|\frac{\partial\mu_1}{\partial\delta_1}\right| |\Delta\delta_1| + \left|\frac{\partial\mu_1}{\partial\delta_2}\right| |\Delta\delta_2| + \left|\frac{\partial\mu_1}{\partial\alpha_{12}}\right| |\Delta\alpha_{12}| \,, \tag{9}$$

where the function $\mu_1 = \mu_1(\delta_1, \delta_2, \alpha_{12})$ is given implicitly by Eq. (5). For the relative errors

$$\Delta_r\mu_1 = \frac{\Delta\mu_1}{\mu_1}, \qquad \Delta_r\delta_i = \frac{\Delta\delta_i}{\delta_i}, \qquad \Delta_r\alpha_{12} = \frac{\Delta\alpha_{12}}{\alpha_{12}}$$

Eq. (9) reads $|\Delta_r\mu_1| = c_{\delta_1}\,|\Delta_r\delta_1| + c_{\delta_2}\,|\Delta_r\delta_2| + c_{\alpha_{12}}\,|\Delta_r\alpha_{12}|$, where

$$c_{\delta_1} = \left|\frac{\partial\mu_1}{\partial\delta_1}\right| \frac{\delta_1}{\mu_1}, \qquad c_{\delta_2} = \left|\frac{\partial\mu_1}{\partial\delta_2}\right| \frac{\delta_2}{\mu_1}, \qquad c_{\alpha_{12}} = \left|\frac{\partial\mu_1}{\partial\alpha_{12}}\right| \frac{\alpha_{12}}{\mu_1} \tag{10}$$

denote *weights of relative errors.* These weights determine the sensitivity of the solution of the inverse problem to the relative errors of the pattern-parameters δ_1, δ_2, and α_{12}.

Figure 6 shows the weights of relative errors for various sets of the pattern-parameters (similar to Fig. 4). Vertically the weight factors c_{δ_1}, c_{δ_2}, $c_{\alpha_{12}}$ are plotted against the interaction angle α_{12} and the ratio δ_1/δ_2 for various specified values of δ_2. We refer to the sum of weight factors $c_{\delta_1} + c_{\delta_2} + c_{\alpha_{12}}$ as the *total weight of relative errors.* The following conclusions can be drawn from these figures.

Negative phase shift Here the total weight of relative errors is bounded approximately by 3.0, which value is approached for very small interaction angles. As the interaction angle increases, the total weight decreases to the lowest possible value of 1.0 which appears for large phase shifts.

The relative difference in phase shifts δ_1, δ_2 (for fixed α_{12}) plays a rather small role since the total weight increases only slowly to the maximal value which is obtained if the amplitudes of the both waves are the same ($\delta_1/\delta_2 = 1$).

Considering the weights of relative errors separately, we conclude that an accurate measurement of the interaction angle is important, while errors in measuring the phase shifts play a secondary role, especially, if the shifts are large.

Positive phase shift Here the total weight of relative errors is not bounded anymore. It becomes very large when the amplitudes of the waves become almost equal (as shown earlier, the interaction of equal amplitude solitons can only exist with negative phase shift).

However, if the phase shifts are rather different, say $\delta_1 < 0.5\delta_2$, the bound of the total weight of relative errors is small (a little above of the minimal value of 1.0). And so, these cases are less sensitive than the

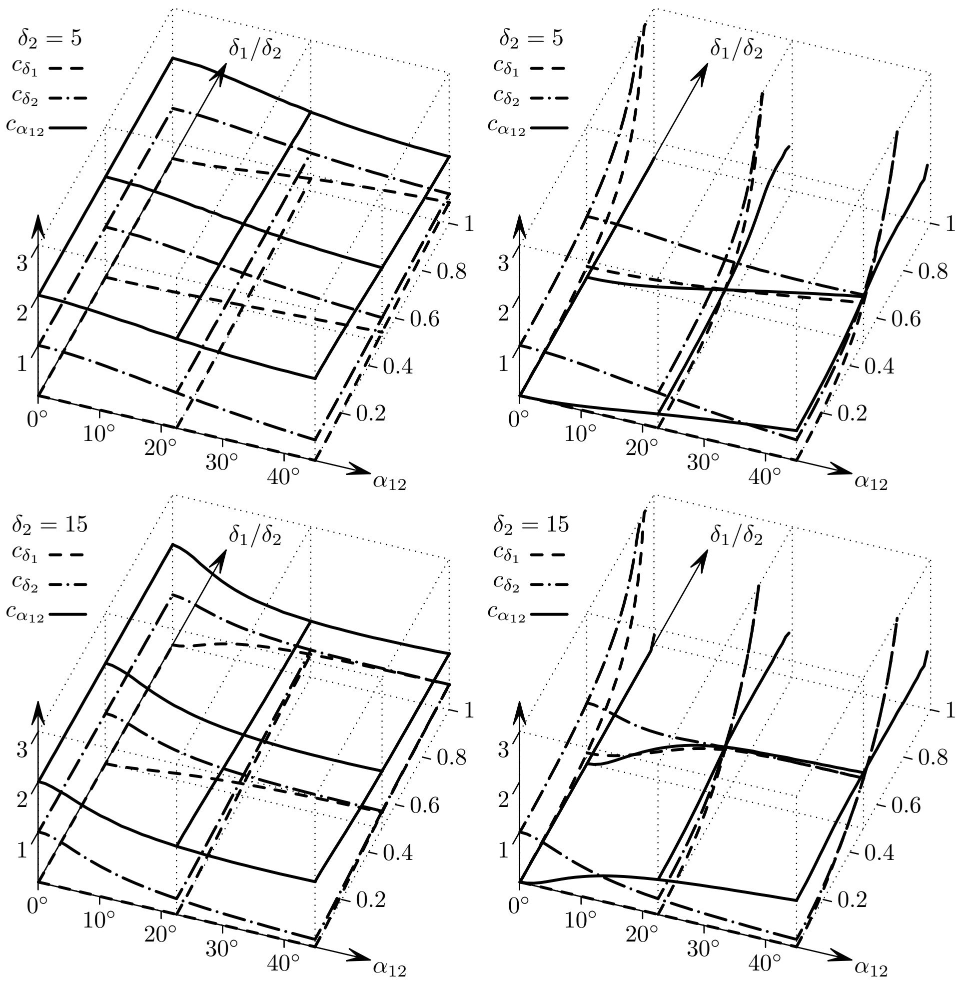

Figure 6: Weights of relative errors for interactions with negative (left) and positive (right) phase shift (see text).

corresponding cases with negative phase shift and smaller phase shift values. As for negative phase shift, the accurate measurement of the interaction angle is important, but for positive phase shift its importance is taken over by the errors of phase shift measurements if they become closer ($\delta_1 \to \delta_2$).

6 Conclusions

In this paper we derived approximate solutions to the inverse problem for wave crests and carried out a complete sensitivity analysis of this method.

As a result, we find that the method is most sensitive to errors in the interaction angle, especially, if it is small. As the main source of errors we recognize the way how the crest-lines, that are used for measuring the pattern-parameters, are drawn on the wave pattern picture. This is because the positions of wave crests are not well-defined as seen in Fig. 1. In a separate paper [5] the reconstruction of crest-lines in spatial wave fields will be investigated in detail.

In conclusion, the method of calculating the heights of the waves from a "photographic" record is applicable if the positions of the waves crests can be well defined from the wave pattern (if they are long enough, for instance) and the pattern-parameters can be determined accurately.

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