Construction of multi-soliton interaction patterns of KdV type equations

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Abstract. To predict the wave parameters from the interaction patterns of waves (the inverse problem), the corresponding direct problem needs to be solved. In this paper interaction patterns of multi-soliton solutions of KdV type equations are constructed. The connection between the multi-soliton interaction patterns and the decomposition of a multi-soliton solution into a linear superposition of solitons and interaction solitons is demonstrated. All new concepts are illustrated for three-soliton solutions of the Kadomtshev-Petviashvili equation.

Key words: Wave interaction patterns; Multi-soliton interaction; Direct problem of wave crests.
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1 Introduction

Consider water-air surface waves which form complicated wave patterns during the propagation. In mathematical terms, a two-dimensional surface and its perturbations that can propagate in different directions are considered. Our ultimate goal is to predict the wave parameters (amplitudes, traveling directions, etc.) from the geometry of the interaction patterns formed by surface waves — the *inverse* problem of wave crests.

In the following we associate wave crests with solitons. Solitons are found to be rather robust phenomena (in the theory) of water waves. More importantly, using solitons as a model, we can actually tackle the inverse problem that would be otherwise too complicated, if not impossible, to solve when using even the simplest (no winds, incompressible fluid, etc.) free surface equations of water waves.

This inverse problem of wave crests was first solved for two wave interactions in [1, 2]. To tackle the inverse problem, the *direct* problem (i.e. constructing wave interaction patterns from predefined wave parameters) needs to be solved first. The interaction patterns are always stationary in the case of two waves. That, however, is not true for more than two waves. Then the interaction patterns are nonstationary, in general, and therefore much more complicated to handle.

In this paper we will solve the direct problem of wave crests for an arbitrary number of waves. As in [1], we assume that the dynamics of waves is governed by some member in the class of KdV type equations (in Hirota sense) [3, 4]. The class of KdV type equations contains large number of nonlinear partial differential equations that support soliton solutions. Multi-soliton solutions of KdV type equations were constructed and analyzed in [5] using Hirota bilinear formalism [6].

There have been other attempts to construct interaction pictures for more than two solitons. See, for example, [7, 8] where authors describe the three-soliton interactions in terms of the motion of two-soliton resonant interactions (resonance triads). However, the authors recognize among other limitations that this description is very much an idealization because at each interaction point the condition for a resonant interaction is not met.

The organization of this paper is as follows. KdV type equations and their multi-soliton solutions are introduced in the following Section 2. In Section 3 a phase pattern set of a multi-soliton solution is defined and a method for its construction is introduced. Actual interaction patterns (real pattern sets) of solitons are constructed in Section 4. All new concepts are illustrated for three-soliton solutions of the Kadomtshev-Petviashvili equation. Finally, conclusions are drawn in Section 5.

2 Multi-soliton solutions of KdV type equations

In this Section a brief overview of multi-soliton solutions of KdV type equations is given. Detailed treatment can be found in [5].

Consider a (2+1)-dimensional KdV type equation for $u(\boldsymbol{x})$, $\boldsymbol{x} = (x, y, t)^{\mathrm{T}}$:

$$K[u] \equiv K(u, u_x, u_y, u_t, \ldots) = 0 \tag{1}$$

that has related Hirota polynomials $P = P(\mu, \nu, \omega)$ and $N = N(\mu, \nu, \omega)$ such that

$$P(D_{\boldsymbol{x}})\theta \cdot \theta = 0 \quad \Rightarrow \quad K[\theta^{-2} N(D_{\boldsymbol{x}})\theta \cdot \theta] = 0$$

holds for all nontrivial positive functions $\theta = \theta(\boldsymbol{x})$, where $D_{\boldsymbol{x}} = (D_x, D_y, D_t)^{\mathrm{T}}$ is a vector of Hirota derivatives (Defs. 1–3 in [5]).

Notation (Index map κ). For $\boldsymbol{\alpha} \in \{0,1\}^g$ a set $\kappa(\boldsymbol{\alpha}) \in \wp(\{1, \ldots, g\})$ is defined such that (i) if $\boldsymbol{\alpha}_i = 1$ then $i \in \kappa(\boldsymbol{\alpha})$ and (ii) $|\kappa(\boldsymbol{\alpha})| = \boldsymbol{\alpha}^{\mathrm{T}}\boldsymbol{\alpha}$. For example, $\kappa((0,1,0,1,1)^{\mathrm{T}}) = \{2,4,5\}$. We also write $A_{\{2,4,5\}}$ as A_{245}, for example.

Genus g soliton solution of the KdV type equation (1) reads $u(\boldsymbol{x}) = U(\mathbf{K}\boldsymbol{x})$ (Theorem III.1 in [5]), where $\mathbf{K} = (\boldsymbol{\mu}, \boldsymbol{\nu}, \boldsymbol{\omega})$, $\boldsymbol{\mu}, \boldsymbol{\nu}, \boldsymbol{\omega} \in \mathbb{R}^g$, and

$$U(\boldsymbol{\varphi}) = \Theta^{-2} N(D_{\boldsymbol{\varphi}}^{\mathrm{T}}\mathbf{K})\Theta \cdot \Theta, \tag{2}$$

$$\Theta(\boldsymbol{\varphi}) = \sum_{\boldsymbol{\alpha} \in \{0,1\}^g} A_{\kappa(\boldsymbol{\alpha})} e^{\boldsymbol{\alpha}^{\mathrm{T}}\boldsymbol{\varphi}}, \tag{3}$$

$$A_{\kappa(\boldsymbol{\alpha})} = \begin{cases} 1 & \text{if } \boldsymbol{\alpha}^{\mathrm{T}}\boldsymbol{\alpha} < 2, \\ -P(\boldsymbol{k}_i - \boldsymbol{k}_j)/P(\boldsymbol{k}_i + \boldsymbol{k}_j) & \text{if } \kappa(\boldsymbol{\alpha}) = \{i, j\}, \\ \prod\limits_{\substack{i,j \in \kappa(\boldsymbol{\alpha}) \\ i<j}} A_{ij} & \text{if } \boldsymbol{\alpha}^{\mathrm{T}}\boldsymbol{\alpha} > 2, \end{cases} \tag{4}$$

where $\boldsymbol{k}_i^{\mathrm{T}}$ is the i-th row of a matrix $\mathbf{K}$; $D_{\boldsymbol{\varphi}}^{\mathrm{T}}\mathbf{K} \equiv (\mathbf{K}^{\mathrm{T}} D_{\boldsymbol{\varphi}})^{\mathrm{T}}$ with $D_{\boldsymbol{\varphi}} = (D_{\varphi_1}, \ldots, D_{\varphi_g})^{\mathrm{T}}$ denoting Hirota derivatives with respect to phase variables $\boldsymbol{\varphi} = (\varphi_1, \ldots, \varphi_g)^{\mathrm{T}}$. In addition the following conditions must hold:

$$\begin{aligned} &\text{dispersion relations:} && P(\boldsymbol{k}_i) = 0, \\ &\text{boundness conditions:} && P(\boldsymbol{k}_i + \boldsymbol{k}_j) P(\boldsymbol{k}_i - \boldsymbol{k}_j) \leqslant 0, \\ &\text{Hirota conditions:} && \sum_{\boldsymbol{\alpha} \in (\boldsymbol{\beta} - \{0,1\}^g) \cap \{0,1\}^g} A_{\kappa(\boldsymbol{\alpha})} A_{\kappa(\boldsymbol{\beta}-\boldsymbol{\alpha})} P((2\boldsymbol{\alpha} - \boldsymbol{\beta})^{\mathrm{T}}\mathbf{K}) = 0, \end{aligned}$$

where $\boldsymbol{\beta} \in \{0,1\}^g$ and $\boldsymbol{\beta}^{\mathrm{T}}\boldsymbol{\beta} > 2$.

Notation (Excluding variables). For a n-sequence of symbols we use a superscript rounded with parenthesis to denote a sequence of symbols that is a subset of the n-sequence containing symbols with indices that are *not* listed in the superscript and preserving the order. For example, $(a, b, c, d)^{(1,3)} = (b, d)$, $(\varphi_1, \varphi_2, \varphi_3)^{(2)} = (\varphi_1, \varphi_3)$.

A multi-soliton solution $U = U(\boldsymbol{\varphi})$ (in phase variables) is defined to have the following property (Definition 5 in [5]): $\lim_{\varphi_i \to \pm\infty} U$ is a genus $(g-1)$ soliton solution such that that

$$(\lim_{\varphi_i \to +\infty} U)(\boldsymbol{\varphi}^{(i)}) = (\lim_{\varphi_i \to -\infty} U)(\boldsymbol{\varphi}^{(i)} - \boldsymbol{\Delta}_i{}^{(i)}), \tag{5}$$

where $\boldsymbol{\varphi}^{(i)} = (\varphi_1, \ldots, \varphi_{i-1}, \varphi_{i+1}, \ldots, \varphi_g)^{\mathrm{T}}$, $\boldsymbol{\Delta}$ is a symmetric $g \times g$-matrix of phase variables $\Delta_{ij} = -\ln A_{ij}$ if $i \neq j$, and $\Delta_{ii} = 0$. The i-th column of $\boldsymbol{\Delta}$ is denoted by $\boldsymbol{\Delta}_i$.

A multi-soliton solution $U = U(\boldsymbol{\varphi})$ can be decomposed into a superposition of soliton and interaction soliton terms (Theorem IV.3 in [5]):

$$U = \sum_{\boldsymbol{\beta} \in \{0,1\}^g} S_{\kappa(\boldsymbol{\beta})}, \tag{6}$$

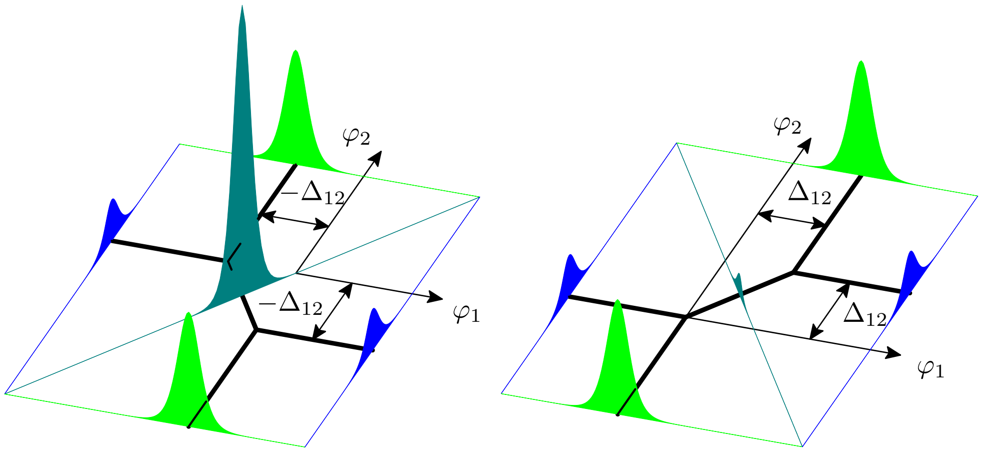

Figure 1: Phase pattern sets (bold lines) of two-soliton solutions with negative and positive phase shifts, respectively. These illustrations correspond to two-soliton solutions of the KP equation.

where

$$S_{\kappa(\boldsymbol{\beta})} = \frac{1}{2\Theta'^2} \sum_{\substack{\boldsymbol{\alpha},\boldsymbol{\alpha}' \in \{0,1\}^g \\ |\boldsymbol{\alpha}-\boldsymbol{\alpha}'|_{\mathrm{ew}}=\boldsymbol{\beta}}} \sqrt{A_{\kappa(\boldsymbol{\alpha})}A_{\kappa(\tilde{\boldsymbol{\alpha}})}A_{\kappa(\boldsymbol{\alpha}')}A_{\kappa(\tilde{\boldsymbol{\alpha}}')}}\, N((\boldsymbol{\alpha}-\boldsymbol{\alpha}')^{\mathrm{T}}\mathbf{K})$$
$$\times \cosh\left((\boldsymbol{\alpha}-\tilde{\boldsymbol{\alpha}}')^{\mathrm{T}}\boldsymbol{\varphi} + \frac{1}{2}\ln\frac{A_{\kappa(\boldsymbol{\alpha})}A_{\kappa(\boldsymbol{\alpha}')}}{A_{\kappa(\tilde{\boldsymbol{\alpha}})}A_{\kappa(\tilde{\boldsymbol{\alpha}}')}}\right), \quad (7)$$

$$\Theta' = \frac{1}{\sqrt{2}} \sum_{\boldsymbol{\alpha}\in\{0,1\}^{\mathrm{T}}} \sqrt{A_{\kappa(\boldsymbol{\alpha})}A_{\kappa(\tilde{\boldsymbol{\alpha}})}}\cosh\frac{(\boldsymbol{\alpha}-\tilde{\boldsymbol{\alpha}})^{\mathrm{T}}\boldsymbol{\varphi} + \ln A_{\kappa(\boldsymbol{\alpha})}/A_{\kappa(\tilde{\boldsymbol{\alpha}})}}{2},$$

$|\cdot|_{\mathrm{ew}} \equiv (|\cdot_1|,\ldots,|\cdot_g|)^{\mathrm{T}}$ denotes the element-wise absolute value operation, and $\tilde{\boldsymbol{\alpha}} \equiv \boldsymbol{i} - \boldsymbol{\alpha}$ with $\boldsymbol{i} \in \{1\}^g$ such that $\boldsymbol{i}^{\mathrm{T}}\boldsymbol{i} = g$. For multi-soliton solutions in real variables, $u = u(\boldsymbol{x})$, we write

$$u = \sum_{\boldsymbol{\beta}\in\{0,1\}^g} s_{\kappa(\boldsymbol{\beta})},$$

where $s_{\kappa(\boldsymbol{\beta})}(\boldsymbol{x}) = S_{\kappa(\boldsymbol{\beta})}(\mathbf{K}\boldsymbol{x})$.

3 Phase pattern sets

In [1] we introduced a geometric representation of soliton interaction pictures. In this simplified description, the so-called *pattern set* is defined to describe the positions of solitons. For example, for a two-soliton solution in phase variables,

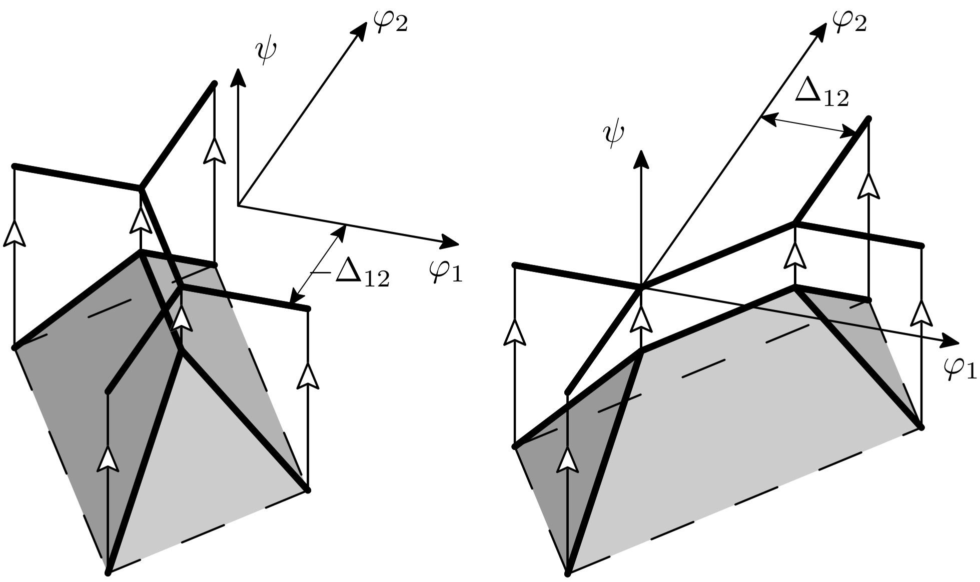

Figure 2: Construction of the phase pattern sets (bold lines on the (φ_1, φ_2)-plane) for two-soliton solutions with negative and positive phase shifts, respectively. White arrows indicate the projection of polyhedron edges to the space of phase variables.

$U = U(\varphi_1, \varphi_2)$, the phase pattern set P_U consists of two pairs of parallel line slices and a connecting line segment as shown in Fig. 1. The parallel line slices correspond to positions of soliton terms S_1 and S_2, respectively, and the connecting line segment to the position of interaction soliton S_{12} [1].

In this section we construct phase pattern sets for multi-soliton solutions. To get a better understanding of the method, we start with the two soliton case and then extend it for the general multi-soliton case.

3.1 Construction of a phase pattern set for two-soliton solution

For two-soliton solutions, there are two types of interactions (with positive and with negative phase shifts, respectively) that have different phase pattern sets as shown in Fig. 1. In order to describe all possible phase pattern sets in an unified manner (also for multi-soliton solutions), we introduce the following construction. Namely, we extend the space of phase variables by defining an auxiliary phase variable, ψ. In this extended space of phase variables $(\varphi_1, \varphi_2, \psi)$ we define a convex polyhedron such that the orthogonal projection of its edges to the original space of phase variables does coincide with the phase pattern set of the corresponding two-soliton solution. This construction is illustrated in Fig. 2.

3.2 Construction of a phase pattern set for multi-soliton solution

In the following we give a general definition for phase pattern sets and introduce a method for their construction. First, we need to introduce some auxiliary notations.

Notation.

- For $A \subset \mathbb{R}^g$, $\boldsymbol{c} \in \mathbb{R}^g$ we define $A + \boldsymbol{c} \equiv \{\boldsymbol{a} + \boldsymbol{c} | \boldsymbol{a} \in A\}$.
- For $A \subset \mathbb{R}^g$, $B \subset \mathbb{R}^{g-1}$, $\boldsymbol{\varphi} \in \mathbb{R}^g$, $i = 1, \dots, g$, we write

  $$A|_{\varphi_i = -\infty} = B$$

  iff $\exists c$ such that $\{\boldsymbol{\varphi}^{(i)} | \boldsymbol{\varphi} \in A \text{ and } \varphi_i = c'\} = B$ holds for all $c' < c$.
- Similarly, we write

  $$A|_{\varphi_i = +\infty} = B$$

  iff $\exists c$ such that $\{\boldsymbol{\varphi}^{(i)} | \boldsymbol{\varphi} \in A \text{ and } \varphi_i = c'\} = B$ holds for all $c' > c$.

Definition 1 (Phase pattern set). A set $P_U \subset \mathbb{R}^g$ is called the *phase pattern set* of a genus g soliton solution $U = U(\boldsymbol{\varphi})$ with the phase shift matrix $\boldsymbol{\Delta}$ iff

1. If $g = 1$, then $P_U = \{0\}$.
2. If $g > 1$, then the following conditions are satisfied for $i = 1, \dots, g$:

$$P_U|_{\varphi_i = -\infty} = P_{U^{(i)}}, \tag{8}$$

$$P_U|_{\varphi_i = +\infty} = P_{U^{(i)}} + \boldsymbol{\Delta}_i{}^{(i)}, \tag{9}$$

where $P_{U^{(i)}}$ is the phase pattern set of a genus $(g-1)$ soliton solution $U^{(i)} \equiv \lim_{\varphi_i \to -\infty} U$.

Theorem 3.1 (Construction of a phase pattern set). *Let $U = U(\boldsymbol{\varphi})$ be a genus g soliton solution with the phase shift matrix $\boldsymbol{\Delta}$. Let H be a convex polyhedron:*

$$H = \left\{ (\boldsymbol{\varphi}, \psi) | \boldsymbol{n}^{\mathrm{T}} \boldsymbol{\varphi} + \psi + c_{\boldsymbol{n}} \leqslant 0, \boldsymbol{n} \in \{-1, 1\}^g \right\}, \tag{10}$$

where

$$c_{\boldsymbol{n}} = c - \frac{1}{4} (\boldsymbol{i} + \boldsymbol{n})^{\mathrm{T}} \boldsymbol{\Delta} (\boldsymbol{i} + \boldsymbol{n}),$$

$c \in \mathbb{R}$ is arbitrary. Let R be a set of ridges of the polyhedron H:

$$R = \bigcup_{\substack{\boldsymbol{n}, \boldsymbol{m} \in \{-1,1\}^g \\ \boldsymbol{n} \neq \boldsymbol{m}}} R_{\boldsymbol{n}, \boldsymbol{m}}, \tag{11}$$

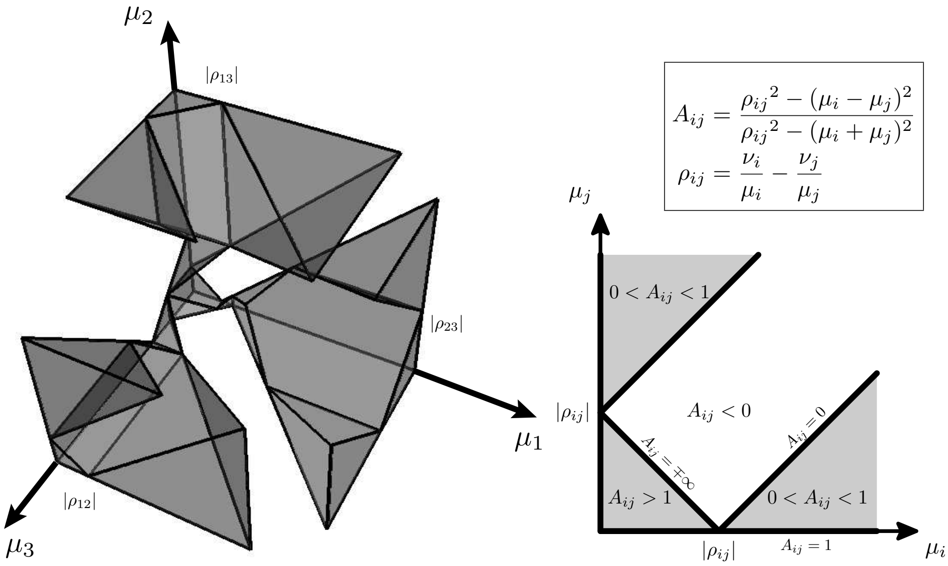

Figure 3: "Map" of genus 3 soliton types for the KP equation. The type is defined by the signs of phase shifts $\Delta_{ij} = -\log A_{ij}$. Shaded regions correspond to bounded solutions. The case corresponds to $\nu_1 = 0$, $\rho_{12} = 0.268$, $\rho_{13} = -0.364$.

where

$$R_{\boldsymbol{n},\boldsymbol{m}} = H \cap \left\{ (\boldsymbol{\varphi}, \psi) \,\middle|\, \begin{array}{c} \boldsymbol{n}^{\mathrm{T}}\boldsymbol{\varphi} + \psi + c_{\boldsymbol{n}} = 0 \\ \boldsymbol{m}^{\mathrm{T}}\boldsymbol{\varphi} + \psi + c_{\boldsymbol{m}} = 0 \end{array} \right\}. \tag{12}$$

Then an orthogonal projection $(\boldsymbol{\varphi}, \psi) \mapsto \boldsymbol{\varphi}$ *of* R *is a phase pattern set of* U*:*

$$P_U = R|_{\boldsymbol{\varphi}}. \tag{13}$$

Proof. See Appendix A.

3.3 Phase pattern set of a three-soliton solution

In the following we illustrate the connection between phase pattern sets and soliton solutions for the KP (Kadomtshev-Petviashvili) equation

$$(u_t + 6uu_x + u_{xxx})_x + 3u_{yy} = 0,$$

that, as a member of the class of KdV type equations, has related Hirota polynomials $P = \mu\omega + \mu^4 + 3\nu^2$ and $N = \mu^2$. In particular, we consider its three-soliton solutions.

The boundness conditions $A_{12,13,23} > 0$ define regions in the space of $\boldsymbol{\mu}$-parameters where KP three-soliton solutions are bounded as shown in Figure 3. In addition, for different combinations of signs of the phase shift parameters

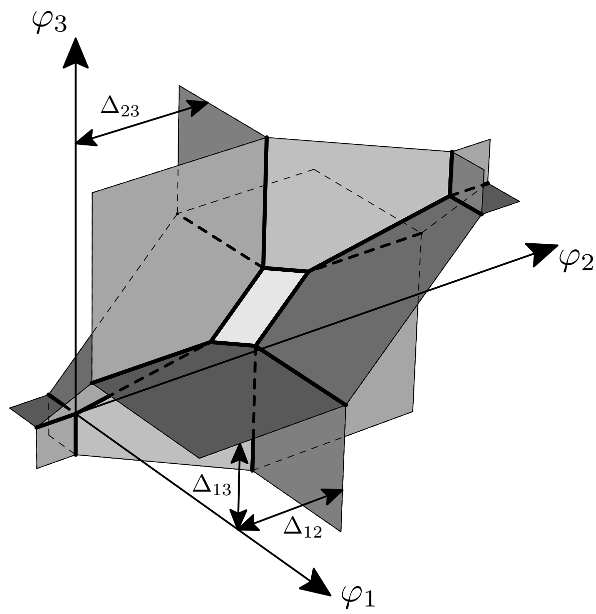

Figure 4: Phase pattern set (grayed regions) of a three-soliton solution constructed according to Theorem 3.1. Only the center part of the pattern set is shown, the grayed regions stretch to infinity. The case corresponds to $0 < \Delta_{13} < \Delta_{12} < \Delta_{23}$.

$\Delta_{ij} = -\log A_{ij}$ we distinguish four types of three-soliton solutions (cf. two types of two-soliton solutions): the phase shifts are 1) all positive, 2) two positive and one negative, 3) one positive and two negative, 4) all negative, respectively.

The signs of the phase shift parameters $\Delta_{12,13,23}$ (of the KP case) define also relative amplitudes of the interaction soliton terms $S_{12,13,23}$: for $\Delta_{ij} < 0$ ($\Delta_{ij} > 0$), the amplitude of S_{ij} is larger (smaller) than the amplitudes of soliton terms S_i and S_j (cf. Fig. 1)[1]. The corresponding rule for the interaction soliton term S_{123} is more complicated for solution types if at least one of the phase shift parameters is positive. However, for $\Delta_{12,13,23} < 0$, the amplitude of S_{123} is largest compared to all other interaction terms.

Figure 4 demonstrates the phase pattern set P_U of a KP three-soliton solution U with all phase shift parameters being positive. The set P_U (shaded plane slices) is obtained according to Theorem 3.1: in the 4-dimensional extended space of phase variables $(\varphi_1, \varphi_2, \varphi_3, \psi)$ a polyhedron (10) is constructed; then its ridges (11) are projected to the 3-dimensional space of phase variables $(\varphi_1, \varphi_2, \varphi_3)$. As a result the phase pattern set P_U is obtained. Phase pattern sets for other types of three-soliton solutions are constructed in the same way — more examples follow below.

Example 1. Figure 5 shows isosurface plot of a KP three-soliton solution U

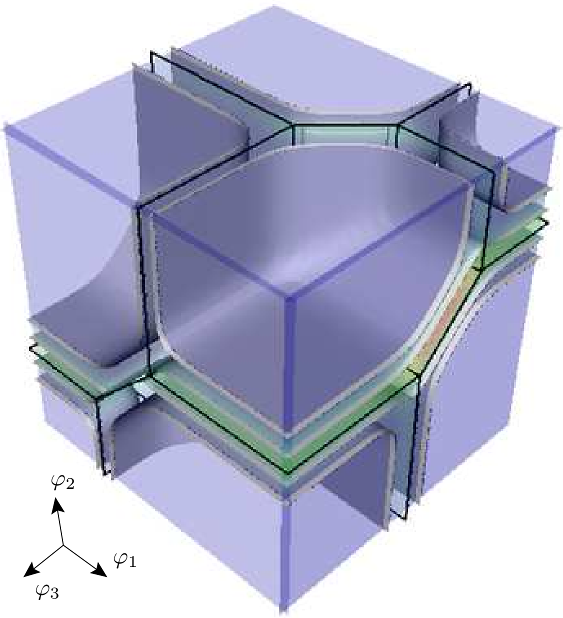

Figure 5: Isosurface plot of a KP three-soliton solution in phase variables. The case corresponds to the point $\boldsymbol{\mu} = (-0.3996\rho_{13}, \rho_{12} - 0.4\rho_{13}, -0.5994\rho_{13})$ in Figure 3.

with $\Delta_{12} > 0$ and $\Delta_{13,23} < 0$. The corresponding phase pattern set is demonstrated in the form of a wireframe in the same plot. Clearly, the phase pattern set represents the supporting regions of the function U, or equivalently, the positions of wave crests in phase variables.

Figure 6 shows isosurface plots of interaction terms $S_{2,3,23,123}$. These plots illustrate close connection between the parts of a phase pattern set P_U and the decomposition (6): each plane slice in P_U represents a supporting region of one of the interaction term in the decomposition.

Example 2. Figure 7 shows another example of a KP three-soliton solution with $\Delta_{12,13} > 0$ and $\Delta_{23} < 0$. In this case the amplitude of the interaction term S_{123} is almost zero. The "hole" in the isosurface plot illustrates just that.

4 Interaction patterns

In this section we complete the direct problem of wave crests by constructing interaction patterns for multi-soliton solutions in real space-time variables.

To demonstrate the dynamics of interaction patterns, in the following we write multi-soliton solutions of the KdV type equation (1) as $u(\boldsymbol{x}, t) = U(\mathbf{K}\boldsymbol{x} + \boldsymbol{\omega}t)$, where $\boldsymbol{x} = (x, y)^{\mathrm{T}}$ and $\mathbf{K} = (\boldsymbol{\mu}, \boldsymbol{\nu})$. With this notation we associate the

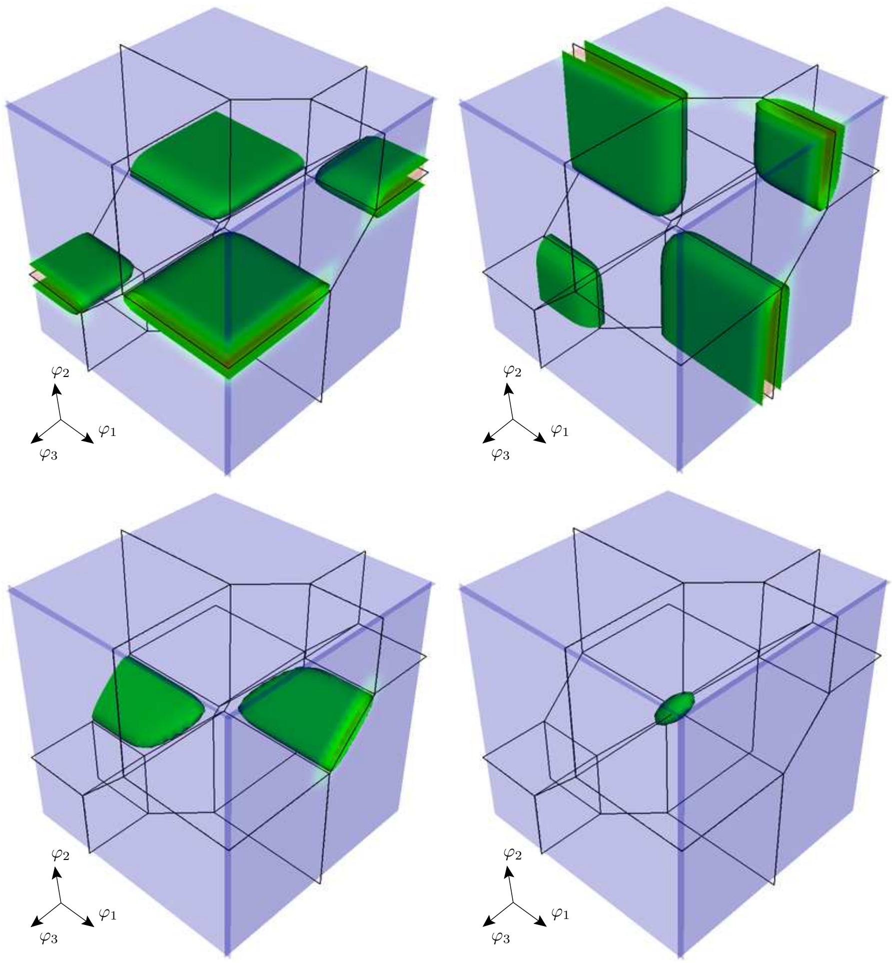

Figure 6: Isosurface plots of interaction terms S_2, S_3 (upper figures), S_{23}, S_{123} (lower figures) of the KP three-soliton solution in phase variables. The isosurfaces correspond to half of the amplitude of the corresponding interaction terms.

variable t as the time parameter.

Definition 2 (Real pattern set [1]**).** A set $P_u(t) \subset \mathbb{R}^2$ is called the *real pattern set* of a genus g soliton solution $u(\boldsymbol{x}, t) = U(\mathbf{K}\boldsymbol{x} + \boldsymbol{\omega}t)$ iff

$$P_u(t) = \{\boldsymbol{x} \in \mathbb{R}^2 | \mathbf{K}\boldsymbol{x} + \boldsymbol{\omega}t \in P_U\}, \tag{14}$$

where P_U is the phase pattern set of the corresponding genus g soliton solution $U = U(\boldsymbol{\varphi})$ in phase variables.

Propositon 4.1. *If $g \geqslant 2$ and the columns of $\mathbf{K}$ are not collinear then the*

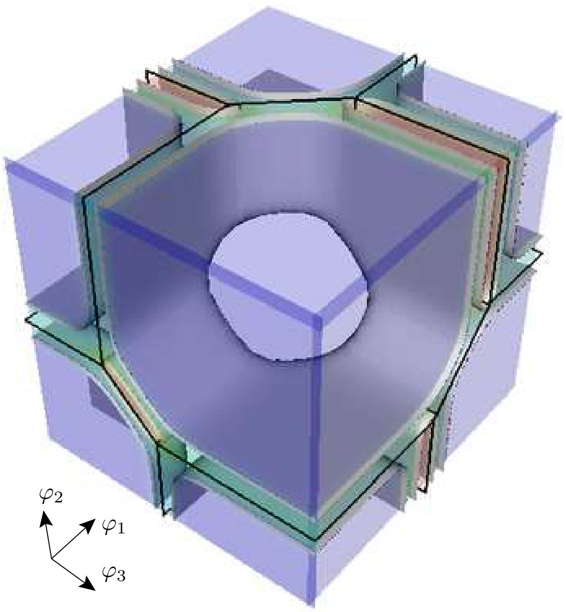

Figure 7: Isosurface plot of a KP three-soliton solution in phase variables. The case corresponds to the point $\boldsymbol{\mu} = (\rho_{12} - \rho_{13}, -0.999\rho_{13}, 0.999\rho_{12})$ in Figure 3.

real pattern set of a genus g soliton solution reads

$$P_u(t) = (\mathbf{K}^{\mathrm{T}}\mathbf{K})^{-1}\mathbf{K}^{\mathrm{T}}(P_U \cap (\mathbf{K}\mathbb{R}^2 + \boldsymbol{\omega} t)) - \boldsymbol{w} t, \tag{15}$$

where $\boldsymbol{w} = (\mathbf{K}^{\mathrm{T}}\mathbf{K})^{-1}\mathbf{K}^{\mathrm{T}}\boldsymbol{\omega}$.

Proof. Expression (14) is equivalent to

$$\mathbf{K}P_u(t) + \boldsymbol{\omega} t = P_U \cap (\mathbf{K}\mathbb{R}^2 + \boldsymbol{\omega} t). \tag{16}$$

Due to the non-collinearity of $\mathbf{K}$ columns, there exists $(\mathbf{K}^{\mathrm{T}}\mathbf{K})^{-1}$. After multiplying the both sides of (16) with $(\mathbf{K}^{\mathrm{T}}\mathbf{K})^{-1}\mathbf{K}^{\mathrm{T}}$, we obtain (15). $\square$

To interpret the real pattern set of a multi-soliton solution, we write it in the following form

$$P_u(t) = (\mathbf{K}^{\mathrm{T}}\mathbf{K})^{-1}\mathbf{K}^{\mathrm{T}}(P_U \cap (\mathbf{K}\mathbb{R}^2 + \boldsymbol{\omega}_\perp t)) - \boldsymbol{w} t, \tag{17}$$

where $\boldsymbol{\omega}_\perp = \boldsymbol{\omega} - \mathbf{K}\boldsymbol{w}$ is perpendicular to the 2-dimensional hyperplane $\mathbf{K}\mathbb{R}^2$ in the g-dimensional space of phase variables: $\mathbf{K}^{\mathrm{T}}\boldsymbol{\omega}_\perp = \mathbf{0}$. Since the phase pattern set P_U represents the supporting regions of the given multi-soliton solution in phase variables, the intersection $P_U \cap (\mathbf{K}\mathbb{R}^2 + \boldsymbol{\omega}_\perp t)$ defines the representation of supporting regions of the multi-soliton in real variables after mapping the intersection to the space of real variables with $(\mathbf{K}^{\mathrm{T}}\mathbf{K})^{-1}\mathbf{K}^{\mathrm{T}}$. If $\boldsymbol{\omega}_\perp \neq 0$, the image of the real space $\mathbf{K}\mathbb{R}^2$ is translated in the direction of $\boldsymbol{\omega}_\perp$ in the space of phase variables as the time parameter t is increased. During this translation, the intersection $P_U \cap (\mathbf{K}\mathbb{R}^2 + \boldsymbol{\omega}_\perp t)$ changes accordingly as the set P_U is fixed in the space of phase variables. So, the first term in (17) is responsible for the nonstationarity of the real pattern set $P_u(t)$. The second term in the expression of $P_u(t)$ defines just a translation of the interaction pattern in the space of real variables. The special case $\boldsymbol{\omega}_\perp = 0$ corresponds to a multi-soliton solution that is stationary regardless how large is the number of solitons.

4.1 Real pattern set of a three-soliton solution

Figures 8 and 9 illustrate the construction of interaction patterns of a KP three-soliton solution with all phase shift parameters being negative. Figure 9 shows two instances of the real pattern set $P_u(t)$ for time moments $t = -150$ and $t = 0$, respectively. They are found as the original regions of the intersection $P_U \cap (\mathbf{K}\mathbb{R}^2 + \boldsymbol{\omega} t)$ with respect to the mapping $\boldsymbol{x} \mapsto \mathbf{K}\boldsymbol{x} + \boldsymbol{\omega} t$. Figure 8 shows this for the time moment $t = 0$. As t is increased, the plane $\mathbf{K}\mathbb{R}^2$ is translated in the opposite direction of the vector $\boldsymbol{\omega}$ in the space of phase variables. In this particular case, the plane $\mathbf{K}\mathbb{R}^2$ in Fig. 8 moves away as observed by a reader. As a result, the real pattern set $P_u(t)$ is nonstationary.

Because each plane slice of the phase pattern set P_U represents one of the interaction soliton terms $S_{1,2,3,12,13,23,123}$ in the superposition (6), so does each line slice of the real pattern set $P_u(t)$ in the space of real variables. However, not all interaction terms may be visible in the real space for all time. Only those interaction solitons are visible that correspond to plane slices of P_U that have common points with the plane $\mathbf{K}\mathbb{R}^2 + \boldsymbol{\omega} t$ at the given time moment t. For this reason there is no line slice corresponding to the interaction soliton term s_{23} in Figure 9. In fact, as time parameter t is increased, the interaction soliton term s_{123} disappears eventually and an interaction pattern forms instead with three subpatterns of pairwise interactions between the three solitons. The same happens for decreasing time parameter. In conclusion, the interaction soliton s_{123} exists only finite amount of time. Interaction solitons $s_{12,13,23}$ may disappear only for finite time period but they eventually reappear to the interaction pattern. Solitons $s_{1,2,3}$ are always present in the interaction patterns.

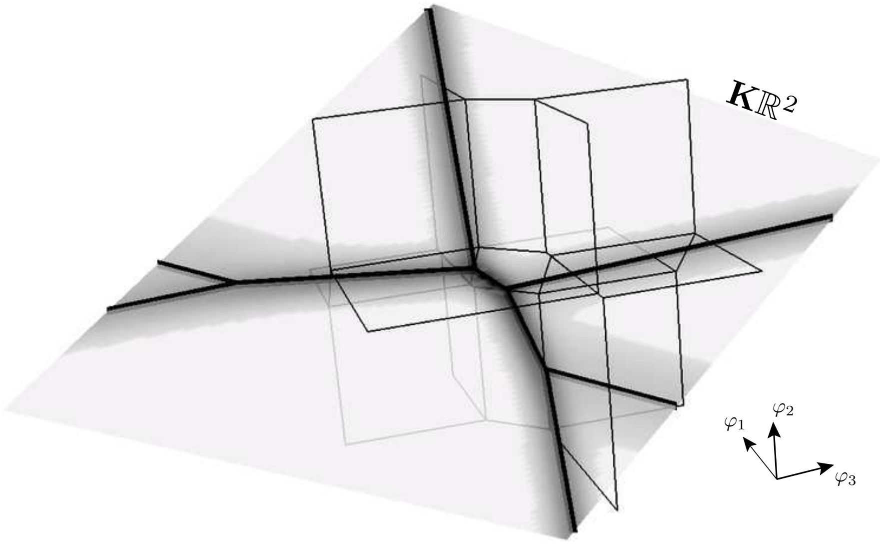

Figure 8: The image of the real pattern set $P_u(t = 0)$ (bold lines) is the intersection of the plane $\mathbf{K}\mathbb{R}^2$ and the phase pattern set P_U (wire-frame). Density plot is shown for $U|_{\mathbf{K}\mathbb{R}^2}$. The case corresponds to the point $\boldsymbol{\mu} = (0.3996\rho_{12}, 0.6\rho_{12}, -\rho_{13} - 0.4\rho_{12})$ in Figure 3.

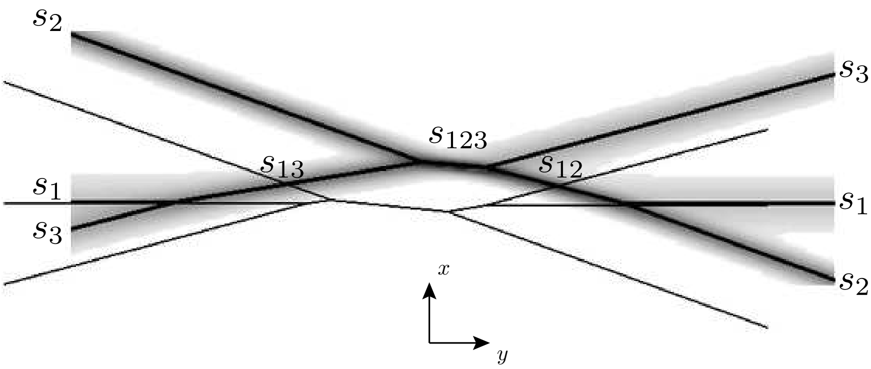

Figure 9: The real pattern set $P_u(t = 0)$ (bold lines) is obtained by mapping the intersection $P_U \cap \mathbf{K}\mathbb{R}^2$ in Figure 8 to the space of real variables. The density plot corresponds to function $u(x, y, t = 0)$. The real pattern set $P_u(t = -150)$ (lines) is shown to illustrate the nonstationarity of $P_u(t)$.

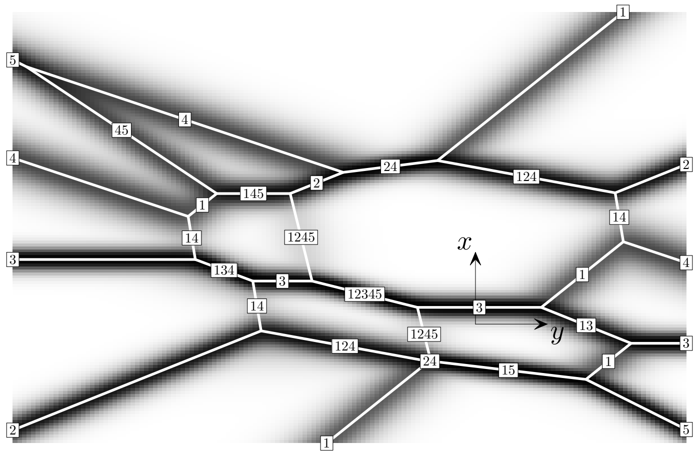

Figure 10: The real pattern set (white lines) of a five-soliton solution (abstract case, corresponds to Fig. 1 in [1]). Labels are the indices of the soliton terms of the corresponding decomposition.

4.2 Real pattern set of a five-soliton solution

Figure 10 illustrates a snapshot of an animated density plot of a five-soliton solution and the corresponding instance of its real pattern set. The latter is constructed according to the following procedure. First, a phase pattern set P_U is constructed according to Theorem 3.1 as a projection of the ridges of a special 6-dimensional polyhedron to the 5-dimensional space of phase variables. Then, for a fixed time parameter t, an intersection of the set P_U and a 2-dimensional hyperplane $\mathbf{K}\mathbb{R}^2 + \boldsymbol{\omega} t$ is found. Finally, this intersection is mapped with $(\mathbf{K}^T\mathbf{K})^{-1}\mathbf{K}^{\mathrm{T}}$ to the space of real variables resulting an instance of a real pattern set $P_u(t)$. As the time parameter t is increased, the hyperplane $\mathbf{K}\mathbb{R}^2 + \boldsymbol{\omega} t$ shifts in the space of phase variables causing changes to the intersection $P_U \cap (\mathbf{K}\mathbb{R}^2 + \boldsymbol{\omega} t)$. These changes are expressed in the nonstationarity of the interaction patterns of multi-soliton solutions.

5 Conclusions

In this paper we constructed interaction patterns for multi-soliton solutions of KdV type equations.

Although, the interaction patterns are, in general, nonstationary and look complicated, their formation can be deterministically constructed for any time moment. For that, we first considered multi-soliton solutions in phase variables in which these solutions have very regular structures. In particular, to represent supporting regions of soliton solutions in phase variables, we defined the so-called phase pattern set. Though, the phase pattern set is defined in a possibly high-dimensional space of phase variables, it can be algorithmically constructed using the tools from the computational geometry. Namely, we discovered that the phase pattern set of a multi-soliton solution is formed by projecting the ridges of a special polyhedron in the extended space of phase variables to the space of phase variables.

To represent actual interaction patterns, we defined the so-called real pattern set that is, for a fixed time moment t, isomorphic to an intersection of the phase pattern set and a certain two-dimensional hyperplane in the space of phase variables. We showed that the nonstationarity of interaction patterns is directly related to the movement of this hyperplane in the space of phase variables as the time parameter t is varied.

In addition, we demonstrated close connection between the pattern sets defined above and the decomposition of multi-soliton solutions into a superposition of interaction solitons. In fact, each linear part of the phase (real) pattern set represents the supporting region of one of the terms in the decomposition in phase (real) variables. We also explained possible emergence and absorption of interaction solitons in the space of real variables.

In conclusion, we solved the direct problem of wave crests for multi-soliton solutions. These results will be used for solving the corresponding inverse problem. Initial analysis shows that the inverse problem can be solved, in principle, also for multi-soliton solutions. This is due to the fact that the number of unknown parameters matches with the number measurable invariants (phase shift and angle relations) of a multi-soliton solution. The details will be reported in a successive paper.

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Appendix A Proof of Theorem 3.1

In the following we show that the phase pattern set, constructed according to Theorem 3.1, satisfies Definition 1. It is straightforward to check that the claim (13) holds for $g = 1$. In the following we consider the case $g > 1$ (cf. also Fig. 2).

(i) It is easy to check that

$$c_{\boldsymbol{n}} = c_{\boldsymbol{n}^{(i)}} - \frac{1}{2}(\boldsymbol{i}+\boldsymbol{n})^{\mathrm{T}}\boldsymbol{\Delta}_i(1+n_i)$$

holds for $\boldsymbol{n} \in \{-1,1\}^g$, where $c_{\boldsymbol{n}^{(i)}}$ are the coefficients of a polyhedron corresponding to a genus $(g-1)$ soliton solution $U^{(i)} = \lim_{\varphi_i\to-\infty} U$.

(ii) Polyhedron H is upper bounded with respect to variable ψ:

$$\max\{\psi|(\boldsymbol{\varphi},\psi) \in H\} < +\infty. \tag{18}$$

(iii) Equivalent representation of ridges $R_{\boldsymbol{n},\boldsymbol{m}}$ is

$$R_{\boldsymbol{n},\boldsymbol{m}} = H \cap \left\{ (\boldsymbol{\varphi},\psi) \;\middle|\; \begin{array}{c} (\boldsymbol{n}-\boldsymbol{m})^{\mathrm{T}}\boldsymbol{\varphi} + c_{\boldsymbol{n}} - c_{\boldsymbol{m}} = 0 \\ \boldsymbol{n}^{\mathrm{T}}\boldsymbol{\varphi} + \psi + c_{\boldsymbol{n}} = 0 \end{array} \right\}. \tag{19}$$

(iv) Let $i = 1,\dots,g$ be fixed. From (iii) follows that the set $R_{\boldsymbol{n},\boldsymbol{m}} \cap \{\varphi_i = C\}$ can be nonempty for $C \to \pm\infty$ only if $n_i = m_i$. From (ii) follows that $R_{\boldsymbol{n},\boldsymbol{m}} \cap \{\varphi_i = C\} = \emptyset$ whenever $n_i = m_i = -1$ and $C \to +\infty$, or $n_i = m_i = +1$ and $C \to -\infty$. Let us denote

$$R_i^- = \bigcup_{\substack{\boldsymbol{n}\neq\boldsymbol{m} \\ n_i=m_i=-1}} R_{\boldsymbol{n},\boldsymbol{m}}, \qquad R_i^+ = \bigcup_{\substack{\boldsymbol{n}\neq\boldsymbol{m} \\ n_i=m_i=+1}} R_{\boldsymbol{n},\boldsymbol{m}}. \tag{20}$$

Ridges in R_i^- and R_i^+ have nonempty intersections with the hyperplane $\varphi_i = C$ if $C \to -\infty$ and $C \to +\infty$, respectively.

(v) We have

$$R \cap \{\varphi_i = C\} = \begin{cases} R_i^- \cap \{\varphi_i = C\} & \text{for } C \to -\infty, \\ R_i^+ \cap \{\varphi_i = C\} & \text{for } C \to +\infty. \end{cases}$$

Consequently,

$$R|_{\boldsymbol{\varphi}}\Big|_{\varphi_i=-\infty} \equiv P_U|_{\varphi_i=-\infty} = R_i^-|_{\boldsymbol{\varphi}}\Big|_{\varphi_i=-\infty}, \qquad P_U|_{\varphi_i=+\infty} = R_i^+|_{\boldsymbol{\varphi}}\Big|_{\varphi_i=+\infty}.$$

(vi) If $n_i = m_i = -1$ then using (i), we have

$$R_{\boldsymbol{n},\boldsymbol{m}} = H \cap \left\{ (\boldsymbol{\varphi},\psi) \;\middle|\; \begin{array}{c} (\boldsymbol{n}^{(i)}-\boldsymbol{m}^{(i)})^{\mathrm{T}}\boldsymbol{\varphi}^{(i)} + c_{\boldsymbol{n}^{(i)}} - c_{\boldsymbol{m}^{(i)}} = 0 \\ {\boldsymbol{n}^{(i)}}^{\mathrm{T}}\boldsymbol{\varphi}^{(i)} + \psi - \varphi_i + c_{\boldsymbol{n}^{(i)}} = 0 \end{array} \right\},$$

that, for a fixed $\varphi_i = C$, has the form of the ridge $R_{\boldsymbol{n}^{(i)},\boldsymbol{m}^{(i)}}$ of a polyhedron corresponding to a genus $(g-1)$ soliton solution $U^{(i)} = \lim_{\varphi_i\to-\infty} U$. Therefore, using (v) we have

$$P_U|_{\varphi_i=-\infty} = P_{U^{(i)}}.$$

(vii) If $n_i = m_i = +1$ then using (i), we have

$$R_{\boldsymbol{n},\boldsymbol{m}} = H \cap \left\{ (\boldsymbol{\varphi}, \psi) \;\middle|\; \begin{array}{c} (\boldsymbol{n}^{(i)} - \boldsymbol{m}^{(i)})^{\mathrm{T}}(\boldsymbol{\varphi}^{(i)} - \boldsymbol{\Delta}_i{}^{(i)}) + c_{\boldsymbol{n}^{(i)}} - c_{\boldsymbol{m}^{(i)}} = 0 \\ \boldsymbol{n}^{(i)\mathrm{T}}(\boldsymbol{\varphi}^{(i)} - \boldsymbol{\Delta}_i{}^{(i)}) + \psi + \varphi_i + \boldsymbol{i}^{\mathrm{T}}\boldsymbol{\Delta}_i + c_{\boldsymbol{n}^{(i)}} = 0 \end{array} \right\},$$

that is exactly a shifted version of the previous case (vi). Consequently, we have

$$P_U\big|_{\varphi_i=+\infty} = R_i^+\big|_{\boldsymbol{\varphi}}\Big|_{\varphi_i=+\infty} = R_i^-\big|_{\boldsymbol{\varphi}}\Big|_{\varphi_i=-\infty} + \boldsymbol{\Delta}_i{}^{(i)} = P_{U^{(i)}} + \boldsymbol{\Delta}_i{}^{(i)}.$$

□

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