Construction and decomposition of multi-soliton solutions of KdV type equations

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Abstract. A detailed description of multi-soliton (more than two) interactions is needed for many practical applications for solving both the direct and inverse problems in multi-directional wave phenomenon (for example, surface waves). In this paper a strict novel formalism for constructing multi-soliton solutions of KdV (Korteweg-de Vries) type 2-D equations is presented. In this formalism solutions are derived in phase variables, in which the analysis of multi-soliton interactions is most natural and simple. Results of this paper include (i) a complete description of KdV 2-D multi-soliton solutions based on the Hirota formalism, and (ii) a novel decomposition of multi-soliton solutions. With this decomposition a multi-soliton solution is interpreted as a linear superposition of solitons and interaction solitons. Finally, expressions for calculating the amplitudes of interaction solitons are derived.

PACS: 05.45.Yv, 04.30.Nk

1 Introduction

Solitons have been proved to be rather robust phenomena in the theory of nonlinear wave propagation in dispersive media. Soliton interactions have been extensively studied for many decades on a large variety of model equations with both analytical and numerical methods. These studies repeatedly report the basic features of soliton phenomena to be (i) the existence of travelling local identities (called solitons) that (ii) interact elastically with their own kind. However, for practical applications, these studies, that often demonstrate soliton phenomenon only for most simple cases, do not provide enough details, especially if more than two solitons are involved. The aim of this paper is to develop a detailed description of multi-soliton interactions for a certain class of equations supporting soliton solutions.

According to the classification of Hietarinta et al [11, 12], nonlinear partial differential equations (nonlinear PDEs) supporting soliton solutions are classified to KdV, MKdV, SG, NLS, or BO type (this list may not be complete) according to how they are bi-linearized to Hirota form [13, 14]. Then, for these bilinear equations explicit analytical solutions can be found which can be transformed to explicit analytical solutions of the original nonlinear PDEs. Each class of the above equations has its own functional form for multi-soliton solutions. In this paper we deal with KdV type 2D equations.

As a possible application, consider an inverse problem of wave crests, i.e. the problem of predicting wave parameters from the geometry of the interaction picture of waves. Such an inverse problem was solved in [1, 2] for a two wave case modelled by two-soliton interactions. For that a full analysis of two-soliton solutions (the direct problem) was presented. As a result, the concept of an interaction soliton was introduced and related to two-soliton solutions by announcing a novel decomposition: a two-soliton soliton solution is written as a sum of three terms with two representing pairs of single solitons (initial ones and their shifted counterparts) and the third representing the interaction soliton that connects the single solitons before and after the interaction. This description facilitated a better geometrical representation of two-soliton interactions.

With a view to tackle the inverse problem for more than two waves, the corresponding direct multi-soliton problem needs to be solved first. As above, we relate waves with solitons by assuming the corresponding model equation to belong to the class of KdV type equations. The direct problem is about constructing multi-soliton interaction patterns from a given specification of solitons (their travelling directions and amplitudes). This is done by finding the relations between an analytically given multi-soliton solution and geometrically represented interaction pictures.

In this paper we develop for that purpose above a detailed description of multi-soliton interactions. In this description, the concept of interaction soliton is generalized for multi-soliton solutions and an interpretation of multi-soliton interactions is given that will facilitate geometrical representation of multi-soliton interactions. As far as for that purpose there is no successive description available, some facts from the basic theory must also be ordered in the way suitable for the further analysis.

For solving bilinear equations, we use *different approach* from the traditional formal self-truncating series approach (see e.g. [13, 15]). Namely, we argue that solutions to a bilinear equation can be written as sums of eigenfunctions (that are exponential functions) of the related bilinear operator. *The advantage of this approach* is that it allows us formally to derive the relations for the coefficients in this sum without assuming any particular form of a bilinear equation. The approach is general and can be used also for deriving multi-periodic soliton solutions expressed in terms of the Riemann theta-function.

Multi-soliton solutions are solutions with a certain genus g, that is, they can be represented as functions of g phase variables which are related to real variables by a linear transform. In this paper we construct multi-soliton solutions of KdV type equations (in particular, of the corresponding Hirota bilinear equations) in terms of phase variables. It means that with a suitable projection we have actually obtained multi-soliton solutions for original equations in real variables. The construction is novel in many aspects. First, the related change of variables for bi-linearization of KdV type equations is universally represented in terms of Hirota operators for ease of algebraic manipulations. Second, a precise mathematical definition of multi-soliton solutions is introduced that allows to construct multi-soliton solutions for the entire class of KdV type equations in an abstract way. Finally, the use of phase variables facilitates simple and explicit proofs of the results.

The organization of this paper is as follows. In the following Section 2 we recall the Hirota bilinear formalism and give a precise definition to KdV type equations. Here we also introduce some auxiliary notations to serve our formalism. On the basis of definitions of Section 2, the set of multi-soliton solutions of KdV type equations is defined and constructed in Section 3. KdV multi-soliton solutions are analyzed in Section 4 which is central in this paper. The main feature of the analysis is *a novel decomposition* of KdV multi-soliton solutions based on the results of Sections 2 and 3. The notion of an interaction soliton is extended for multi-soliton interactions, and finally, an interpretation of multi-soliton interactions in phase variables is given. Conclusions are drawn in Section 5. Proofs to theorems and propositions are collected in Appendices.

2 Preliminaries

In the following we consider $(2+1)$-dimensional nonlinear PDEs for $u = u(\boldsymbol{x})$, $\boldsymbol{x} = (x, y, t)^T$: $K(u, u_x, u_y, u_t, \ldots) = 0$ or $K[u] = 0$ for short. In particular, we are interested in KdV type equations (in the Hirota sense) [11] that are defined in terms of Hirota bilinear formalism [13]. The members of this rather large class of equations, recognized by Hietarinta et al [11, 12], support multi-soliton solutions that are special genus g solutions (g is a positive integer) of the form $u(\boldsymbol{x}) = U(\boldsymbol{K}\boldsymbol{x})$, where U is a function with g (*phase*) variables $\boldsymbol{\varphi} = (\varphi_1, \ldots, \varphi_g)^T \in \mathbb{R}^g$ and $\boldsymbol{K} = (\boldsymbol{\mu}, \boldsymbol{\nu}, \boldsymbol{\omega})$, $\boldsymbol{\mu}, \boldsymbol{\nu}, \boldsymbol{\omega} \in \mathbb{R}^g$, is a matrix with pairwise non-collinear rows. A more precise definition to multi-soliton solutions is given in the following Section 3.

The rest of this section is devoted for introducing preliminary definitions and notations necessary for the further analysis.

Notation. We use strict matrix formalism: vectors are column vectors and denoted by bold lowercase letters or symbols, e.g., $\boldsymbol{x}$, $\boldsymbol{\mu}$; matrices are formed by column vectors and denoted by bold uppercase letters, e.g., $\boldsymbol{K}$. We denote functions in real ($\boldsymbol{x}$) and in phase ($\boldsymbol{\varphi}$) variables with lower and upper case letters or symbols, respectively. For example, $u = u(\boldsymbol{x})$, $U = U(\boldsymbol{\varphi})$.

The following two definitions are due to Hirota [13] but adjusted here for our notation.

Definition 1 (Hirota polynomial). In this paper a polynomial $B = B(\mu, \nu, \omega)$ is called the *Hirota polynomial* iff $B(0, 0, 0) = 0$ and $B(-\mu, -\nu, -\omega) = B(\mu, \nu, \omega)$. We denote Hirota polynomials also as $B = B(\boldsymbol{k})$, where $\boldsymbol{k} = (\mu, \nu, \omega)^T$.

(Note that we do not require that Hirota polynomials should satisfy Hirota conditions (11) as in [16].)

Definition 2 (Hirota derivative). A symbol D_x is called the *Hirota derivative* with respect to variable x and defined to act on a pair of functions (f, g) as follows:

$$(D_x f \cdot g)(x) \equiv (\partial_x - \partial_{x'}) f(x) g(x')\big|_{x'=x}\,, \tag{1}$$

where ∂_x denotes partial derivative with respect to x. Similarly are defined Hirota derivatives with respect to y and t variables: D_y and D_t, respectively. The composition of Hirota derivatives is defined as follows:

$$\begin{aligned}&(D_x^n D_y^m D_t^k f \cdot g)(\boldsymbol{x}) \equiv \\ &\qquad (\partial_x - \partial_{x'})^n (\partial_y - \partial_{y'})^m (\partial_t - \partial_{t'})^k f(\boldsymbol{x}) g(\boldsymbol{x}')\Big|_{\boldsymbol{x}'=\boldsymbol{x}}\,.\end{aligned} \tag{2}$$

Hirota derivatives are bilinear operators. If B is a polynomial then $B(D_x, D_y, D_t)$ is called the *Hirota bilinear operator*. We denote Hirota bilinear operators also as $B(D_{\boldsymbol{x}})$, where $D_{\boldsymbol{x}} = (D_x, D_y, D_t)^T$.

See Appendix A for the properties of Hirota bilinear operators that are relevant to this paper.

The following definition is based on the original work of Hietarinta et al [11, 12] but given here in a more precise form.

Definition 3 (KdV type equation). An equation $K[u] = 0$ is called the *KdV type equation* (in the Hirota sense) iff there exist two nontrivial Hirota polynomials $P = P(\mu, \nu, \omega)$ and $N = N(\mu, \nu, \omega)$ such that

$$P(D_{\boldsymbol{x}})\theta \cdot \theta = 0 \quad \Rightarrow \quad K\left[\theta^{-2} N(D_{\boldsymbol{x}})\theta \cdot \theta\right] = 0 \tag{3}$$

holds for all nontrivial positive functions $\theta = \theta(\boldsymbol{x})$. Equation $P(D_{\boldsymbol{x}})\theta \cdot \theta = 0$ is called the *Hirota bilinear form* of the corresponding equation $K[u] = 0$.

The following remarks reason the use of phase variables instead of real (space-time) variables throughout of this paper.

Remark 1. If $K[u] = 0$ is a KdV type equation then its genus g solutions of the form $u = \theta^{-2}N(D_{\boldsymbol{x}})\theta \cdot \theta$ are defined by the existence of a function $\Theta = \Theta(\boldsymbol{\varphi})$ and a matrix $\boldsymbol{K}$ such that $\theta(\boldsymbol{x}) = \Theta(\boldsymbol{K}\boldsymbol{x})$. Indeed, if we define for a given function $\Theta = \Theta(\boldsymbol{\varphi})$ a function $U(\boldsymbol{\varphi}) = \Theta^{-2}N(D_{\boldsymbol{\varphi}}^T\boldsymbol{K})\Theta \cdot \Theta$, where $D_{\boldsymbol{\varphi}} = (D_{\varphi_1}, \ldots, D_{\varphi_g})^T$ denotes Hirota derivatives with respect to phase variables $\boldsymbol{\varphi}$, then $u(\boldsymbol{x}) = U(\boldsymbol{K}\boldsymbol{x})$ is a solution to $K[u] = 0$.

For brevity, we define the operator $L : \Theta \to U = L[\Theta]$ as follows

$$L[\Theta] = \frac{N(D_{\boldsymbol{\varphi}}^T\boldsymbol{K})\Theta \cdot \Theta}{\Theta^2}. \tag{4}$$

While using this notation it is assumed that $N = N(\boldsymbol{k})$ is a Hirota polynomial. Fundamental properties of the operator L are proved in Appendix B.

Clearly, if Θ is a solution to $P(D_{\boldsymbol{\varphi}}^T\boldsymbol{K})\Theta \cdot \Theta = 0$ then a genus g solution to the corresponding KdV type equation $K[u] = 0$ is $u(\boldsymbol{x}) = L[\Theta](\boldsymbol{K}\boldsymbol{x})$.

The following notations below have an important role in the formalism developed in this paper. They shorten the representation of coming formulas considerably, and not less importantly, they facilitate software implementation of the results for practical applications.

Notation (Excluding variables). For a n-sequence of symbols we use a superscript rounded with parenthesis to denote a sequence of symbols that is a subset of the n-sequence and contains symbols with indices that are *not* listed in the superscript and preserving the order. For example, $(a, b, c, d)^{(1,3)} = (b, d)$, $(\varphi_1, \varphi_2, \varphi_3)^{(2)} = (\varphi_1, \varphi_3)$.

Notation (Index map κ). For every $\boldsymbol{\alpha} \in \{0, 1\}^g$ we define a set $\kappa(\boldsymbol{\alpha}) \in \wp(\{1, \ldots, g\})$ ($\wp(S)$ denotes the set of all subsets of a set S) such that (i) if $\alpha_i = 1$ then $i \in \kappa(\boldsymbol{\alpha})$ and (ii) $|\kappa(\boldsymbol{\alpha})| = \boldsymbol{\alpha}^T\boldsymbol{\alpha}$. For example, $\kappa((1, 0, 1, 1, 0)^T) = \{1, 3, 4\}$. We also write $A_{\{1,3,4\}}$ as A_{134}, for example.

This completes the basis for the formalism presented below.

Finally, we illustrate the introduced notation with the following examples.

Example 1 *(Kadomtshev-Petviashvili (KP) equation [17]).* As an example of a KdV type 2D equation, consider the KP equation $(u_t + 6uu_x + u_{xxx})_x + 3u_{yy} = 0$. The change of variables $u = 2\partial_x^2 \ln \theta = \theta^{-2}N(D_{\boldsymbol{x}})\theta \cdot \theta$ with $N(\mu, \nu, \omega) = \mu^2$ leads to the corresponding Hirota bilinear form $(D_tD_x + D_x^4 + 3D_y^2)\theta \cdot \theta = 0$ with the related Hirota polynomial $P(\mu, \nu, \omega) = \omega\mu + \mu^4 + 3\nu^2$ (see [18]).

Example 2 *(One-soliton solution).* Let us take $g = 1$ and

$$\Theta(\varphi_1) = 1 + e^{\varphi_1}.$$

We find that $P(D_{\boldsymbol{\varphi}}^T\boldsymbol{K})\Theta \cdot \Theta = 2P(\mu_1, \nu_1, \omega_1)e^{\varphi_1}$ after using (27) and the properties of the Hirota polynomial P. Similarly we find

$$L[\Theta] = \frac{2N(\mu_1, \nu_1, \omega_1)e^{\varphi_1}}{(1 + e^{\varphi_1})^2} = \frac{1}{2}N(\mu_1, \nu_1, \omega_1)\operatorname{sech}^2\frac{\varphi_1}{2}.$$

Therefore, if we require $P(\mu_1, \nu_1, \omega_1) = 0$ then the genus 1 solution of the KdV type equation $K[u] = 0$ is

$$u(x, y, t) = \frac{1}{2} N(\mu_1, \nu_1, \omega_1) \mathrm{sech}^2 \frac{\mu_1 x + \nu_1 y + \omega_1 t}{2}.$$

3 Multi-soliton solutions

In what follows we give a precise mathematical definition for multi-soliton solutions and then construct them explicitly for KdV type equations. Our definition is novel in that it is (i) minimal but sufficient to summarize the well-known basic properties of soliton phenomenon, and (ii) constructive. Although, the functional form of the multi-soliton solution for KdV type equations is well-known for decades[13], the proofs are available only for the simplest (e.g. the classical KdV) cases and yet with lengthy manipulations. Our construction, however, is novel in this sense that it facilitates most general and yet explicit proof for the construction of multi-soliton solutions to all members in the class of KdV type equations.

Let $K[u] = 0$ be a nonlinear PDE. Let us denote a set of all trivial (constant in $\boldsymbol{x}$) solutions of the equation $K[u] = 0$ by symbol $\mathcal{S}_0$ and call it the *set of vacuum-soliton solutions* of the equation $K[u] = 0$.

Definition 4 (Multi-soliton solutions). A set $\mathcal{S}_g$ is called the *set of genus g soliton solutions* (in phase variables) of the equation $K[u] = 0$ iff it consists of functions $U = U(\boldsymbol{\varphi}) \in \mathcal{S}_g$, $\boldsymbol{\varphi} \in \mathbb{R}^g$ that satisfy the following conditions:

1. $u(\boldsymbol{x}) = U(\boldsymbol{K}\boldsymbol{x})$ is a genus g solution of $K[u] = 0$;
2. $\lim\limits_{\varphi_i \to \pm\infty} U \in \mathcal{S}_{g-1}$ for all $i = 1, \ldots, g$;
3. there exists symmetric $g \times g$-matrix $\boldsymbol{\Delta}$ such that

$$(\lim_{\varphi_i \to +\infty} U)(\boldsymbol{\varphi}^{(i)}) = (\lim_{\varphi_i \to -\infty} U)(\boldsymbol{\varphi}^{(i)} - \boldsymbol{\Delta}_i{}^{(i)}), \tag{5}$$

   where $\boldsymbol{\varphi}^{(i)} \in \mathbb{R}^{g-1}$, $\boldsymbol{\varphi}^{(i)} = (\varphi_1, \ldots, \varphi_{i-1}, \varphi_{i+1}, \ldots, \varphi_g)^T$, for all $i = 1, \ldots, g$, and $\boldsymbol{\Delta}_i$ denotes the i-th column of $\boldsymbol{\Delta}$.

Constants Δ_{ij} are called the *phase shift parameters*. We define diagonal elements $\Delta_{ii} = 0$.

Remark 2. Generally speaking, there are three basic conditions for soliton phenomena: (i) there must exist some identities, called solitons, (ii) that are local, and (iii) survive the interactions with other solitons that cause only phase shifts in soliton evolutions. Definition 4 is very much inspired by this statement. For example, let us consider a two-soliton solution. As shown in [1], the row vectors of $\boldsymbol{K}$ correspond to wave vectors (traveling direction+speed) of solitons. Also, when finding the limit, say, $\varphi_2 \to -\infty$ on a two-soliton solution, then the result is an one-soliton solution in variable φ_1. On the other hand, the limit process $\varphi_2 \to +\infty$ produces the same one-soliton solution but is shifted only by some amount.

In the following we construct multi-soliton solutions for KdV type equations.

Theorem 3.1 (KdV multi-soliton solutions (cf. [19])**).** *If $K[u]=0$ is a KdV type equation (with Hirota polynomials P and N) then its bounded genus g soliton solutions are $U = L[\Theta]$, where*

$$\Theta = \sum_{\boldsymbol{\alpha}\in\{0,1\}^g} A_{\kappa(\boldsymbol{\alpha})} e^{\boldsymbol{\alpha}^T \boldsymbol{\varphi}} \tag{6}$$

and the following conditions hold for the coefficients $A_{\kappa(\boldsymbol{\alpha})}$ and the matrix $\boldsymbol{K}$:

$$A_{\kappa(\boldsymbol{\alpha})} = \begin{cases} 1 & if\, \boldsymbol{\alpha}^T\boldsymbol{\alpha} < 2, \\ \prod_{\substack{i,j\in\kappa(\boldsymbol{\alpha}) \\ i<j}} A_{ij} & if\, \boldsymbol{\alpha}^T\boldsymbol{\alpha} > 2, \end{cases} \tag{7}$$

Dispersion relations:	$P(\boldsymbol{k}_i) = 0,$	(8)
Relations for A_{ij}:	$A_{ij}P(\boldsymbol{k}_i+\boldsymbol{k}_j) + P(\boldsymbol{k}_i-\boldsymbol{k}_j) = 0,$	(9)
Boundness conditions:	$P(\boldsymbol{k}_i+\boldsymbol{k}_j)P(\boldsymbol{k}_i-\boldsymbol{k}_j) \leqslant 0,$	(10)
Hirota conditions:	$\sum_{\boldsymbol{\alpha}\in(\boldsymbol{\beta}-\{0,1\}^g)\cap\{0,1\}^g} A_{\kappa(\boldsymbol{\alpha})}A_{\kappa(\boldsymbol{\beta}-\boldsymbol{\alpha})} P((2\boldsymbol{\alpha}-\boldsymbol{\beta})^T\boldsymbol{K}) = 0,$	(11)

where $\boldsymbol{\beta}\in\{0,1\}^g$, $\boldsymbol{\beta}^T\boldsymbol{\beta} > 2$, and $\boldsymbol{k}_i^T$ is the i-th row of the matrix $\boldsymbol{K}$. In addition, the off-diagonal phase shift parameters read

$$\Delta_{ij} = -\ln A_{ij}. \tag{12}$$

Proof. See Appendix C. □

Example 3 *(Two-soliton solution).* Taking $g=2$, the theta-function (6) reads

$$\Theta = 1 + e^{\varphi_1} + e^{\varphi_2} + A_{12}e^{\varphi_1+\varphi_2}.$$

After some algebraic manipulations[1] we find that the function $U = L[\Theta]$ can be written as

$$\begin{aligned} U(\varphi_1,\varphi_2) = \Theta'^{-2}\Big(2\sqrt{A_{12}}\big(&N(\boldsymbol{k}_1)\cosh(\varphi_2+\ln\sqrt{A_{12}}) \\ &+N(\boldsymbol{k}_2)\cosh(\varphi_1+\ln\sqrt{A_{12}})\big) \\ &+N(\boldsymbol{k}_1-\boldsymbol{k}_2) + A_{12}N(\boldsymbol{k}_1+\boldsymbol{k}_2)\Big) \end{aligned} \tag{13}$$

where

$$\Theta' = \sqrt{2}\cosh\frac{1}{2}(\varphi_1 - \varphi_2) + \sqrt{2A_{12}}\cosh\frac{1}{2}(\varphi_1 + \varphi_2 + \ln A_{12}). \qquad (14)$$

If we require that the rows of $\boldsymbol{K}$ satisfy $P(\boldsymbol{k}_1) = 0$, $P(\boldsymbol{k}_2) = 0$ (the dispersion relations), $P(\boldsymbol{k}_1 + \boldsymbol{k}_2)P(\boldsymbol{k}_1 - \boldsymbol{k}_2) < 0$ (the boundness condition), and the coefficient A_{12} is found as

$$A_{12} = -P(\boldsymbol{k}_1 - \boldsymbol{k}_2)/P(\boldsymbol{k}_1 + \boldsymbol{k}_2)$$

then the genus 2 soliton solution of the KdV type equation $K[u] = 0$ is

$$u(x, y, t) = U(\mu_1 x + \nu_1 y + \omega_1 t, \mu_2 x + \nu_2 y + \omega_2 t).$$

The special cases $P(\boldsymbol{k}_1 + \boldsymbol{k}_2) = 0$ and $P(\boldsymbol{k}_1 - \boldsymbol{k}_2) = 0$ lead to a trivial solution and a resonant soliton solution (see also [20, 21, 3]), respectively, as shown in [1].

The following Corollary is well-known (see e.g. [13, 11]) but given here for the completeness of the following discussion.

Corollary 1. *Every KdV type equation supports genus 1 and 2 soliton solutions.*

Proof. See examples 2 and 3. □

Remark 3. There exist KdV type equations that support also genus 3 and more soliton solutions. For example, the classical KdV equation [19] and the KP equation [22]. For those equations all Hirota conditions (11) are trivial if the dispersion relations (8) and relations for A_{ij} (9) are fulfilled. KdV type equations for which all Hirota conditions are trivial for arbitrary genus g are said to *support* multi-soliton solutions. In addition, it is conjectured that such equations are completely integrable [11].

However, there also exist KdV type equations (e.g. the 2D Boussinesq equation [23]) for which Hirota conditions (11) are nontrivial and some additional constraints are provided for the matrix $\boldsymbol{K}$. Such equations are said to *not support* multi-soliton solutions. We note that this *does not* mean that such equations *do not* have multi-soliton solutions (as often mistakenly assumed) — one cannot just vary the parameters of individual solitons arbitrarily, and the physical meaning of nontrivial Hirota conditions is unclear (as opposite to dispersion relations).

4 Decomposition of KdV multi-soliton solution

This section holds the main result of this paper. In particular, we introduce a novel decomposition of KdV multi-soliton solutions that will be used for describing multi-soliton interactions. We proceed as follows. First, we illustrate the main idea for the simplest and yet nontrivial case. Then we present the main theorem with a consequent proposition that proves the usefulness and sensibility of the result. Finally, we attach the result with an interpretation for multi-soliton interactions.

Let us first consider the simplest soliton interaction that is described by a two-soliton solution. It is shown in [1] that a two-soliton solution can be written as a superposition of two solitons and an interaction soliton:

$$U = S_1 + S_2 + S_{12},$$

respectively, where all terms are functions of (φ_1, φ_2). When two solitons interact, they will undergo a phase shift. So, the terms S_1 and S_2 contain initial single solitons [1]:

$$U_1(\varphi_1) = \lim_{\varphi_2 \to -\infty} S_1(\varphi_1, \varphi_2), \qquad U_2(\varphi_2) = \lim_{\varphi_1 \to -\infty} S_2(\varphi_1, \varphi_2),$$

and their shifted counterparts:

$$U_1(\varphi_1 - \Delta_{12}) = \lim_{\varphi_2 \to +\infty} S_1(\varphi_1, \varphi_2), \quad U_2(\varphi_2 - \Delta_{12}) = \lim_{\varphi_1 \to +\infty} S_2(\varphi_1, \varphi_2),$$

respectively, where U_1, U_2 denote one-soliton solutions representing the corresponding single solitons. As shown in [1], the leftover term S_{12} contributes to the two-soliton solution only in the interaction region where it connects the single solitons and their shifted counterparts. The interaction soliton S_{12} has a soliton-like profile but only locally — far from the interaction region the interaction soliton vanishes:

$$\lim_{\varphi_i \to \pm\infty} S_{12}(\varphi_1, \varphi_2) = 0.$$

In the following we extend this description of two-soliton interactions for multi-soliton interactions. We start by decomposing multi-soliton solutions in the same way. For that we need the following definitions and auxiliary results:

Definition 5 (Equivalent theta-functions). Functions Θ and Θ' are called the *equivalent theta-functions* iff $L[\Theta] = L[\Theta']$.

Propositon 4.1. *Functions Θ and $e^{\boldsymbol{\alpha}^T \boldsymbol{\varphi}}\Theta$ are the equivalent theta-functions.*

Proof. Follows directly from Proposition B.1. □

Notation (Negation operator˜). For every $\boldsymbol{\alpha} \in \{0,1\}^g$ we define $\tilde{\boldsymbol{\alpha}} \equiv \boldsymbol{i} - \boldsymbol{\alpha}$, where $\boldsymbol{i} \in \{1\}^g$ such that $\boldsymbol{i}^T \boldsymbol{i} = g$.

Propositon 4.2. *Functions Θ in (6) and*

$$\Theta' = \frac{1}{\sqrt{2}} \sum_{\boldsymbol{\alpha} \in \{0,1\}^g} \sqrt{A_{\kappa(\boldsymbol{\alpha})} A_{\kappa(\tilde{\boldsymbol{\alpha}})}} \cosh \frac{(\boldsymbol{\alpha} - \tilde{\boldsymbol{\alpha}})^T \boldsymbol{\varphi} + \ln A_{\kappa(\boldsymbol{\alpha})}/A_{\kappa(\tilde{\boldsymbol{\alpha}})}}{2} \tag{15}$$

are the equivalent theta-functions provided that they share the same set of coefficients $A_{\kappa(\boldsymbol{\alpha})}$.

Proof. See Appendix D. □

Next, we present the central theorem of this paper.

Theorem 4.1 (Decomposition of KdV multi-soliton solution). *If U is a genus g soliton solution of a KdV type equation as given in Theorem 3.1 then it can be written as a superposition:*

$$U = \sum_{\boldsymbol{\beta} \in \{0,1\}^g} S_{\kappa(\boldsymbol{\beta})}, \tag{16}$$

where

$$S_{\kappa(\boldsymbol{\beta})} = \frac{1}{2\Theta'^2} \sum_{\substack{\boldsymbol{\alpha}, \boldsymbol{\alpha}' \in \{0,1\}^g \\ |\boldsymbol{\alpha} - \boldsymbol{\alpha}'|_{ew} = \boldsymbol{\beta}}} \sqrt{A_{\kappa(\boldsymbol{\alpha})} A_{\kappa(\tilde{\boldsymbol{\alpha}})} A_{\kappa(\boldsymbol{\alpha}')} A_{\kappa(\tilde{\boldsymbol{\alpha}}')}} N((\boldsymbol{\alpha} - \boldsymbol{\alpha}')^T \boldsymbol{K})$$
$$\times \cosh\Big((\boldsymbol{\alpha} - \tilde{\boldsymbol{\alpha}}')^T \boldsymbol{\varphi} + \frac{1}{2} \ln \frac{A_{\kappa(\boldsymbol{\alpha})} A_{\kappa(\boldsymbol{\alpha}')}}{A_{\kappa(\tilde{\boldsymbol{\alpha}})} A_{\kappa(\tilde{\boldsymbol{\alpha}}')}}\Big), \tag{17}$$

Θ' is defined by (15), and $|\cdot|_{ew} \equiv (|\cdot_1|, \ldots, |\cdot_g|)^T$ denotes the element-wise absolute value operation. In particular, $S_\emptyset = 0$ for all g.

Proof. To obtain the representation (16), we use the following identity

$$\begin{aligned} &2N(D_{\boldsymbol{\varphi}}^T) \cosh(\boldsymbol{\alpha}^T \boldsymbol{\varphi} + \varphi_0) \cdot \cosh(\boldsymbol{\alpha}'^T \boldsymbol{\varphi} + \varphi_0') \\ &= N((\boldsymbol{\alpha} - \boldsymbol{\alpha}')^T \boldsymbol{K}) \cosh((\boldsymbol{\alpha} + \boldsymbol{\alpha}')^T \boldsymbol{\varphi} + \varphi_0 + \varphi_0') \\ &+ N((\boldsymbol{\alpha} + \boldsymbol{\alpha}')^T \boldsymbol{K}) \cosh((\boldsymbol{\alpha} - \boldsymbol{\alpha}')^T \boldsymbol{\varphi} + \varphi_0 - \varphi_0') \end{aligned}$$

that can be easily checked using Corollary A.2. Let us find

$$N(D_{\boldsymbol{\varphi}}^T)\Theta' \cdot \Theta' = \frac{1}{4} \sum_{\boldsymbol{\alpha}, \boldsymbol{\alpha}' \in \{0,1\}^g} \sqrt{A_{\kappa(\boldsymbol{\alpha})} A_{\kappa(\tilde{\boldsymbol{\alpha}})} A_{\kappa(\boldsymbol{\alpha}')} A_{\kappa(\tilde{\boldsymbol{\alpha}}')}}$$
$$\times \Big(N((\boldsymbol{\alpha} - \boldsymbol{\alpha}')^T \boldsymbol{K}) \cosh\big((\boldsymbol{\alpha} - \tilde{\boldsymbol{\alpha}}')^T \boldsymbol{\varphi} + \frac{1}{2} \ln \frac{A_{\kappa(\boldsymbol{\alpha})} A_{\kappa(\boldsymbol{\alpha}')}}{A_{\kappa(\tilde{\boldsymbol{\alpha}})} A_{\kappa(\tilde{\boldsymbol{\alpha}}')}}\big)$$
$$+ N((\boldsymbol{\alpha} - \tilde{\boldsymbol{\alpha}}')^T \boldsymbol{K}) \cosh\big((\boldsymbol{\alpha} - \boldsymbol{\alpha}')^T \boldsymbol{\varphi} + \frac{1}{2} \ln \frac{A_{\kappa(\boldsymbol{\alpha})} A_{\kappa(\tilde{\boldsymbol{\alpha}}')}}{A_{\kappa(\tilde{\boldsymbol{\alpha}})} A_{\kappa(\boldsymbol{\alpha}')}}\big)\Big).$$

After splitting this sum into two sums over the terms in the parenthesis and changing the summation order in the second sum by the replacement $\boldsymbol{\alpha}' \to \tilde{\boldsymbol{\alpha}}'$ (as a result of the two sums become identical), we obtain

$$N(D_{\boldsymbol{\varphi}}^T)\Theta' \cdot \Theta' = \frac{1}{2} \sum_{\boldsymbol{\alpha}, \boldsymbol{\alpha}' \in \{0,1\}^g} \sqrt{A_{\kappa(\boldsymbol{\alpha})} A_{\kappa(\tilde{\boldsymbol{\alpha}})} A_{\kappa(\boldsymbol{\alpha}')} A_{\kappa(\tilde{\boldsymbol{\alpha}}')}} N((\boldsymbol{\alpha} - \boldsymbol{\alpha}')^T \boldsymbol{K})$$
$$\times \cosh\big((\boldsymbol{\alpha} - \tilde{\boldsymbol{\alpha}}')^T \boldsymbol{\varphi} + \frac{1}{2} \ln \frac{A_{\kappa(\boldsymbol{\alpha})} A_{\kappa(\boldsymbol{\alpha}')}}{A_{\kappa(\tilde{\boldsymbol{\alpha}})} A_{\kappa(\tilde{\boldsymbol{\alpha}}')}}\big).$$

Collecting the terms in this result according to the condition $|\boldsymbol{\alpha} - \boldsymbol{\alpha}'|_{ew} = \boldsymbol{\beta}$ and dividing it by Θ'^2, we obtain $S_{\kappa(\boldsymbol{\beta})}$ as defined in the superposition (16).

$S_\emptyset = 0$ follows from the fact that N is a Hirota polynomial. □

Example 4 *(Decomposition of KdV one-soliton solutions).* Taking $g = 1$, we have a trivial decomposition (cf. example 2):

$$U = S_1 = \frac{N(\mu_1, \nu_1, \omega_1)}{2\cosh^2 \frac{\varphi_1}{2}} = \frac{N(\mu_1, \nu_1, \omega_1)}{2} \mathrm{sech}^2 \frac{\varphi_1}{2}.$$

Example 5 *(Decomposition of KdV two-soliton solutions).* Taking $g = 2$, the theta-function in (15) reads (14) and the terms of the corresponding decomposition are (cf. example 3):

$$S_1(\varphi_1, \varphi_2) = \frac{2\sqrt{A_{12}}N(\mu_1, \nu_1, \omega_1)}{\Theta'^2} \cosh(\varphi_2 + \ln\sqrt{A_{12}}), \tag{18}$$

$$S_2(\varphi_1, \varphi_2) = \frac{2\sqrt{A_{12}}N(\mu_2, \nu_2, \omega_2)}{\Theta'^2} \cosh(\varphi_1 + \ln\sqrt{A_{12}}), \tag{19}$$

$$\begin{aligned} S_{12}(\varphi_1, \varphi_2) &= \frac{1}{\Theta'^2}(N(\mu_1 - \mu_2, \nu_1 - \nu_2, \omega_1 - \omega_2) \\ &+ A_{12}N(\mu_1 + \mu_2, \nu_1 + \nu_2, \omega_1 + \omega_2)). \end{aligned} \tag{20}$$

The following result shows that the decomposition (16) is indeed "interference free", that is, the supporting regions of its terms are pair-wisely non-inclusive, and consequently, each term represents an unique part of the multi-soliton solution. Therefore, the decomposition (16) forms a suitable basis for describing multi-soliton interactions.

Propositon 4.3 (Asymptotic behavior of $S_{\kappa(\boldsymbol{\beta})}$). *If $S_{\kappa(\boldsymbol{\beta})}$, $\boldsymbol{\beta} \in \{0,1\}^g$, are the terms of the decomposition of the KdV multi-soliton solution as given in (17) then the following equations hold:*

$$\lim_{\varphi_i \to \pm\infty} S_{\kappa(\boldsymbol{\beta})} = 0, \qquad \text{whenever } \beta_i = 1, \tag{21}$$

$$(\lim_{\varphi_i \to -\infty} S_{\kappa(\boldsymbol{\beta})})(\boldsymbol{\varphi}^{(i)}) = S^{(i)}_{\kappa(\boldsymbol{\beta}^{(i)})}(\boldsymbol{\varphi}^{(i)}), \qquad \text{if } \beta_i = 0, \tag{22}$$

$$(\lim_{\varphi_i \to +\infty} S_{\kappa(\boldsymbol{\beta})})(\boldsymbol{\varphi}^{(i)}) = S^{(i)}_{\kappa(\boldsymbol{\beta}^{(i)})}(\boldsymbol{\varphi}^{(i)} - \boldsymbol{\Delta}_i{}^{(i)}), \qquad \text{if } \beta_i = 0, \tag{23}$$

where $i = 1, \ldots, g$, $\boldsymbol{\varphi}^{(i)} = (\varphi_1, \ldots, \varphi_{i-1}, \varphi_{i+1}, \ldots, \varphi_g)^T$, $\boldsymbol{\beta}^{(i)} = (\beta_1, \ldots, \beta_{i-1}, \beta_{i+1}, \ldots, \beta_g)^T$, and $S^{(i)}_{\kappa(\boldsymbol{\beta}^{(i)})}$ are the terms of the decomposition for $U^{(i)} = \lim_{\varphi_i \to -\infty} U$.

Proof. See Appendix E. □

For interpreting the main result, we introduce the following notation.

Notation (Solitons and interaction solitons). The terms in the decomposition (16) that do *not* vanish in the limit process over $(g-1)$ phase variables approaching to infinity, are called the *soliton terms.* All other terms are called the *interaction soliton terms.*

Using the properties (21)–(23), it is easy to see that $S_{\kappa(\boldsymbol{\beta})}$ is a soliton term only if $\boldsymbol{\beta}^T\boldsymbol{\beta} = 1$. For instance,

$$\lim_{\substack{\varphi_j \to -\infty \\ j\in\{1,\dots,g\}\setminus\{i\}}} S_i(\boldsymbol{\varphi}) = U_i(\varphi_i) = \frac{N(\boldsymbol{k}_i)}{2}\operatorname{sech}^2\frac{\varphi_i}{2}, \tag{24}$$

$$\lim_{\substack{\varphi_j \to +\infty \\ j\in\{1,\dots,g\}\setminus\{i\}}} S_i(\boldsymbol{\varphi}) = U_i(\varphi_i - \sum_{\substack{j=1 \\ j\neq i}}^{g} \Delta_{ij}), \tag{25}$$

where we have denoted $U_i = S_i^{(\{1,\dots,g\}\setminus\{i\})}$. All interaction soliton terms vanish:

$$\lim_{\substack{\varphi_j \to -\infty \\ j\in\{1,\dots,g\}\setminus\{i\}}} S_{\kappa(\boldsymbol{\beta})} = 0, \qquad \text{whenever } \boldsymbol{\beta}^T\boldsymbol{\beta} > 1.$$

Remark 4 *(Interpretation of multi-soliton decomposition).* To interpret the decomposition of multi-soliton solution, we proceed from the following statement:

> Interaction of two solitons (of any kind) involves an interaction soliton that connects the two solitons and their shifted counterparts.

This statement is shown to hold for two-soliton interactions in [1].

Now, in the interactions with more than two solitons, interactions between solitons (with $\boldsymbol{\beta}^T\boldsymbol{\beta} = 1$) and interaction solitons (say, with $\boldsymbol{\beta}^T\boldsymbol{\beta} = 2$) are possible (e.g. the third soliton interacting with the interaction soliton of the first and second soliton). This interaction introduces a new interaction soliton (with $\boldsymbol{\beta}^T\boldsymbol{\beta} = 3$), that is, in addition to interaction solitons from the pairwise interactions of solitons. New interaction solitons (say, with $\boldsymbol{\beta}^T\boldsymbol{\beta} = 4$) are introduced from the interactions between interaction solitons (e.g. the interaction soliton of the third and fourth soliton interacting with the interaction soliton of the first and second soliton). This description can be easily prolonged. The decomposition of multi-soliton solutions contains all solitons and all possible interaction solitons.

Finally, we have

Propositon 4.4 (Amplitudes of $S_{\kappa(\boldsymbol{\beta})}$). *The amplitude of a soliton term S_i is $N(\boldsymbol{k}_i)/2$. The amplitude of an interaction soliton term $S_{\kappa(\boldsymbol{\beta})}$ ($\boldsymbol{\beta}^T\boldsymbol{\beta} > 1$) is*

$$(\lim_{\substack{\varphi_j \to -\infty \\ j\in\kappa(\tilde{\boldsymbol{\beta}})}} S_{\kappa(\boldsymbol{\beta})})((\frac{1}{2}\sum_{i\in\kappa(\boldsymbol{\beta})\setminus\{j\}} \Delta_{ij})_{j\in\kappa(\boldsymbol{\beta})}).$$

Proof. See Appendix F. □

5 Conclusions

In this paper multi-soliton solutions of KdV type equations were constructed and analyzed using Hirota bilinear formalism. We have used parallel description in real and phase variables, that is why some basic information is rewritten using a suitable notation. As a result, *a novel decomposition* of KdV multi-soliton solutions was introduced. This decomposition is useful for interpreting multi-soliton interactions. According to this interpretation, a multi-soliton solution is a superposition of solitons and interaction solitons. Interaction solitons are soliton-like identities with a local support and they are formed in relation with soliton interactions — the interaction solitons connect interacting solitons with their shifted counterparts. Interaction solitons interact with other solitons in the same fashion and generate new interaction solitons. Thus the full interaction picture is always rather complicated.

Note that the introduced *decomposition is different, both conceptually and functionally, from the well-known GGKM* (Gardner, Green, Kruskal, Miura) "eigenvalue and eigenfunction" decomposition [4]. According to the GGKM decomposition, a soliton is identified by the corresponding term in the decomposition for all time, even during interaction [5]. This description facilitates various formal definitions for the paths (the trajectories of centers of masses) of single solitons so that solitons can be tracked down also in interaction regions (see [6, 7] and [8, 9] for two different definitions). But note that such definitions of soliton paths are plausible only far from interaction regions — near or inside of an interaction region conclusions drawn can be contradictory. For example, the authors in [7] find that during the two-soliton interaction of the KdV (Korteweg-de Vries) equation the smaller soliton can move, for a while, with a negative velocity, although the corresponding model describes strictly unidirectional wave motion. The same conclusion is drawn in [9] with the different path definition; even more drastic example is given in [10] where it is shown that the velocities of smaller solitons (calculated as the velocities of their center of the masses) may become infinite. These results show that the GGKM decomposition is not well suited for describing the structure of soliton interactions.

The results of this more or less technical study will be used in a successive paper[24] where the geometric representation of multi-soliton interactions will be considered in order to construct multi-soliton interaction pictures. Then the relevance of the projection of solutions in phase variables to solutions in real variables will explicitly show up, demonstrating the advantage of this approach. Our further intention is to use these results for determining the velocities and amplitudes of surface water waves from their interaction patterns.

6 Acknowledgments

The author is grateful to Professor Erik van Groesen (Enschede, The Netherlands) for critical reading of the manuscript and instructive guidance throughout of this work, and Professor Jüri Engelbrecht (Tallinn, Estonia) for continuous support of this work.

Part of this research is sponsored by the Royal Netherlands Academy of Arts and Sciences in the program for scientific cooperation The Netherlands-Indonesia. Support from the Faculty of Mathematical Sciences of the University Twente is greatly appreciated. Financial support from the Estonian Science Foundation is acknowledged (grants 2631, 4068).

Appendix A Properties of Hirota bilinear operators

Propositon A.1 (Eigenvalue problem of a Hirota bilinear operator). *If $B = B(\boldsymbol{k})$ is a Hirota polynomial then*

$$B(D_{\boldsymbol{x}})e^{\boldsymbol{k}^T\boldsymbol{x}+\varphi_0}\cdot e^{\boldsymbol{k}'^T\boldsymbol{x}+\varphi_0'} = B(\boldsymbol{k}-\boldsymbol{k}')e^{(\boldsymbol{k}+\boldsymbol{k}')^T\boldsymbol{x}+\varphi_0+\varphi_0'}, \tag{26}$$

where $\boldsymbol{k} = (\mu,\nu,\omega)^T, \boldsymbol{k}' = (\mu',\nu',\omega')^T \in \mathbb{R}^3$, and $\varphi_0, \varphi_0' \in \mathbb{R}$.

Proof. It is sufficient to show that (26) holds for a monomial of Hirota derivatives:

$$\begin{aligned} &D_x^n D_y^m D_t^k e^{\mu x+\nu y+\omega t+\varphi_0}\cdot e^{\mu' x+\nu' y+\omega' t+\varphi_0'} \\ &= (\partial_x-\partial_{x'})^n(\partial_y-\partial_{y'})^m(\partial_t-\partial_{t'})^k e^{\mu x+\mu' x'+\nu y+\nu' y'+\omega t+\omega' t'}\Big|_{\boldsymbol{x}'=\boldsymbol{x}} e^{\varphi_0+\varphi_0'} \\ &= (\mu-\mu')^n(\nu-\nu')^m(\omega-\omega')^k e^{(\mu+\mu')x+(\nu+\nu')y+(\omega+\omega')t+\varphi_0+\varphi_0'} \end{aligned}$$

because $(\partial_x-\partial_{x'})e^{\mu x+\mu' x'} = (\mu-\mu')e^{\mu x+\mu' x'}$, for example. ∎

Propositon A.2 (Proposition A.1 in phase variables). *If $B = B(\boldsymbol{k})$ is a Hirota polynomial then*

$$B(D_{\boldsymbol{\varphi}}^T\boldsymbol{K})e^{\boldsymbol{\alpha}^T\boldsymbol{\varphi}+\varphi_0}\cdot e^{\boldsymbol{\alpha}'^T\boldsymbol{\varphi}+\varphi_0'} = B((\boldsymbol{\alpha}-\boldsymbol{\alpha}')^T\boldsymbol{K})e^{(\boldsymbol{\alpha}+\boldsymbol{\alpha}')^T\boldsymbol{\varphi}+\varphi_0+\varphi_0'}, \tag{27}$$

where $\boldsymbol{\alpha}, \boldsymbol{\alpha}' \in \mathbb{R}^g$, $\varphi_0, \varphi_0' \in \mathbb{R}$, $D_{\boldsymbol{\varphi}} = (D_{\varphi_1},\ldots,D_{\varphi_g})^T$ denotes Hirota derivatives with respect to phase variables $\boldsymbol{\varphi}$, and $D_{\boldsymbol{\varphi}}^T\boldsymbol{K} \equiv (\boldsymbol{K}^T D_{\boldsymbol{\varphi}})^T$.

Appendix B Properties of the operator L

The following propositions illustrate two special properties of the operator L that simplify the analysis of multi-soliton solutions considerably.

Propositon B.1 (Gauge invariance of L). *If* $\Theta = \Theta(\boldsymbol{\varphi})$ *is a positive function then*

$$L[e^{\boldsymbol{\alpha}^T\boldsymbol{\varphi}+\varphi_0}\Theta] = L[\Theta] \tag{28}$$

for all $\boldsymbol{\alpha} \in \mathbb{R}^g$ *and* $\varphi_0 \in \mathbb{R}$.

Proof. Equation (28) follows from the definition of L and equation

$$N(D_{\boldsymbol{\varphi}}^T\boldsymbol{K})e^{\boldsymbol{\alpha}^T\boldsymbol{\varphi}}\Theta \cdot e^{\boldsymbol{\alpha}^T\boldsymbol{\varphi}}\Theta = e^{2\boldsymbol{\alpha}^T\boldsymbol{\varphi}}N(D_{\boldsymbol{\varphi}}^T\boldsymbol{K})\Theta \cdot \Theta. \tag{29}$$

(We have taken $\varphi_0 = 0$ for simplicity; otherwise it is canceled already in the definition of L.) For proof, it is sufficient to consider (29) for a monomial of Hirota operators:

$$\begin{aligned}
&D_{\varphi_1}^{n_1}\cdots D_{\varphi_g}^{n_g}e^{\alpha_1\varphi_1+\ldots+\alpha_g\varphi_g}\Theta \cdot e^{\alpha_1\varphi_1+\ldots+\alpha_g\varphi_g}\Theta \\
&= (\partial_{\varphi_1} - \partial_{\varphi_1'})^{n_1}\cdots(\partial_{\varphi_g} - \partial_{\varphi_g'})^{n_g}e^{\alpha_1(\varphi_1+\varphi_1')+\ldots+\alpha_g(\varphi_g+\varphi_g')}\Theta\Theta'\Big|_{\boldsymbol{\varphi}'=\boldsymbol{\varphi}} \\
&= e^{\alpha_1(\varphi_1+\varphi_1')+\ldots+\alpha_g(\varphi_g+\varphi_g')}(\partial_{\varphi_1} - \partial_{\varphi_1'})^{n_1}\cdots(\partial_{\varphi_g} - \partial_{\varphi_g'})^{n_g}\Theta\Theta'\Big|_{\boldsymbol{\varphi}'=\boldsymbol{\varphi}} \\
&= e^{2\alpha_1\varphi_1+\ldots+2\alpha_g\varphi_g}D_{\varphi_1}^{n_1}\cdots D_{\varphi_g}^{n_g}\Theta \cdot \Theta,
\end{aligned}$$

where we have denoted $\Theta' = \Theta(\boldsymbol{\varphi}')$ and used the following equation

$$(\partial_\varphi - \partial_{\varphi'})^n e^{\alpha(\varphi+\varphi')}\Theta(\varphi)\Theta(\varphi') = e^{\alpha(\varphi+\varphi')}(\partial_\varphi - \partial_{\varphi'})^n\Theta(\varphi)\Theta(\varphi').$$

This equation follows from the equality $(\partial_\varphi - \partial_{\varphi'})e^{\alpha\varphi+\alpha\varphi'}\Theta\Theta' = e^{\alpha(\varphi+\varphi')}(\partial_\varphi - \partial_{\varphi'})\Theta\Theta'$ that can be easily checked to hold. $\square$

Propositon B.2 ($\lim L = L\lim$). *If* $\Theta = \Theta(\boldsymbol{\varphi})$ *is a positive function such that*

$$\exists \lim_{\varphi_i\to\pm\infty}\Theta \neq 0 \qquad \text{and} \qquad \lim_{\varphi_i\to\pm\infty}\partial_{\varphi_i}^n\Theta = 0$$

for all positive integers n, then

$$\lim_{\varphi_i\to\pm\infty}L[\Theta] = L[\lim_{\varphi_i\to\pm\infty}\Theta]. \tag{30}$$

Proof. Let us find

$$\lim_{\varphi_i\to+\infty}L[\Theta] = \lim_{\varphi_i\to+\infty}\frac{N(D_{\boldsymbol{\varphi}}^T\boldsymbol{K})\Theta\cdot\Theta}{\Theta^2} = \frac{\lim_{\varphi_i\to+\infty}N(D_{\boldsymbol{\varphi}}^T\boldsymbol{K})\Theta\cdot\Theta}{\Theta'^2},$$

where we have denoted $\Theta' = \lim_{\varphi_i\to+\infty}\Theta$. Note that

$$\lim_{\varphi_i\to+\infty}D_{\varphi_1}^{n_1}\cdots D_{\varphi_i}^{n_i}\cdots D_{\varphi_g}^{n_g}\Theta\cdot\Theta = D_{\varphi_1}^{n_1}\cdots D_{\varphi_{i-1}}^{n_{i-1}}D_{\varphi_{i+1}}^{n_{i+1}}\cdots D_{\varphi_g}^{n_g}\Theta'\cdot\Theta'$$

if $n_i = 0$, and equals to 0 if $n_i > 0$. That is, we have $\lim_{\varphi_i\to+\infty}N(D_{\boldsymbol{\varphi}}^T\boldsymbol{K})\Theta\cdot\Theta = N(D_{\boldsymbol{\varphi}}^T\boldsymbol{K})\Theta'\cdot\Theta'$, and consequently, $\lim_{\varphi_i\to+\infty}L[\Theta] = L[\Theta']$, which proves (30) for $\varphi_i \to +\infty$.

Proof for the case with $\varphi_i \to -\infty$ is identical. $\square$

Appendix C Proof of Theorem 3.1

We shall present the proof in two steps. First, we show that (7) follows from the Definition 4 of multi-soliton solutions. In addition, Eq. (12) is found. Second, we show that the conditions (8)–(11) follow if Θ in (6) solves the corresponding bilinear form of the equation $K[u] = 0$ (in phase variables): $P(D_{\boldsymbol{\varphi}}^T \boldsymbol{K})\Theta \cdot \Theta = 0$.

Step 1. Coefficients of the theta-function

1. For a function $U = L[\Theta]$ to be bounded, we require that Θ is a positive function. This is satisfied if all coefficients $A_{\kappa(\boldsymbol{\alpha})}$ are non-negative and at least one of them is non-zero.

2. Since $\kappa(\boldsymbol{\alpha})$ equals to $\kappa(\boldsymbol{\alpha}^{(i)}) \cup \{i\}$ or $\kappa(\boldsymbol{\alpha}^{(i)})$ if $\alpha_i = 1$ or $\alpha_i = 0$, respectively, we can write (6) in the following form

$$\Theta = \sum_{\boldsymbol{\alpha}^{(i)} \in \{0,1\}^{g-1}} A_{\kappa(\boldsymbol{\alpha}^{(i)})} e^{\boldsymbol{\alpha}^{(i)T} \boldsymbol{\varphi}^{(i)}} + e^{\varphi_i} \sum_{\boldsymbol{\alpha}^{(i)} \in \{0,1\}^{g-1}} A_{\kappa(\boldsymbol{\alpha}^{(i)}) \cup \{i\}} e^{\boldsymbol{\alpha}^{(i)T} \boldsymbol{\varphi}^{(i)}}, \tag{31}$$

   where both sums are independent with respect to the phase variable φ_i.

3. Functions Θ and $e^{-\varphi_i}\Theta$ satisfy the assumptions of Proposition B.2 for the limit processes $\varphi_i \to -\infty$ and $\varphi_i \to +\infty$, respectively. Indeed, using the representation (31), we have

$$\lim_{\varphi_i \to -\infty} \Theta = \sum_{\boldsymbol{\alpha}^{(i)} \in \{0,1\}^{g-1}} A_{\kappa(\boldsymbol{\alpha}^{(i)})} e^{\boldsymbol{\alpha}^{(i)T} \boldsymbol{\varphi}^{(i)}},$$

$$\lim_{\varphi_i \to +\infty} e^{-\varphi_i}\Theta = \sum_{\boldsymbol{\alpha}^{(i)} \in \{0,1\}^{g-1}} A_{\kappa(\boldsymbol{\alpha}^{(i)}) \cup \{i\}} e^{\boldsymbol{\alpha}^{(i)T} \boldsymbol{\varphi}^{(i)}},$$

   and

$$\lim_{\varphi_i \to -\infty} \partial_{\varphi_i}^n \Theta = 0, \qquad \lim_{\varphi_i \to +\infty} \partial_{\varphi_i}^n e^{-\varphi_i}\Theta = 0$$

   for an arbitrary positive integer n.

4. According to Definition 4 of genus g soliton solutions, we have

$$(\lim_{\varphi_i \to +\infty} L[\Theta])(\boldsymbol{\varphi}^{(i)}) = (\lim_{\varphi_i \to -\infty} L[\Theta])(\boldsymbol{\varphi}^{(i)} - \boldsymbol{\Delta}_i{}^{(i)}).$$

   Using Propositions B.1 and B.2, we find

$$(\lim_{\varphi_i \to +\infty} L[\Theta])(\boldsymbol{\varphi}^{(i)}) = (\lim_{\varphi_i \to +\infty} L[e^{-\varphi_i}\Theta])(\boldsymbol{\varphi}^{(i)})$$
$$= (L[\lim_{\varphi_i \to +\infty} e^{-\varphi_i}\Theta])(\boldsymbol{\varphi}^{(i)}) = L[\sum_{\boldsymbol{\alpha}^{(i)} \in \{0,1\}^{g-1}} A_{\kappa(\boldsymbol{\alpha}^{(i)}) \cup \{i\}} e^{\boldsymbol{\alpha}^{(i)T} \boldsymbol{\varphi}^{(i)}}],$$

$$(\lim_{\varphi_i\to-\infty} L[\Theta])(\boldsymbol{\varphi}^{(i)} - \boldsymbol{\Delta}_i{}^{(i)}) = (L[\lim_{\varphi_i\to-\infty}\Theta])(\boldsymbol{\varphi}^{(i)} - \boldsymbol{\Delta}_i{}^{(i)})$$
$$= L[\sum_{\boldsymbol{\alpha}^{(i)}\in\{0,1\}^{g-1}} A_{\kappa(\boldsymbol{\alpha}^{(i)})} e^{\boldsymbol{\alpha}^{(i)T}(\boldsymbol{\varphi}^{(i)}-\boldsymbol{\Delta}_i{}^{(i)})}].$$

As a result, we have

$$A_{\kappa(\boldsymbol{\alpha}^{(i)})\cup\{i\}} = A_{\kappa(\boldsymbol{\alpha}^{(i)})} e^{-\boldsymbol{\alpha}^{(i)T}\boldsymbol{\Delta}_i{}^{(i)}} \tag{32}$$

for $\boldsymbol{\alpha}^{(i)} \in \{0,1\}^{g-1}$ and $i = 1, \ldots, g$. Below we study this relation in detail.

5. For $\boldsymbol{\alpha}^{(i)} = (0, \ldots, 0)$ Eq. (32) is $A_i = A_\emptyset$. Without the loss of generality, we take $A_\emptyset = 1$. This gives (7) for $\boldsymbol{\alpha}^T\boldsymbol{\alpha} < 2$.

6. For $\alpha_j^{(i)} = 1$ and ${\boldsymbol{\alpha}^{(i)}}^T\boldsymbol{\alpha}^{(i)} = 1$ Eq. (32) reads $A_{ji} = e^{-\Delta_{ij}}$. This gives a formula for phase shift parameters: Eq. (12).

7. For $\alpha_j^{(i)} = \alpha_k^{(i)} = 1$ and ${\boldsymbol{\alpha}^{(i)}}^T\boldsymbol{\alpha}^{(i)} = 2$ Eq. (32) simplifies as follows
$$A_{jki} = A_{jk}e^{-\Delta_{ij}-\Delta_{ik}} = A_{jk}A_{ji}A_{ki} = \prod_{\substack{i',j'\in\{j,k,i\}\\ i'<j'}} A_{i'j'}.$$
Continuing in the same way, we find that all multi-indexed coefficients are the products of double indexed coefficients as given in (7).

Step 2. Hirota conditions

Let $P(D_{\boldsymbol{\varphi}}^T\boldsymbol{K})$ be the corresponding Hirota bilinear operator of the KdV type equation $K[u] = 0$ and let us apply this operator to the pair of Θ functions defined in (6):

$$P(D_{\boldsymbol{\varphi}}^T\boldsymbol{K})\Theta\cdot\Theta = \sum_{\boldsymbol{\alpha},\boldsymbol{\alpha}'\in\{0,1\}^g} A_{\kappa(\boldsymbol{\alpha})}A_{\kappa(\boldsymbol{\alpha}')}P(D_{\boldsymbol{\varphi}}^T\boldsymbol{K})e^{\boldsymbol{\alpha}^T\boldsymbol{\varphi}}\cdot e^{\boldsymbol{\alpha}'^T\boldsymbol{\varphi}}$$
$$= \sum_{\boldsymbol{\alpha},\boldsymbol{\alpha}'\in\{0,1\}^g} A_{\kappa(\boldsymbol{\alpha})}A_{\kappa(\boldsymbol{\alpha}')}P((\boldsymbol{\alpha}-\boldsymbol{\alpha}')^T\boldsymbol{K})e^{(\boldsymbol{\alpha}+\boldsymbol{\alpha}')^T\boldsymbol{\varphi}}$$
$$= \sum_{\boldsymbol{\beta}\in\{0,1,2\}^g}\left(\sum_{\boldsymbol{\alpha}\in(\boldsymbol{\beta}-\{0,1\}^g)\cap\{0,1\}^g} A_{\kappa(\boldsymbol{\alpha})}A_{\kappa(\boldsymbol{\beta}-\boldsymbol{\alpha})}P((2\boldsymbol{\alpha}-\boldsymbol{\beta})^T\boldsymbol{K})\right)e^{\boldsymbol{\beta}^T\boldsymbol{\varphi}},$$

where Proposition A.2 and the change of indices $\boldsymbol{\beta} = \boldsymbol{\alpha} + \boldsymbol{\alpha}'$ were used. So, the function Θ is a solution to the Hirota bilinear form $P(D_{\boldsymbol{\varphi}}^T\boldsymbol{K})\Theta\cdot\Theta = 0$ if equations

$$\sum_{\boldsymbol{\alpha}\in(\boldsymbol{\beta}-\{0,1\}^g)\cap\{0,1\}^g} A_{\kappa(\boldsymbol{\alpha})}A_{\kappa(\boldsymbol{\beta}-\boldsymbol{\alpha})}P((2\boldsymbol{\alpha}-\boldsymbol{\beta})^T\boldsymbol{K}) = 0$$

hold for all $\boldsymbol{\beta} \in \{0,1,2\}^g$. However, a detailed analysis shows that there are 2^g independent equations for $\boldsymbol{\beta} \in \{0,1\}^g$. These are the Hirota conditions in (11). Equations (8) and (9) are the special cases of the Hirota conditions if $\boldsymbol{\beta}^T\boldsymbol{\beta}$ equals to 1 and 2, respectively. Conditions (10) guarantee that genus g soliton solutions $U = L[\Theta]$ are bounded functions if double-indexed coefficients A_{ij} are evaluated according to Eq. (9).

Appendix D Proof of Proposition 4.2

Let us find

$$\Theta = \frac{1}{2} \sum_{\boldsymbol{\alpha}\in\{0,1\}^g} A_{\kappa(\boldsymbol{\alpha})} e^{\boldsymbol{\alpha}^T\boldsymbol{\varphi}} + A_{\kappa(\tilde{\boldsymbol{\alpha}})} e^{\tilde{\boldsymbol{\alpha}}^T\boldsymbol{\varphi}}$$

$$= \frac{1}{2} e^{\boldsymbol{i}^T\boldsymbol{\varphi}/2} \sum_{\boldsymbol{\alpha}\in\{0,1\}^g} A_{\kappa(\boldsymbol{\alpha})} e^{(\boldsymbol{\alpha}-\boldsymbol{i}/2)^T\boldsymbol{\varphi}} + A_{\kappa(\tilde{\boldsymbol{\alpha}})} e^{-(\boldsymbol{\alpha}-\boldsymbol{i}/2)^T\boldsymbol{\varphi}}$$

$$= e^{\boldsymbol{i}^T\boldsymbol{\varphi}/2} \sum_{\boldsymbol{\alpha}\in\{0,1\}^g} \sqrt{A_{\kappa(\boldsymbol{\alpha})} A_{\kappa(\tilde{\boldsymbol{\alpha}})}} \cosh \frac{(\boldsymbol{\alpha}-\tilde{\boldsymbol{\alpha}})^T\boldsymbol{\varphi} + \ln A_{\kappa(\boldsymbol{\alpha})}/A_{\kappa(\tilde{\boldsymbol{\alpha}})}}{2}$$

$$= \sqrt{2} e^{\boldsymbol{i}^T\boldsymbol{\varphi}/2} \Theta',$$

where Θ' is given by (15) and according to Proposition 4.1 is equivalent to the theta-function Θ.

Appendix E Proof of Proposition 4.3

First, Eq. (21) follows from the facts that $\lim_{\varphi_i \pm\infty} \Theta'^{-1} = 0$ and that the sum in (17) is independent with respect to φ_i because $\beta_i = 1 = |\boldsymbol{\alpha} - \boldsymbol{\alpha}'|_i \Rightarrow (\boldsymbol{\alpha} - \tilde{\boldsymbol{\alpha}}')_i = 0$. Second, we expand the superposition (16) as follows

$$U = \sum_{\boldsymbol{\beta}^{(i)}\in\{0,1\}^{g-1}} S_{\kappa(\boldsymbol{\beta}^{(i)})} + S_{\kappa(\boldsymbol{\beta}^{(i)})\cup\{i\}}.$$

Taking the limit $\varphi_i \to -\infty$, we obtain zero for the second term according to (21), and we have

$$U^{(i)}(\boldsymbol{\varphi}^{(i)}) = \sum_{\boldsymbol{\beta}^{(i)}\in\{0,1\}^{g-1}} (\lim_{\varphi_i\to-\infty} S_{\kappa(\boldsymbol{\beta}^{(i)})})(\boldsymbol{\varphi}^{(i)}).$$

On the other hand, a genus $(g-1)$ soliton solution $U^{(i)}$ reads

$$U^{(i)}(\boldsymbol{\varphi}^{(i)}) = \sum_{\boldsymbol{\beta}'\in\{0,1\}^{g-1}} S_{\kappa(\boldsymbol{\beta}')}(\boldsymbol{\varphi}^{(i)}).$$

Equating the right hand sides of the last two equations, we obtain (22). Similarly, for the limit $\varphi_i \to +\infty$, Eq. (23) is derived.

Appendix F Proof of Proposition 4.4

First, the amplitudes of soliton terms (with $\boldsymbol{\beta}^T\boldsymbol{\beta} = 1$) are defined at the maximum points of the corresponding single solitons. So, from (24) follows that the amplitude of a soliton S_i is $N(\boldsymbol{k}_i)/2$.

Second, we note that the maximum point of the highest order interaction soliton term (with $\boldsymbol{\beta}^T\boldsymbol{\beta} = g$) $S_{\kappa(\boldsymbol{i})}$ is at the minimum point of the theta-function Θ' (15) because the sum in (17) does not depend on phase variables if $\boldsymbol{\beta}^T\boldsymbol{\beta} = g$. (Indeed, from $|\boldsymbol{\alpha} - \boldsymbol{\alpha}'|_{ew} = \boldsymbol{i}$ follows that $\boldsymbol{\alpha} - \tilde{\boldsymbol{\alpha}}' = \mathbf{o}$.) The minimum point of the theta-function Θ' is found to be at $(\frac{1}{2}\sum_{i=1,i\neq j}^{g} \Delta_{ij})_{j=1}^{g}$. This point turns out to be also the center of the interaction region (e.g. for the two-soliton solution the center of the interaction region is the maximum point of an interaction soliton [1]).

Finally, in order to find the amplitudes of the remaining interaction soliton terms, we use the same procedure that was used for soliton terms: (i) find a non-vanishing limit of $S_{\kappa(\boldsymbol{\beta})}$ over $(g - \boldsymbol{\beta}^T\boldsymbol{\beta})$ phase variables approaching to infinity and (ii) then find the maximum point of the result. Such a non-vanishing limit is possible only if taken over phase variables φ_j with $j \in \{1, \ldots, g\} \setminus \kappa(\boldsymbol{\beta}) = \kappa(\tilde{\boldsymbol{\beta}})$. So, we find

$$\lim_{\substack{\varphi_j \to -\infty \\ j \in \kappa(\tilde{\boldsymbol{\beta}})}} S_{\kappa(\boldsymbol{\beta})}(\boldsymbol{\varphi}) = S_{\kappa(\boldsymbol{i}')}^{(\kappa(\tilde{\boldsymbol{\beta}}))}(\boldsymbol{\varphi}^{(\kappa(\tilde{\boldsymbol{\beta}}))}),$$

where $\kappa(\boldsymbol{\beta}^{(\kappa(\tilde{\boldsymbol{\beta}}))}) = \kappa(\boldsymbol{i}')$ with $\boldsymbol{i}' \in \{1\}^{\boldsymbol{\beta}^T\boldsymbol{\beta}}$ is used. Note that $S_{\kappa(\boldsymbol{i}')}^{(\kappa(\tilde{\boldsymbol{\beta}}))}$ is the highest order interaction soliton term of the genus $\boldsymbol{\beta}^T\boldsymbol{\beta} = \boldsymbol{i}'^T\boldsymbol{i}'$ soliton solution. Therefore we can use the result above obtained for the highest order interaction solitons.

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