On the Propagation of Solitary Pulses in Microstructured Materials
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Abstract

KdV-type evolution equation, including the third- and the fifth order dispersive and the fourth order nonlinear terms, is used for modelling the wave propagation in microstructured solids like martensitic-austenitic alloys. The character of the dispersion depends on the signs of the third and the fifth order dispersion parameters. In the present paper the model equation is solved numerically under localised initial conditions in the case of mixed dispersion, i.e., the character of dispersion is normal for some wavenumbers and anomalous for others. Two types of solution are defined and discussed. Relatively small solitary waves result in irregular solution. However, if the amplitude exceeds a certain threshold a solution having regular time-space behaviour energes. The latter has tree sub-types: “plaited” solitons, two solitary waves and single solitary wave. Depending on the value of the amplitude of the initial pulse these sub-types can appear alone or in a certain sequence.

Key words: KdV-type equation, Solitary waves, Higher order dispersion, Higher order nonlinearity
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1 Introduction

Over the recent decade microstructured materials (polycrystalline solids, ceramic composites, functionally graded materials, shape-memory alloys, etc.)

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have found wide application in many fields of contemporary engineering and therefore the attention to such materials has been arisen enormously. It is well known that (i) the wave propagation in microstructured media is influenced by nonlinear and dispersive effects, and (ii) the simplest model describing the wave propagation in nonlinear dispersive media is the celebrated Korteweg–de Vries (KdV) equation

$$u_t + [P(u)]_x + du_{3x} = 0, \qquad P(u) = \frac{u^2}{2}. \tag{1}$$

The KdV equation is able to take into account only the lowest order effects, i.e., the quadratic nonlinearity and cubic dispersion while in microstructured solids higher order nonlinear as well as higher order dispersive effects are of importance [1–5]. Recent studies have shown that considering higher order dispersive and nonlinear terms, dramatic changes result in behaviour of the wave propagation (see for example [5] and references therein).

In the present paper the 1D longitudinal wave propagation in microstructured solids is studied making use of higher order KdV-like evolution equation

$$u_t + +[P(u)]_x + du_{3x} + bu_{5x} = 0, \qquad P(u) = -\frac{u^2}{2} + \frac{u^4}{4}. \tag{2}$$

In Eqs. (1) and (2) the independent variables t and x correspond usually to the moving frame [2,3], u is the excitation, d and b the third- and the fifth-order dispersion parameters, respectively and elastic potential $P(u)$ introduces the nonlinearity (quadratic for Eq. (1) and quartic for Eq. (2)). Because of the orders of nonlinearity and dispersion, the model equation (2) is shortly referred as KdV435 below. The sources of higher order effects can be dislocations in the crystal structure of martensitic-austenitic shape-memory alloys [1,3,6]. For the present model the character of dispersion depends on the signs of parameters d and b. For $db < 0$ one has normal dispersion, however for $db > 0$ the dispersion is normal for some wavenumbers and anomalous for others [7]. The latter is refered below as a mixed dispersion case.

Equation (2) is nonintegrable and therefore analytical methods (inverse scattering method for example) are not applicable for for detecting the properties of the model. This is the case when numerical experiments are practically the only way in order to analyse the behaviour of the model. In [7–9] model Eq. (2) is solved numerically under harmonic initial conditions. It is shown that in the cases of normal as well as mixed dispersion a train of interacting solitons can emerge from the initial sine-wave. From the practical viewpoint the case of localised initial conditions may be even more important than that of the harmonic. In [10–12] the propagation of sech^2-type localised pulses (KdV solitons) is simulated numerically in the case of model (2). In these papers the normal dispersion case with $d > 0$ and $b < 0$ is studied. The main result of these studies is the following: if the amplitude of the initial wave exceeds a certain threshold, then such a solitary wave can propagate with nearly constant speed and amplitude, i.e. conserve the energy. However, the pilot studies have shown that interaction of solitary waves having different amplitudes and therefore different speeds is not perfectly elastic. Due to a certain transfer of energy and/or mass the faster solitary wave moves more faster and the slower solitary wave even slower after interaction.

In the present paper the propagation of localised initial pulses is studied in the the case of $d > 0$ and $b > 0$, i.e., in the mixed dispersion case. The problem is stated in Section 2 and numerical method is shortly introduced in Section 3. Results are presented and discussed in Section 4 while conclusion are drawn in Section 5.

2 Statement of the problem

In order to simulate the 1D wave propagation in microstructured solids (characterised by quartic nonlinearity and positive values of third- and fifth order dispersion) the KdV-type Eq. (2), shortly the KdV435 equation, is integrated numerically under localised initial conditions

$$u(x,0) = A\,\mathrm{sech}^2 \frac{x}{\Delta}, \qquad \Delta = \sqrt{\frac{12d}{A}}. \tag{3}$$

Here A is the amplitude and Δ the width of the initial solitary pulse. In fact, the proposed localised initial excitation is the analytical solution of the well-known KdV equation (2) and is known as the KdV soliton [13]. In our case the initial wave (3) is related to the model Eq. (2) through the third order dispersion parameter d (neglecting the quartic nonlinearity and the fifth order dispersion). Logarithmic dispersion parameters

$$d_l = -\log d \quad \text{and} \quad b_l = -\log(b) \tag{4}$$

are used instead of d and b for the analysis herein after.

In order to apply the pseudo-spectral method for the numerical integration the boundary conditions are considered to be periodic, i.e.,

$$u(x,t) = u(x + 2kn\pi, t), \quad n = \pm 1, \pm 2, \ldots, \quad k = 2, 4, 8, 16. \tag{5}$$

Goals of our present study are the following:

- to find numerical solutions for the proposed model equation (2) over wide range of dispersion parameters;

- to describe the time-space behaviour of numerical solutions and establish solution types;
- to examine whether initial localised wave (3) can propagate or not with constant speed and amplitude in media where wave motion is governed by the KdV435 equation (Eq. (2));
- to analyse the interaction of solitary waves in order to detect solitonic behaviour of the solution.

3 Numerical method

Our long experience in the field of numerical simulation of wave propagation in microstructured media has shown that an efficient method for numerical integrations is the pseudospectral method. The essence of this method is based on discrete Fourier Transform (DFT) — space derivatives are found by making use of DFT, for integration with respect to time standard ODE solvers are used [14–17]. In the present study the Fastest Fourier Transform in the West (FFTW) algorithm (for DFT) and implicit Adams solver are applied for this purpose.

For analyses of numerical results discrete spectral analysis is used, i.e., in order to characterise the space-time behaviour of the solution Fourier transform related spectral quantities are used [18]. If the discrete Fourier transform (DFT) is defined by

$$U(\omega, t) = \mathrm{F}u = \sum_{j=0}^{n-1} u(j\Delta x, t) e^{\left(\frac{-2\pi i j \omega}{n}\right)} \tag{6}$$

where n is the number of space-grid points, Δx the space step, i the imaginary unit and $\omega = 0, \pm 1, \pm 2, \ldots, \pm n/2 - 1, -n/2$, then spectral amplitudes

$$S_k(t) = \frac{2\,|U(k,t)|}{N}, \quad \omega = 1, ..., \frac{N}{2} - 1. \tag{7}$$

These spectral characteristics carry additional information about the internal structure of waves. This information is related to the existence of local and global minima and maxima in time dependence of spectral curves.

4 Numerical results and discussion

4.1 General remarks

A large number of numerical simulations over wide range of dispersion parameters

$$0.8 < d_l < 2.4, \qquad 2.0 < b_l < 4.8 \tag{8}$$

were carried out in order to detect the behaviour of the solutions of the proposed model equation. The number of space-grid points was varied according to the behaviour of density

$$C = \int_0^{2\pi} u^2 dx \tag{9}$$

of the second conservation law. The target was to keep the relative error of density C below 0.0001 and it was sufficient to consider only values $n \leq 128$ in the present case. Numerical integrations were carried out in the range $0 \leq t \leq t_{\text{final}}$ until sufficient information for detecting solution behaviour was gathered (usually $t_{\text{final}} = 100...30000$). One has to notice that in the case of higher values of the initial wave amplitude A the formation of the solution is faster than in the case of small values of A. Therefore, longer numerical experiments ($t_{\text{final}} = 5000...30000$) were carried out in the case of smaller values of initial wave amplitude ($A = 0.001 \dots 0.1$).

Based on the analysis of numerical results one can introduce two types of the solution:

(1) A solution with irregular time-space behaviour (irregular solution) characterised by chaotic spread of initial energy, i.e., the initial localised excitation decays into irregular (chaotic) train of waves.
(2) A solution with regular time-space behaviour (regular solution). This solution type has three sub-types:
   (a) ,,plaited“ solitons;
   (b) two solitary waves;
   (c) single solitary wave.

All types and sub-types of the solution are discussed in detail in next subsections.

For fixed values of the dispersion parameters the behaviour of the solution depends on the value of the initial wave amplitude A. A convenient criterion

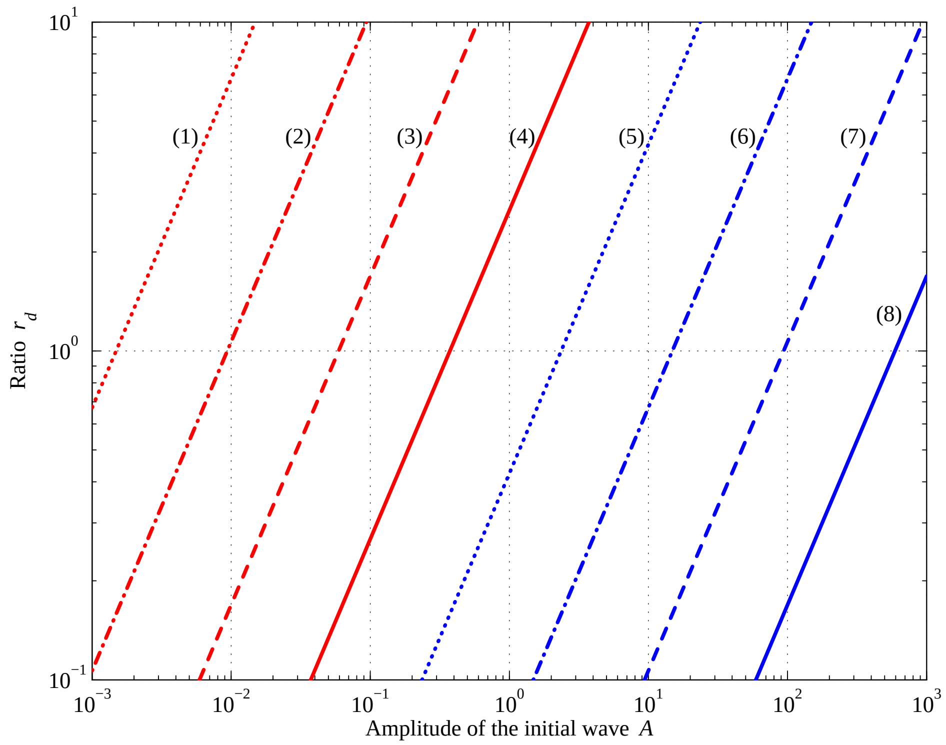

Fig. 1. Quantity $\log r_d$ against $\log A$ for different values of parameter $D = b_l - 2d_l$. Line (1) corresponds to the case $D = -2.8$ ($d_l = 2.4$ and $b_l = 2.0$), line (2) to $D = -2.0$ ($d_l = 2.0$ and $b_l = 2.0$), line (3) to $D = -1.2$ ($d_l = 1.6$ and $b_l = 2.0$), line (4) to $D = -0.4$ ($d_l = 1.2$ and $b_l = 2.0$), line (5) to $D = 0.4$ ($d_l = 0.8$ and $b_l = 2.0$), line (6) to $D = 1.2$ ($d_l = 0.8$ and $b_l = 2.8$), line (7) to $D = 2.0$ ($d_l = 0.8$ and $b_l = 3.6$), and line (8) to $D = 2.8$ ($d_l = 0.8$ and $b_l = 4.4$).

for determining different solution types is the ratio

$$r_d = \frac{\max_x |bu_{5x}(x,0)|}{\max_x |du_{3x}(x,0)|} \tag{10}$$

where $\max_x |bu_{5x}(x,0)|$ represents the maximum of the fifth order term and $\max_x |du_{3x}(x,0)|$ the maximum of the third order term for the initial wave. The value of the ratio r_d depends on the value of the initial amplitude A as well as on the values of dispersion parameters d and b. For the fixed value of the amplitude A the ratio r_d is constant along straight lines

$$b_l = 2d_l + D \tag{11}$$

For fixed values $d = d^*$ and $b = b^*$, i.e., for a certain value $D^* = b_l^* - 2d_l^*$ the quantity $\log r_d$ depends linearly on the $\log A$. One can detect that the higher the amplitude of the initial wave A the higher the radio r_d, i.e., the increase in initial amplitude A increases also the contribution of the fifth order term in the proposed model equation. In Fig. 1 corresponding parallel straight lines are presented for different values of the parameter D in the range of $0.1 \leq r_d \leq 10$

and $10^{-3} \leq A \leq 10^3$. Note that the log − log scale is used and therefore these lines represent linear dependence between $\log r_d$ and $\log A$. On the other hand, numerical experiments demonstrate that for fixed value D^*, i.e., along the line $b_l^* = 2d_l^* + D^*$ the properties of the solution are the same. Therefore it is sufficient to examine the nature of the solution in the case of one pair of dispersion parameters to detect the behaviour of the solution for the whole set of dispersion parameters, described by the same constant D.

According to numerical simulations one can conclude that changes in solutions' behaviour are taking place in the case $r_d \approx 1$, i.e., in the domain where the third- and the fifth-order dispersive terms have nearly equal contribution. Therefore for fixed value of D the most interesting solutions correspond to values of amplitude A that results in $r_d \approx 1$. From the viewpoint of numerical simulation it is reasonable to keep the initial amplitude in the range $0.01 \ldots 1$. Therefore we consider two cases of dispersion parameters in the present paper: (i) $d_l = 1.6$ and $b_l = 2.0$ ($D = -1.2$), and (ii) $d_l = 1.2$ and $b_l = 2.0$ ($D = -0.4$).

4.2 First solution type: solutions with irregular behaviour

In the case of solutions with irregular behaviour, the initial solitary wave falls apart into a train of waves having irregular time-space behaviour. Namely, in course of time, the amplitude of the initial solitary wave starts to decrease and numbers of small amplitude waves appear. The number of waves depends on the value of the initial wave amplitude A. The smaller the value of A the smaller number of waves is evolving from the initial excitation (see Fig. 2 and 3). However, in case of high values of initial wave amplitude, the number of waves in the evolving wave-train can be rather high (see Figs. 2 and 4). Due to irregular behaviour one cannot find any regularity in behaviour of spectral amplitudes (Fig. 5). The phenomenon, covered by the first solution type, can be described as shortage of the initial energy to form a solitary wave(s), which can propagate with constant shape and speed, i.e. conserving the initial energy. In the case of the irregular solution, the third order dispersive term is dominating over the fifth order term, i.e. value of ratio $r_d < 1$.

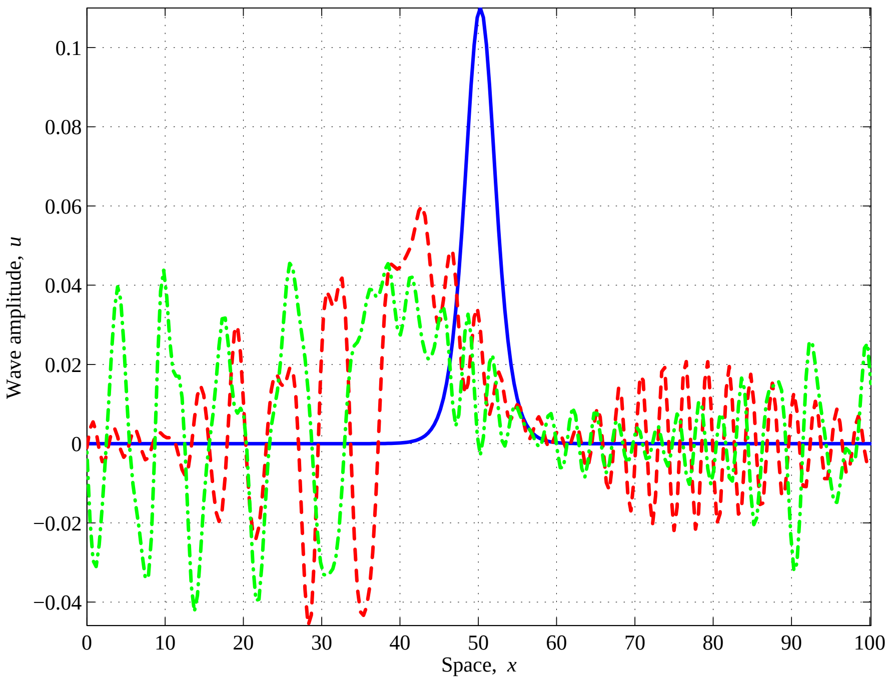

Fig. 2. Irregular solution. Single wave profiles in the case of $d_l = 1.2$, $b_l = 2.0$ and $A = 0.11$. Solid line corresponds to the initial wave profile at $t = 0$, dashed line to time moment $t = 184$ and dotted line to $t = 344$, respectively.

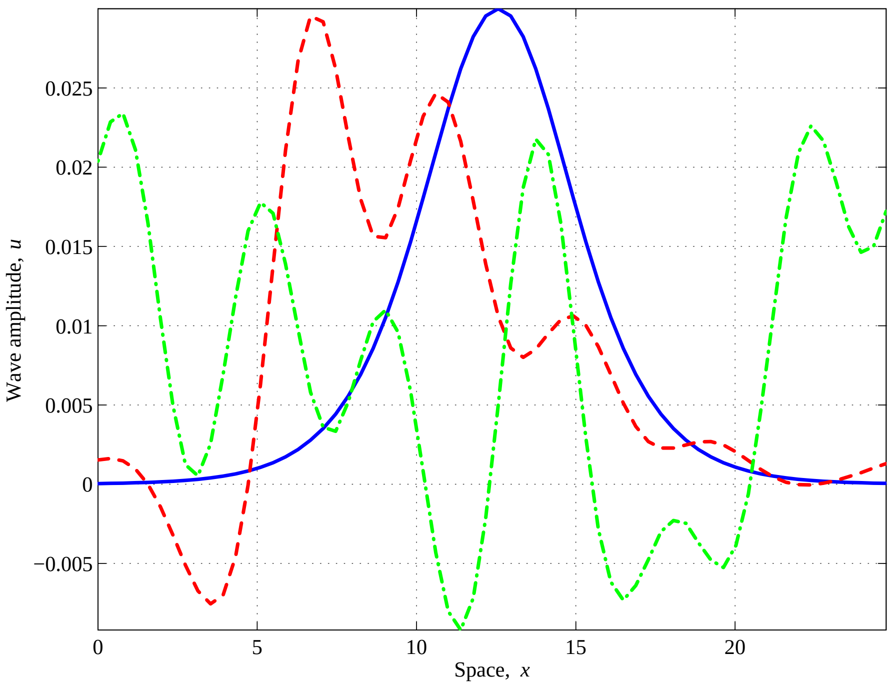

Fig. 3. Irregular solution. Single wave profiles in the case of $d_l = 1.6$, $b_l = 2.0$ and $A = 0.03$. Solid line corresponds to the initial wave profile at $t = 0$, dashed line to time moment $t = 184$ and dotted line to $t = 4034$, respectively.

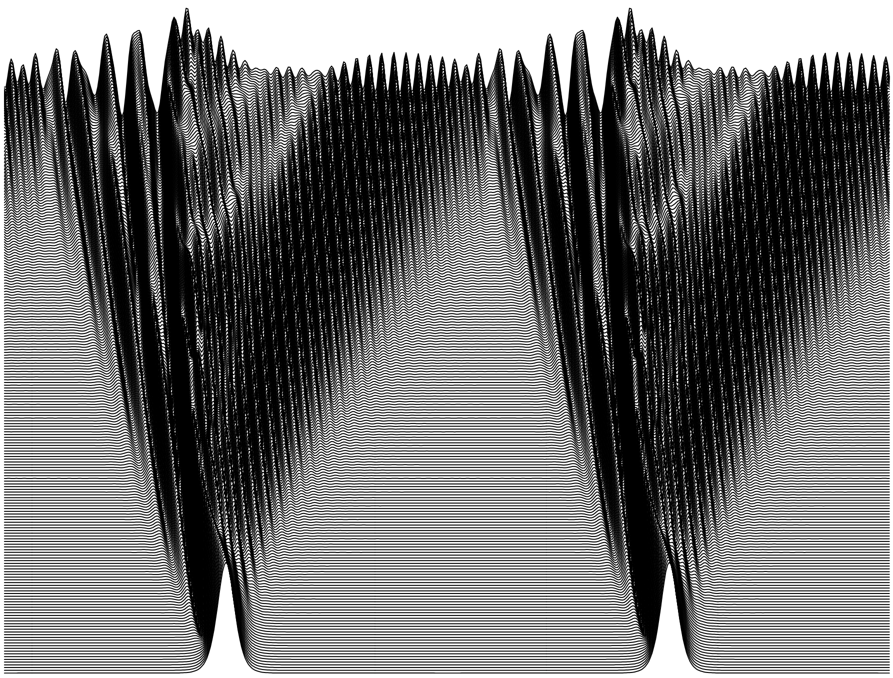

Fig. 4. Irregular solution. Time-slice plot over two space periods for $d_l = 1.2$, $b_l = 2.0$, $A = 0.11$ and $0 < t < 220$.

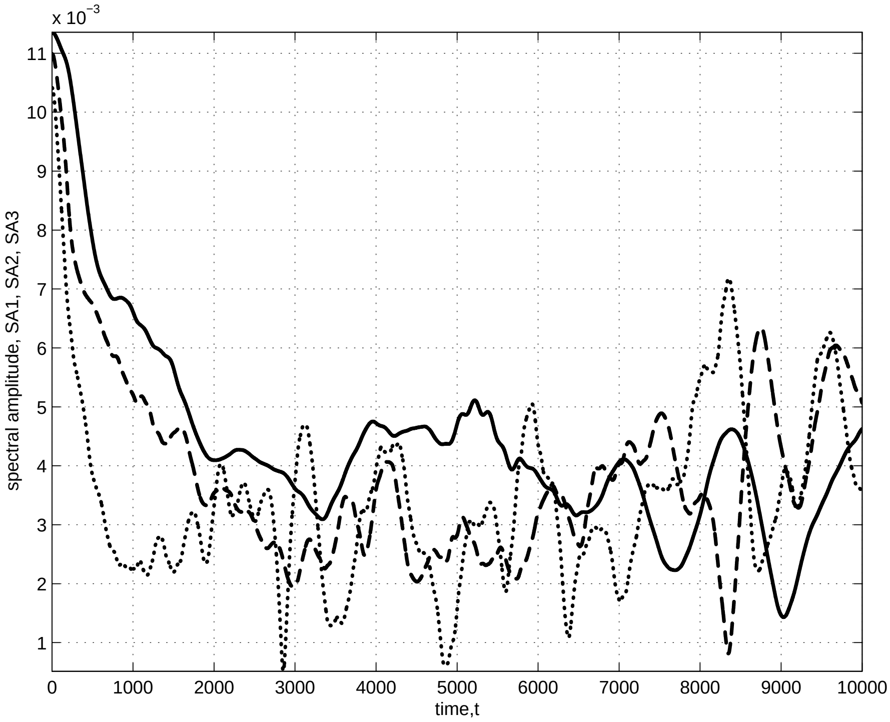

Fig. 5. Irregular solution. First three spectral amplitudes for $d_l = 1.2$, $b_l = 2.0$ and $A = 0.11$. Solid line corresponds to the first, dashed line to the second and dotted line to the third spectral amplitude, respectively.

Table 1
Values of the critical amplitude A^* and corresponding value of the ratio r_d for two different values of the constant $D = b_l - 2d_l$.

D	Critical amplitude A^*	Ratio r_d
-0.4	0.37	0.9907
-1.2	0.063	1.0643

4.3 Second solution type: solutions with regular behaviour

In the case of the second type of the solution a regularly behaving wave-structure forms – one or two solitary wave(s) and an oscillating tail evolve from the initial excitation. It is interesting to notice that transition from the first solution type to the second is quite fast. For example, in the case of $d_l = 1.2$ and $b_l = 2.0$ the difference between two type of the solutions (in terms of initial wave amplitudes) is only $\Delta A = 0.01$. Table 1 presents the critical values of initial amplitude A^* and corresponding values of the ratio of r_d against d. The critical amplitude A^* is the lowest value of amplitude of the initial excitation (detected in our numerical experiments) that results the regular solution. As one can notice, the value of the ratio r_d is very close to 1, i.e., the third and fifth order terms have equal contribution into the proposed model at critical amplitude. Here it is important to notice that ratio r_d corresponds to the initial wave profile (t=0).

Three different sub-types of the regular solution — (a) "plaited" solitons, (b) two solitary waves, and (c) a single solitary wave — were introduced in Subsection 4.1. For fixed values of parameters d_l, b_l and A these three sub-types can appear alone or in a certain sequence. This issue is discussed in detail in Subsection 4.4 — at first properties of sub-types are discussed separately.

4.3.1 "Plaited" solitons

In the case of "plaited" solitons a pair of solitary waves and an oscillating tail evolves from the initial excitation. These two solitary waves interact (i) with each other, and (ii) have the oscillating tail. Furthermore, these solitary waves form a plaited solitary entity propagating with constant group velocity. On can call these solitary waves solitons, because they propagate with constant speed, and restore their amplitude after interacting with each other (see Figs. 6–13). Once evolved from the initial excitation, these solitons continue to interact with each other regularly (Figs. 8, 10–13). By our interpretation these interactions take place due to the oscillating tail – two large waves are ,,travelling" over the tail. However, once formed, there is no remarcable energy exchange between "plaited" structure and small-amplitude oscillating tail anymore and the "plaited" structure can exist over long time intervals.

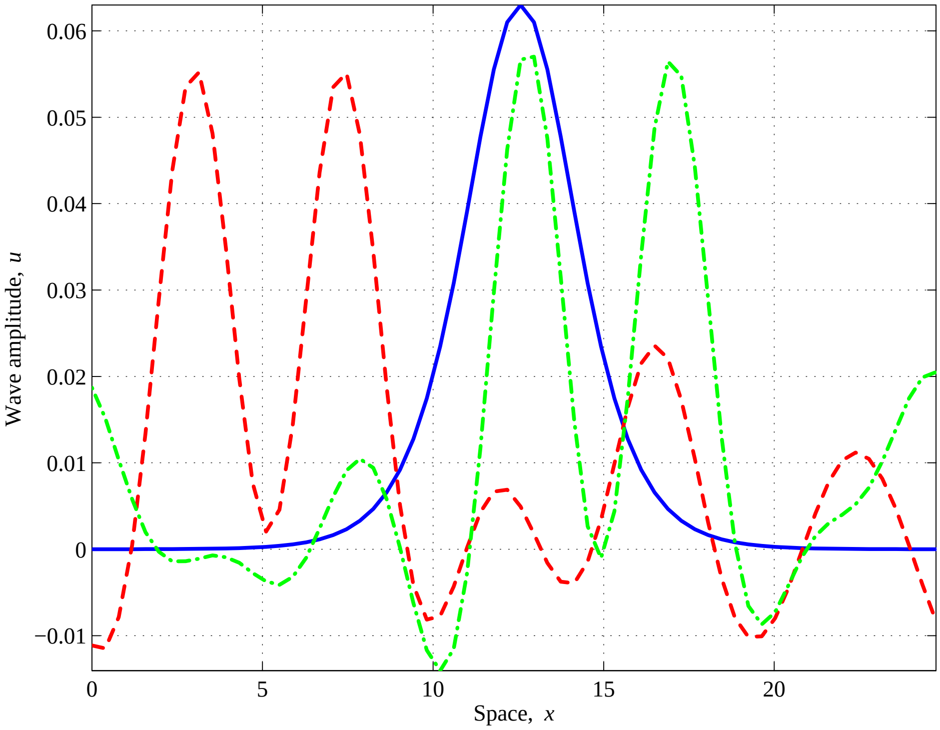

Fig. 6. "Plaited" solitons. Single wave profiles in the case of $d_l = 1.6$, $b_l = 2.0$ and $A = 0.063$. Solid line corresponds to initial wave profile at $t = 0$, dashed line to time moment $t = 1205$ and dash-dotted line to $t = 3629$, respectively.

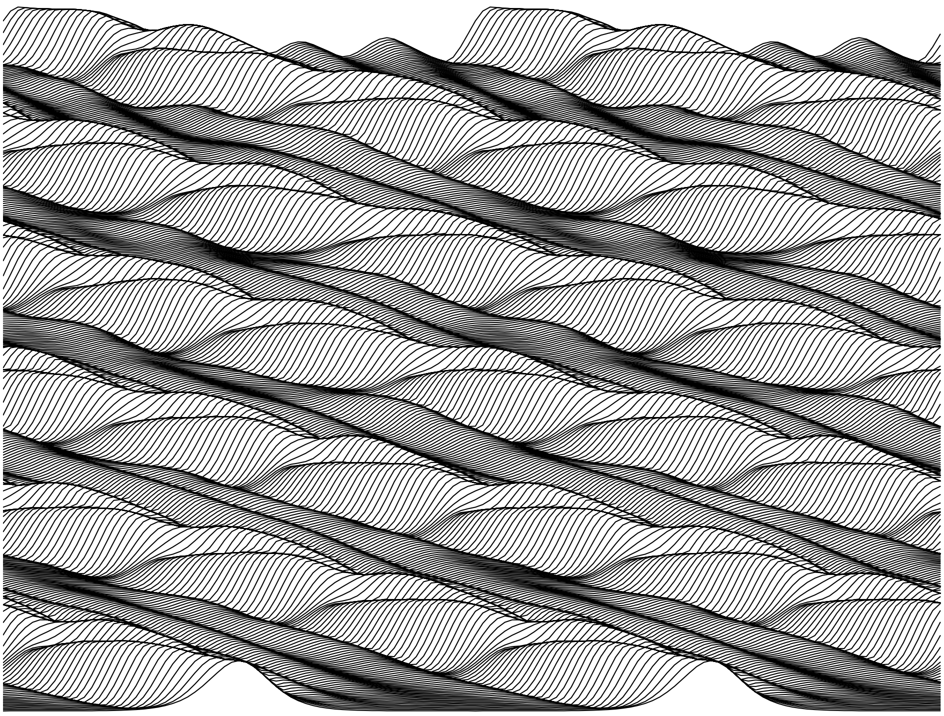

Fig. 7. “Plaited” solitons. Time-slice plot over two space periods in the case of $d_l = 1.6$, $b_l = 2.0$, $A = 0.063$ and $0 < t < 5000$.

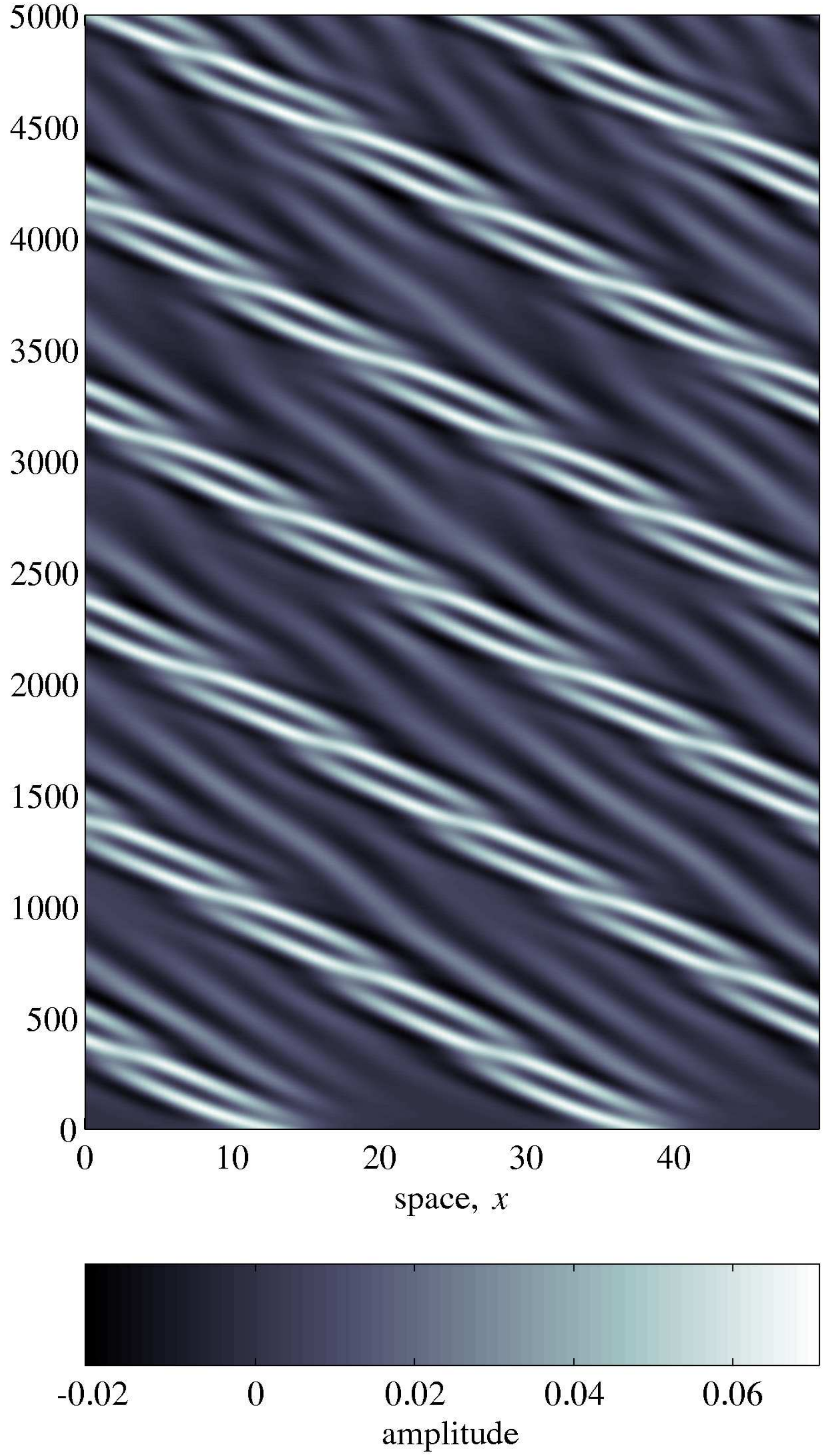

Fig. 8. "Plaited" solitons. Pseudocolor plot over two space periods in the case of $d_l = 1.6$, $b_l = 2.0$, $A = 0.063$ and $0 < t < 5000$.

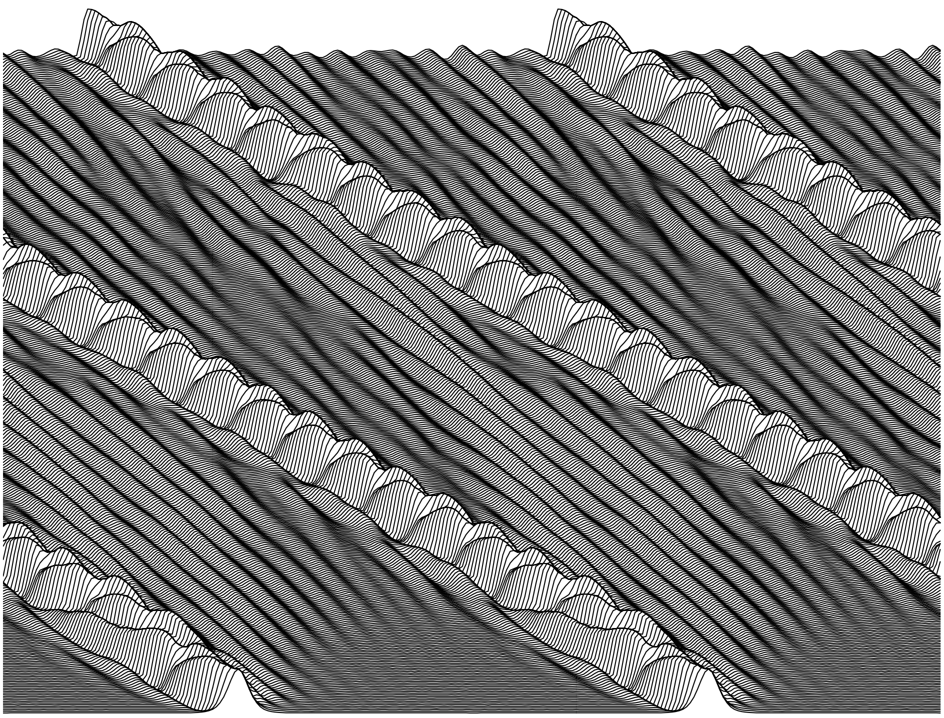

Fig. 9. "Plaited" solitons. Time-slice plot over two space periods in the case of $d_l = 1.2$, $b_l = 2.0$, $A = 0.37$ and $0 < t < 800$.

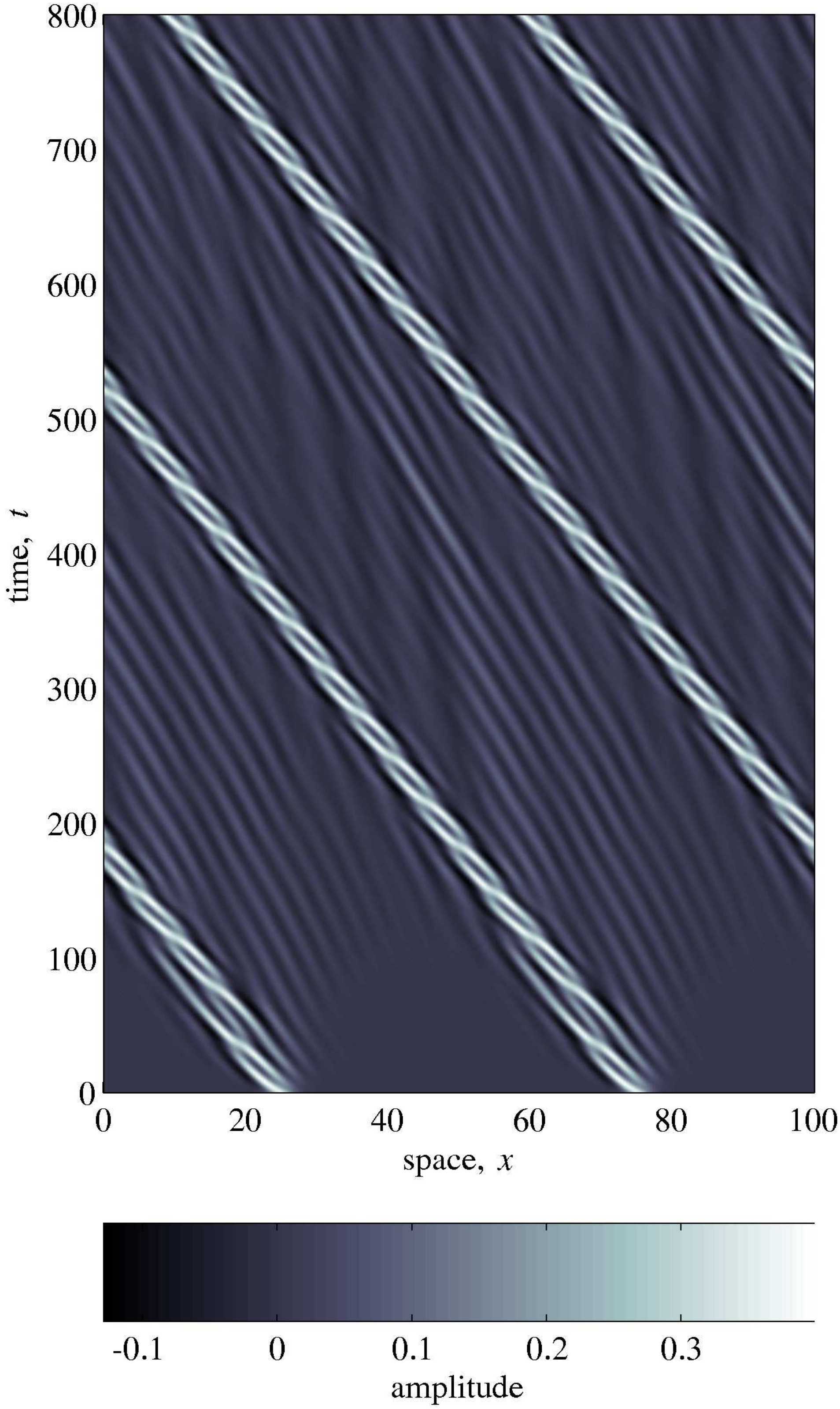

Fig. 10. "Plaited" solitons. Pseudocolor plot over two space periods in the case of $d_l = 1.2$, $b_l = 2.0$, $A = 0.37$ and $0 < t < 800$.

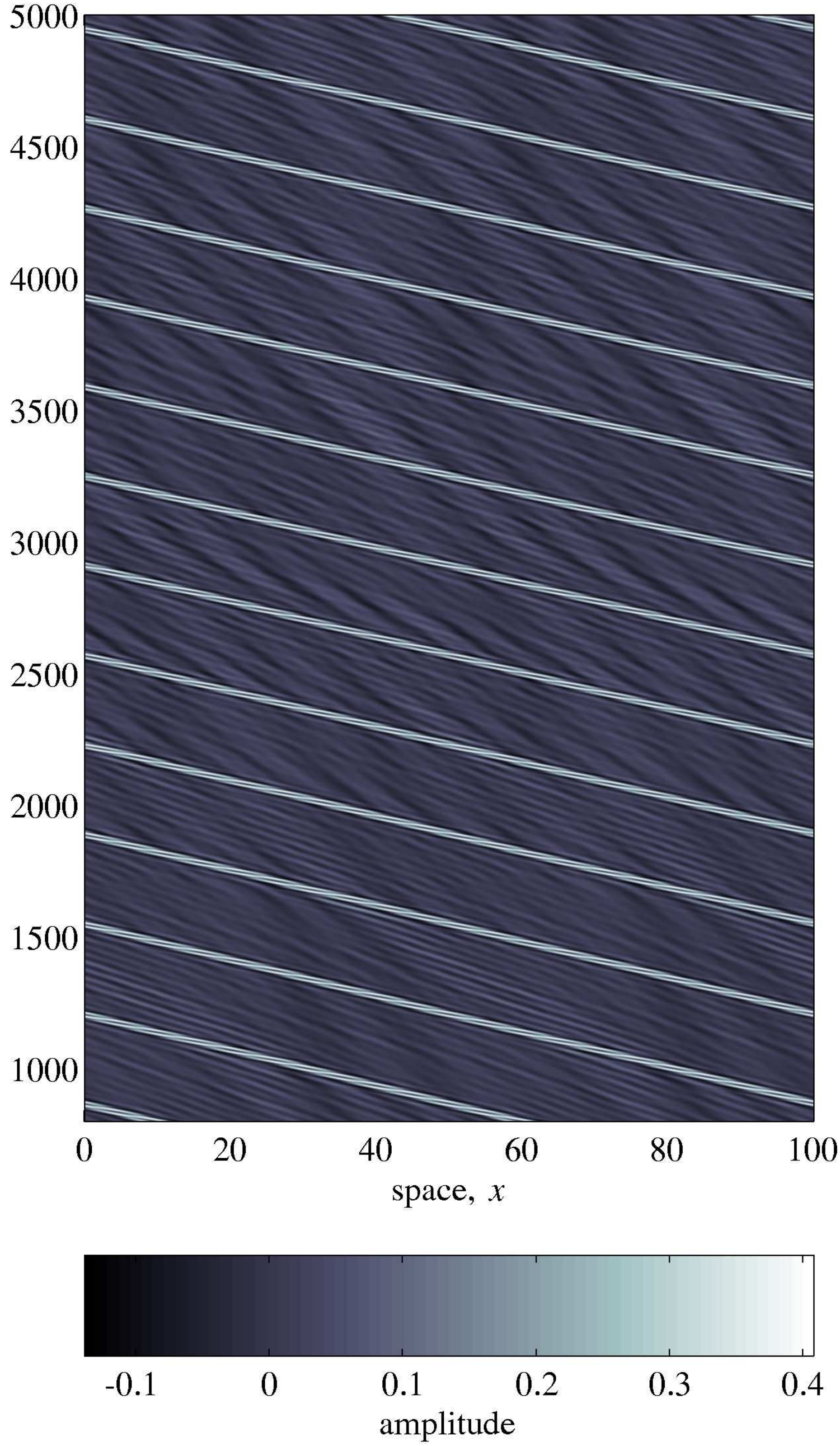

Fig. 11. "Plaited" solitons. Pseudocolor plot over two space periods in the case of $d_l = 1.2$, $b_l = 2.0$, $A = 0.37$ and $800 < t < 5000$.

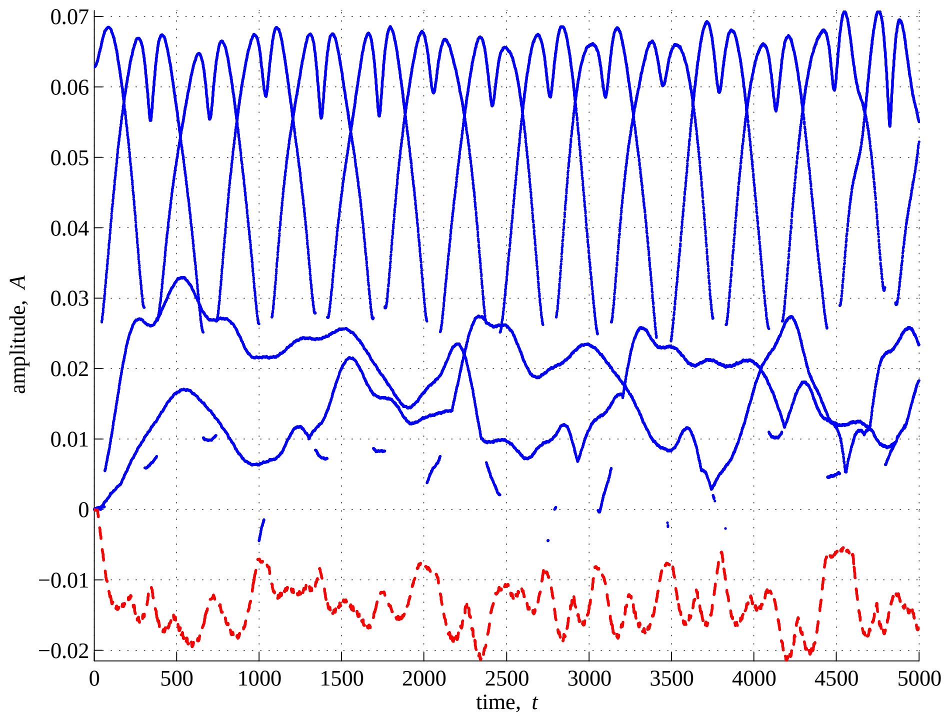

Fig. 12. "Plaited" solitons. Waveprofile maxima and minimum against time in the case of $d_l = 1.6$, $b_l = 2.0$ and $A = 0.063$.

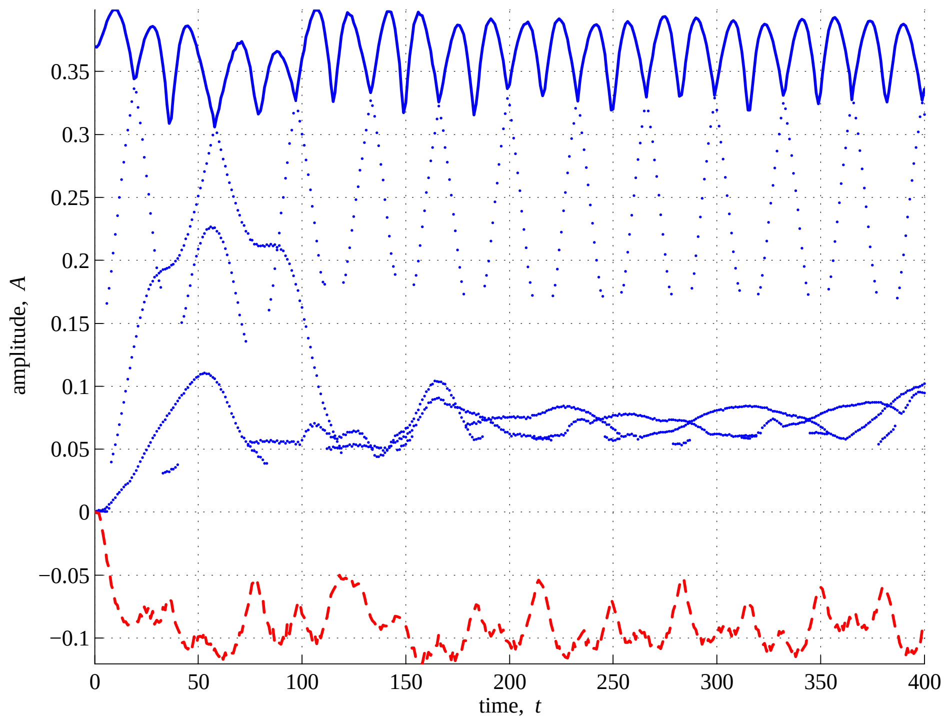

Fig. 13. "Plaited" solitons. Waveprofile maxima and minimum against time in the case of $d_l = 1.2$, $b_l = 2.0$ and $A = 0.37$.

4.3.2 Two solitary waves

In the present case two solitary waves travelling with different speeds evolve from the initial excitation. Due to different speeds these solitary waves can interact with each other (a solitary wave with higher speed overruns a solitary wave with lower speed). Their interaction is not elastic (amplitude of the higher solitary wave increases whereas amplitude of the lower solitary wave decreases) – therefore one can not call them solitons (see Figs. 14–18).

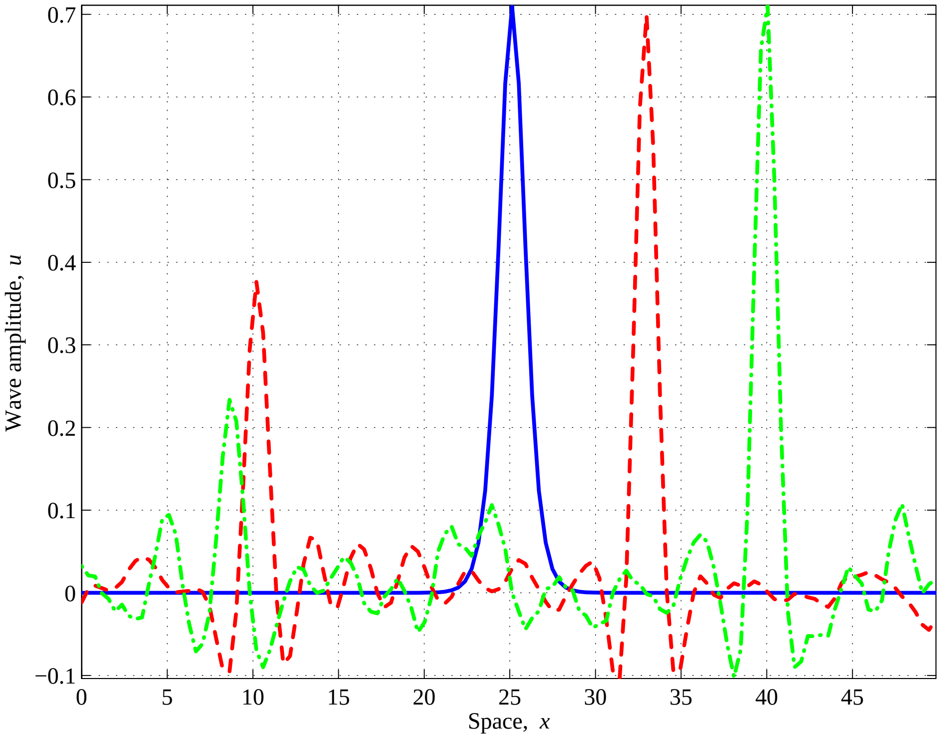

Fig. 14. Two solitary waves. Single wave profiles in the case of $d_l = 1.2$, $b_l = 2.0$ and $A = 0.71$. Solid line corresponds to initial wave profile at t=0, dashed line to time moment $t = 333$ and dash-dotted line to $t = 2984$, respectively.

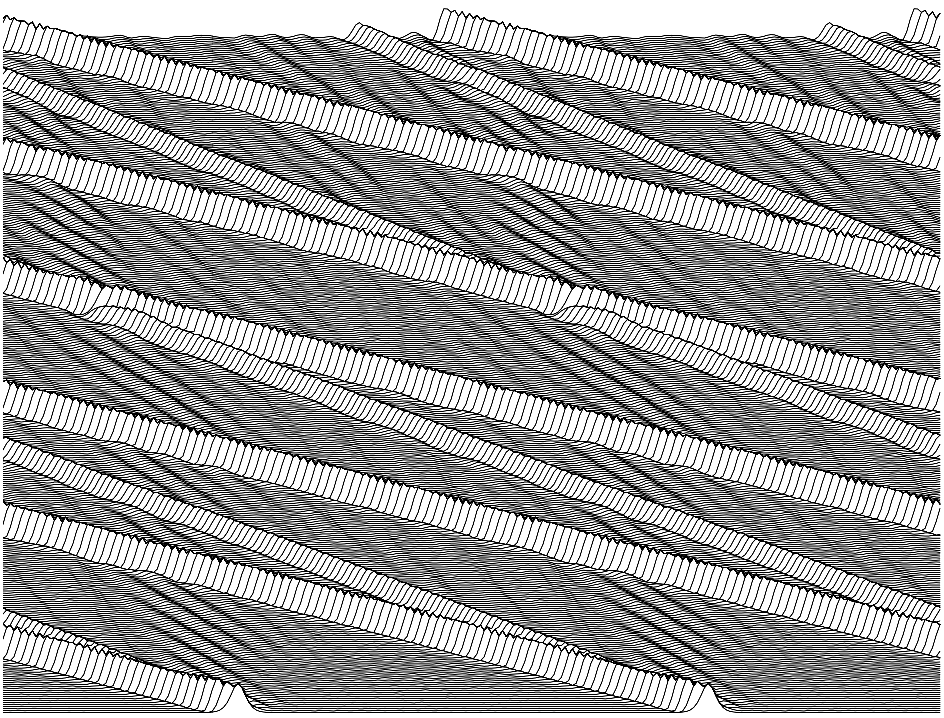

Fig. 15. Two solitary waves. Time-slice plot over two space periods in the case of $d_l = 1.2$, $b_l = 2.0$, $A = 0.71$ and $0 < t < 1000$.

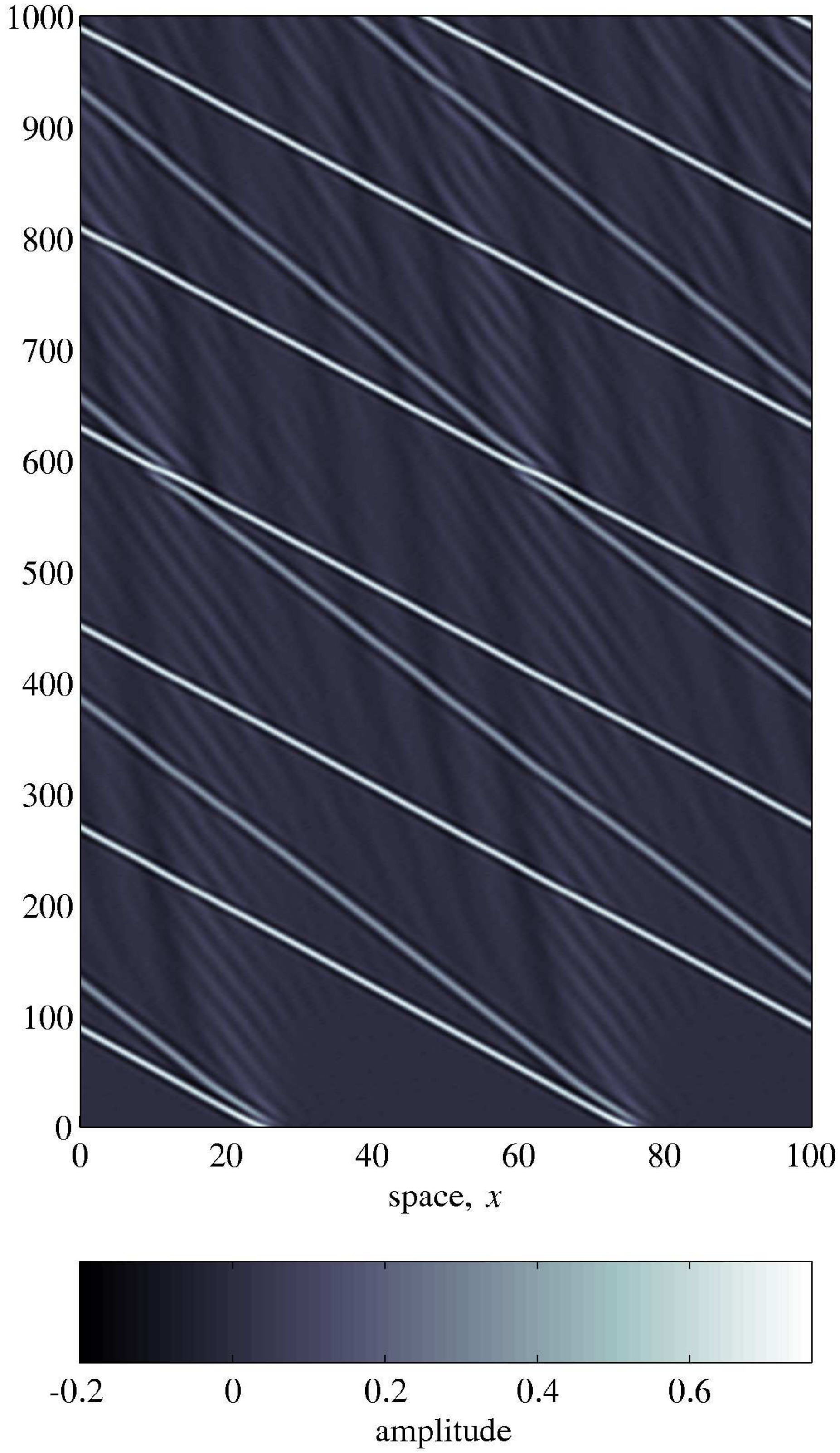

Fig. 16. Two solitary waves. Pseudocolor plot over two space periods in the case of $d_l = 1.2$, $b_l = 2.0$, $A = 0.71$ and $0 < t < 1000$.

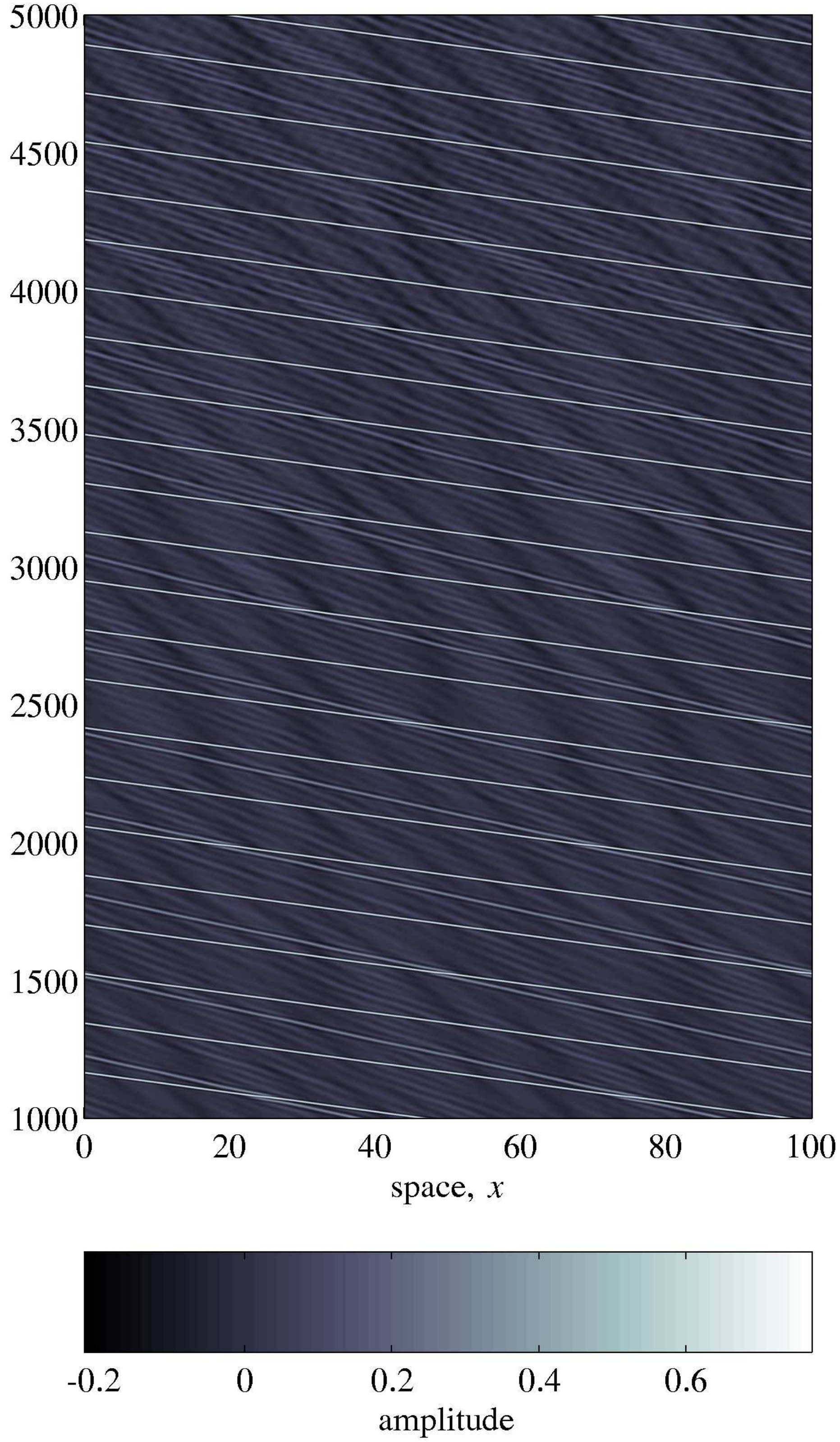

Fig. 17. Two solitary waves. Pseudocolor plot over two space periods in the case of $d_l = 1.2$, $b_l = 2.0$, $A = 0.71$ and $1000 < t < 5000$.

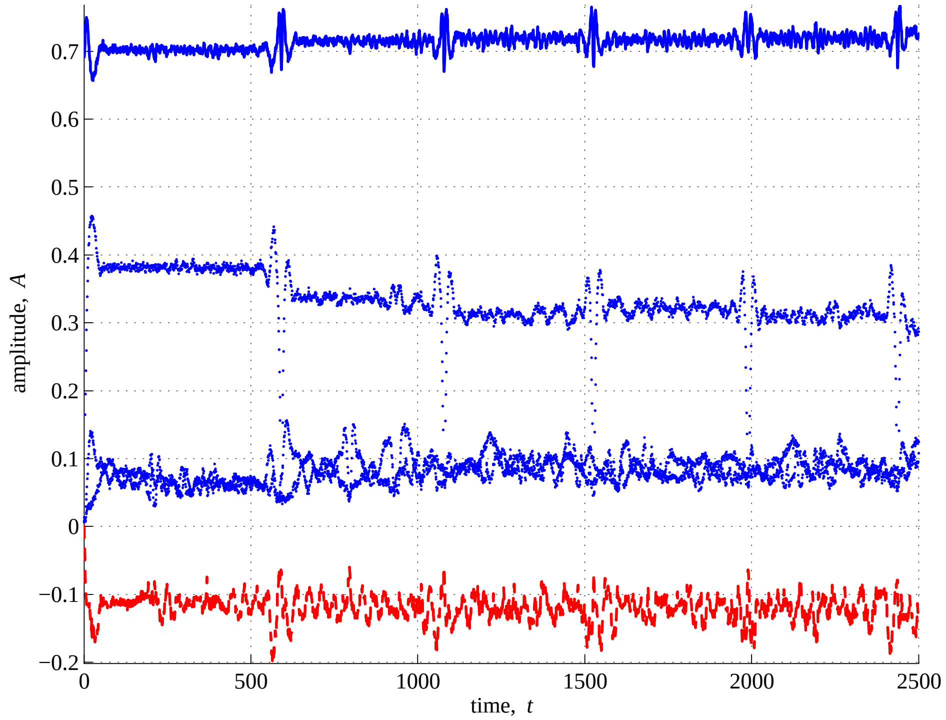

Fig. 18. Two solitary waves. Waveprofile maxima and minimum against time in the case of $d_l = 1.2$, $b_l = 2.0$ and $A = 0.71$.

4.3.3 Single solitary wave

In the case of sub-type (c) one solitary wave together with an oscillating tail forms from the initial excitation. This single solitary wave travels throughout all integration period (Fig. 19) with a constant speed. The amplitude of the travelling solitary wave is oscillating around a level slightly lower than amplitude of the initial excitation (Figs. 20 and 21), because some of the energy of the initial pulse is transfered into the oscillating tail and the solitary wave and the tail are in the permanent interaction. The speed of the propagating wave depends on the value of the amplitude of the initial wave — the higher the amplitude A, the faster the solitary wave is going to the left.

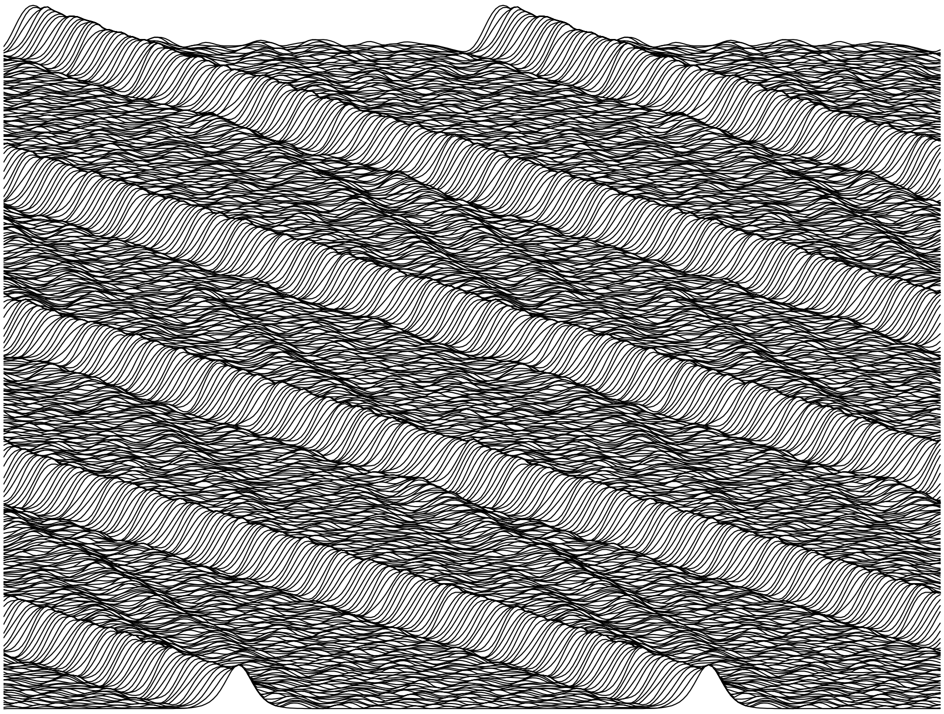

Fig. 19. Single solitary wave. Time-slice plot over two space periods in the case of $d_l = 1.6$, $b_l = 2.8$, $A = 1.25$ and $0 < t < 300$.

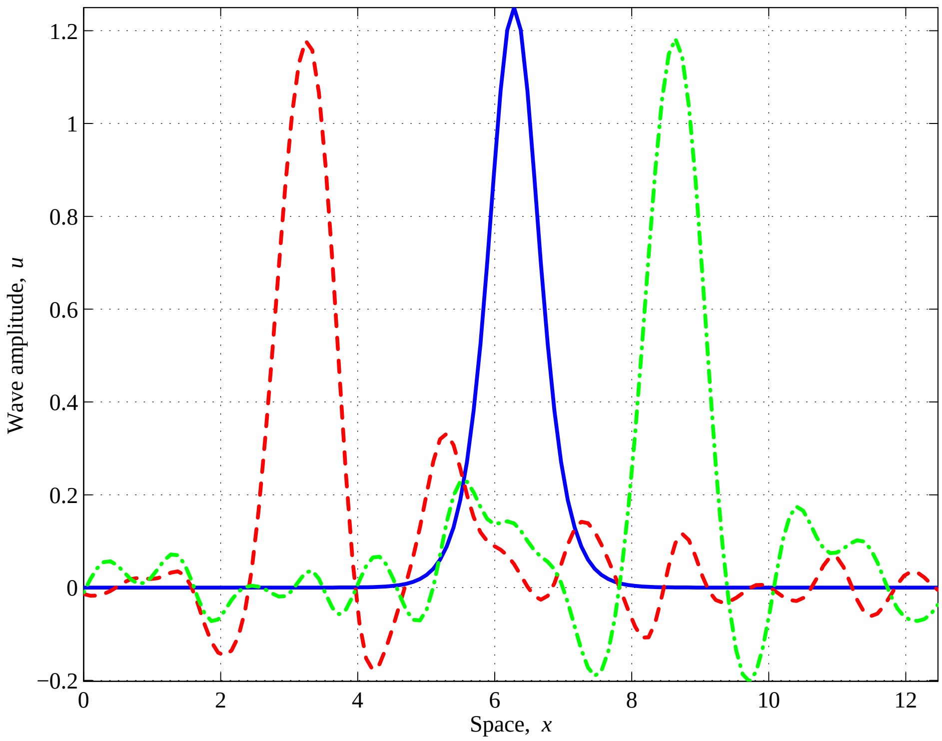

Fig. 20. Single solitary wave. Single wave profiles in the case of $d_l = 1.6$, $b_l = 2.8$ and $A = 1.25$. Solid line corresponds to initial wave profile at $t = 0$, dashed line to time moment $t = 10.5$ and dotted line to $t = 261.9$, respectively.

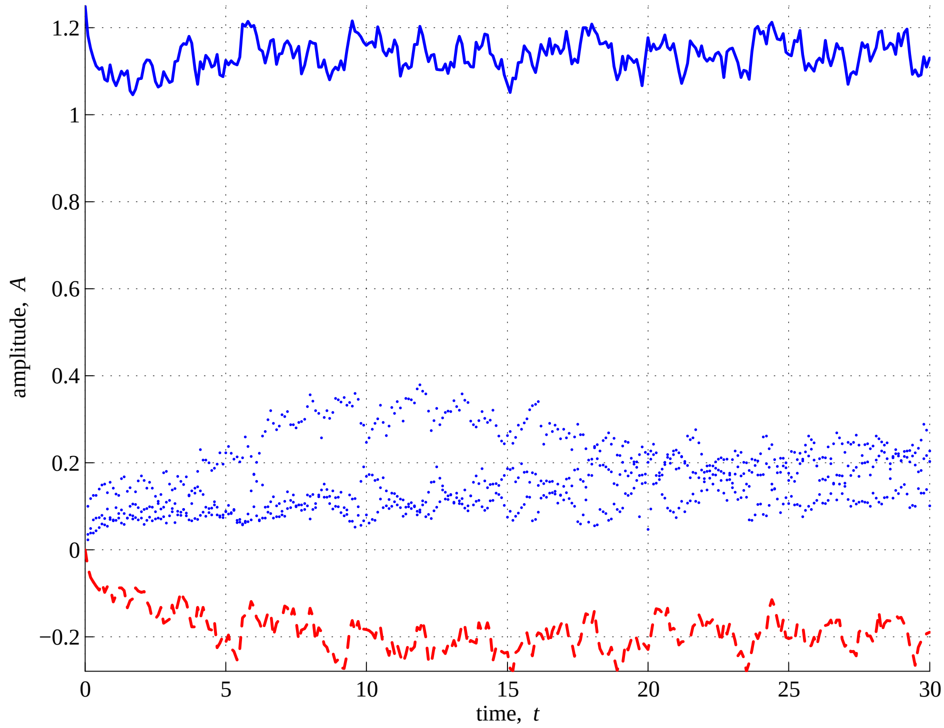

Fig. 21. Single solitary wave. Waveprofile maxima and minimum against time in the case of$d_l = 1.6$, $b_l = 2.8$ and $A = 1.25$.

According to results of numerical simulations over wide range of amplitude A of the initial wave one can conclude that in the case of regular solution all three sub-types (“plaited” solitons, two solitary waves and single solitary wave) can be observed within one solution. In the other words, depending on the value of the amplitude A sub-types (a)–(c) can evolve in a certain order and therefore can be considered as different stages of the solution.

The process can be described in the following way (see Figs. 22–25 and Table 2). Once there is enough energy available in the initial excitation (amplitude of the initial wave exceeds a certain threshold for particular set of dispersion parameters), a plaited wave structure of two solitary waves and a tail forms, i.e., in the case of the first stage two close solitons form from the initial excitation. This sub-type of the solution can last for a long time (see Table 2). However, at some point one of the solitons (with lower amplitude) separates from the “colleague”, i.e., suddenly there is a change in energy balance and one of these starts to propagate on its own. After the separation the higher soliton continues to travel to the left with speed close to the group speed of the plaited stucture while the lower one slows down. This stage corresponds to the sub-type (b) (two solitary waves). Due to different speeds separated solitary waves begin to interact. These interctions are not elastic (see Subsection 4.3.2) and therefore each such interaction decreases the the amplitude of the slower solitary wave until one can not distinguish the latter from the oscillating tail. This solution stage corresponds to the sub-type (c) (single solitary wave).

However, increasing the amplitude A, and consequently the energy, of the initial excitation makes the period of the time for the “plaited” solitons solution shorter. If the initial amplitude exceeds the next threshold one can not observ the “plaited” solitons for the first stage and two solitary waves forms from the initial excitation. Still, this is followed by one solitary wave (Figs. 26–28). Further increase in initial energy makes the two solitary wave solution to disappear as well, and only one solitary wave remains (Table 2).

One can examine the behaviour of the regular solution in Table 2 where durations of different solution stages are presented against the amplitude A. Note, that these durations are approximate, i.e., notation ,,500+“ means that the change from one sub-type to another is taking place between t=500 and t=750, etc. One has to mention that the process described above is not regular. For example, in some cases one can detect situations where the period of time for “plaited” solitons is increased, despite the fact that amplitude of initial wave was increased (see Table 2). One has to mention that there exists always an oscillating tail. This is formed already during formation process and consumes some of the energy from the initial excitation.

Table 2
Duration of different solution stages against the amplitude of the initial pulse A in the case of $d_l = 1.2$ and $b_l = 2.0$. Note that meaning of TTE is ,,till the end" (in the present case until the end of numerical simulation at $t = 5000$).

Amplitude of the initial excitation	Solutions with irregular behaviour	Solutions with regular behaviour		
	Chaotic spread of energy	Plaited solitons	Two solitary waves	One solitary wave
0.34	TTE			
0.36	TTE			
0.37		TTE		
0.39		500-	TTE	
0.41		1000-		TTE
0.43		1500-		TTE
0.45		1500-		TTE
0.47		500+		TTE
0.49		500+		TTE
0.51		500		TTE
0.53		500-	2500	TTE
0.55		500	1500	TTE
0.57		1000-	3500	TTE
0.59		500+	3500-	TTE
0.61		250	2000	TTE
0.63		150	2000	TTE
0.65		100-	1000+	TTE
0.67			4000-	TTE
0.69			1500	TTE
0.71			3000+	TTE

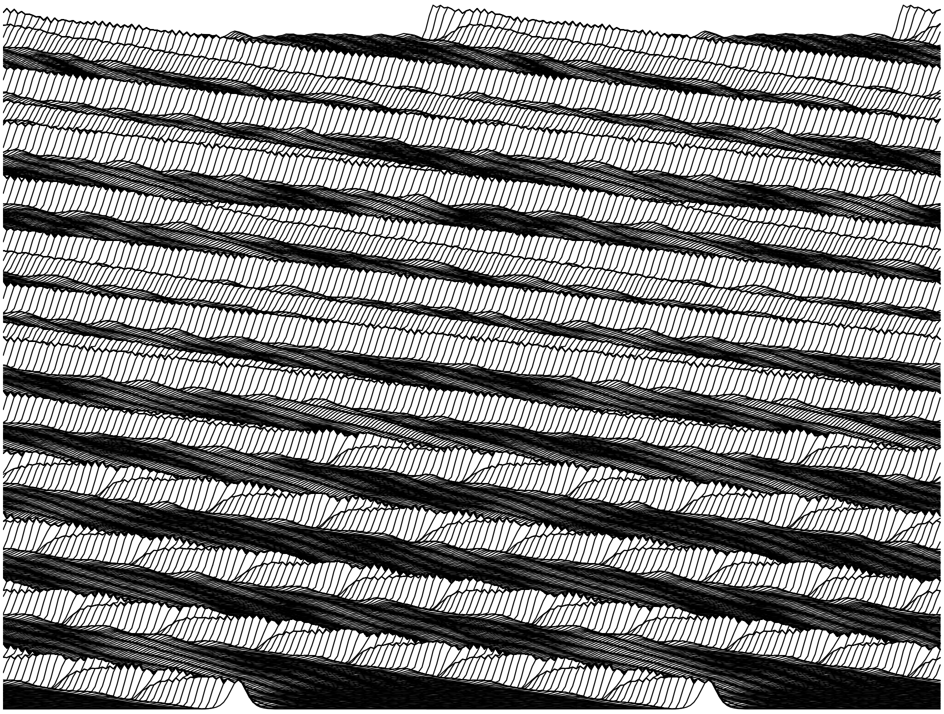

Fig. 22. Time-slice plot over two space periods in the case of $d_l = 1.2$, $b_l = 2.0$, $A = 0.57$ and $0 < t < 2500$.

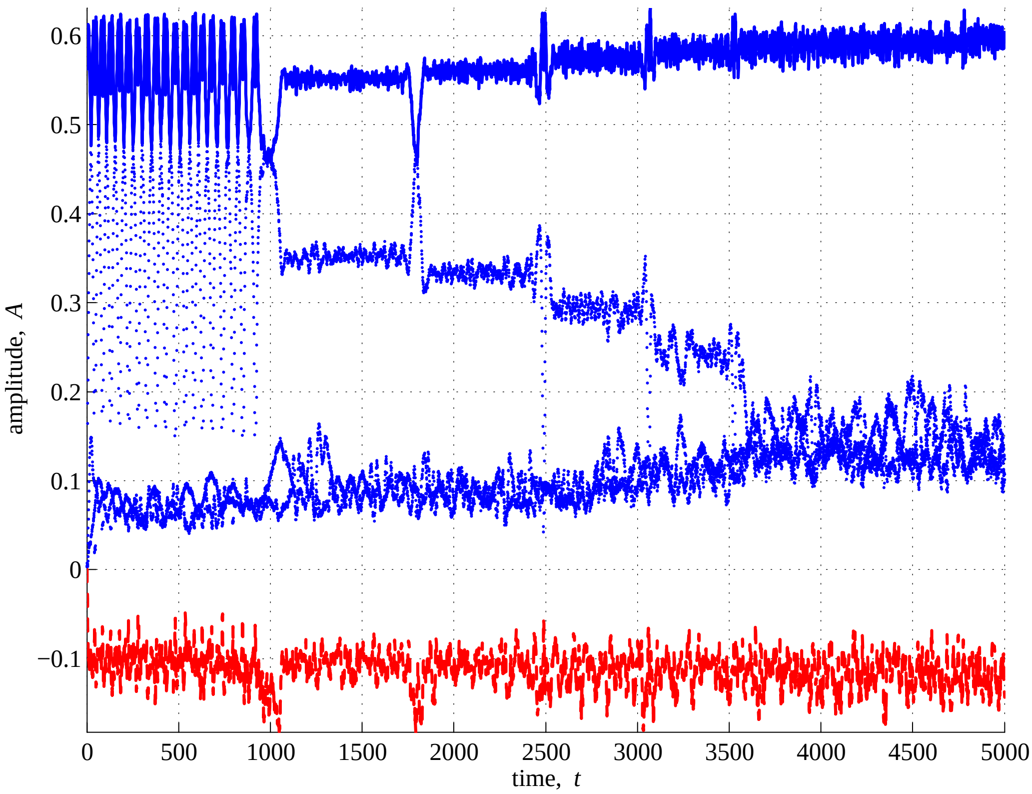

Fig. 23. Maxima of the solution, case $d_l = 1.2$, $b_l = 2.0$ and $A = 0.57$.

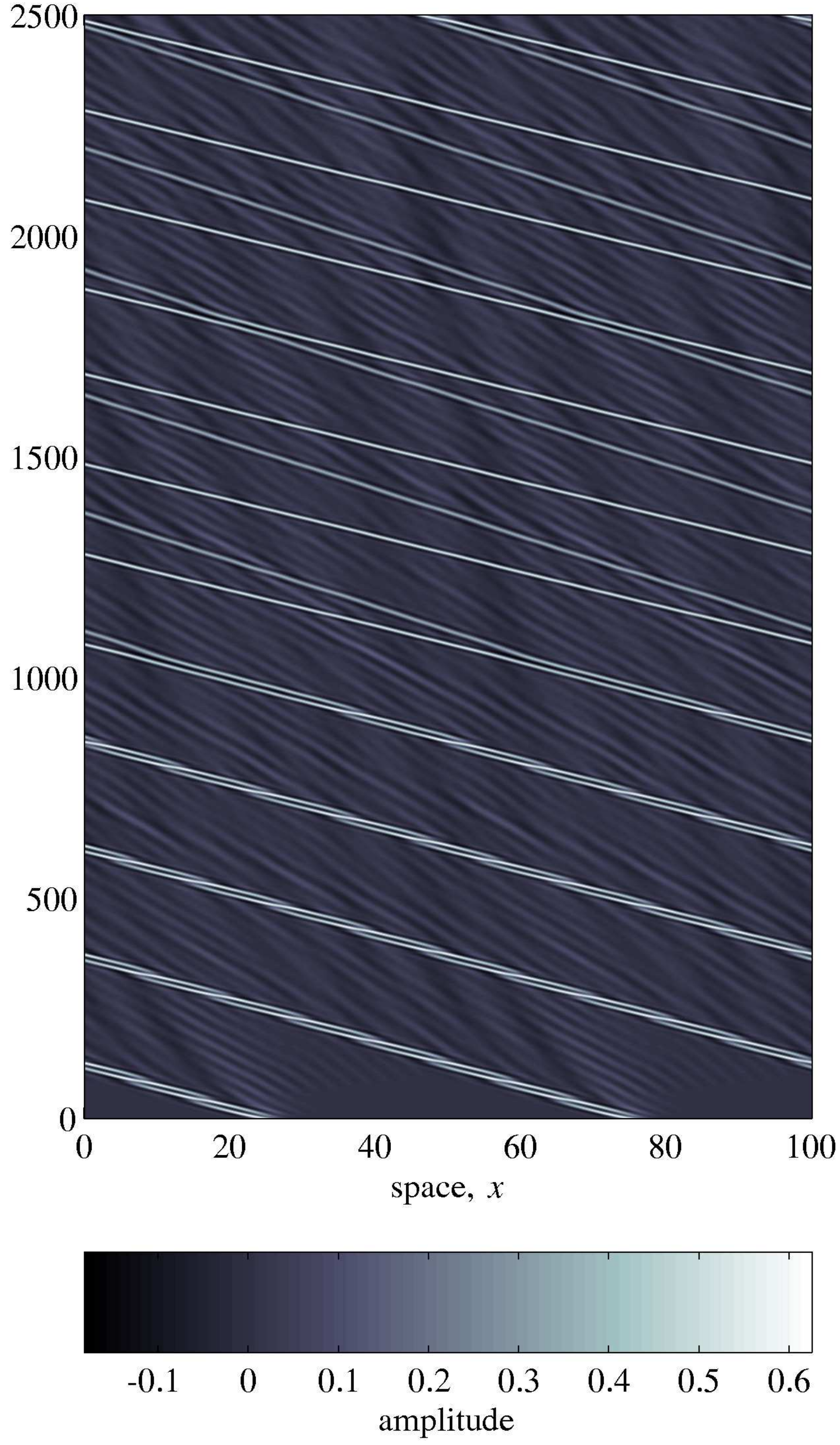

Fig. 24. Pseudocolor plot over two space periods in the case of $d_l = 1.2$, $b_l = 2.0$, $A = 0.57$ and $0 < t < 2500$.

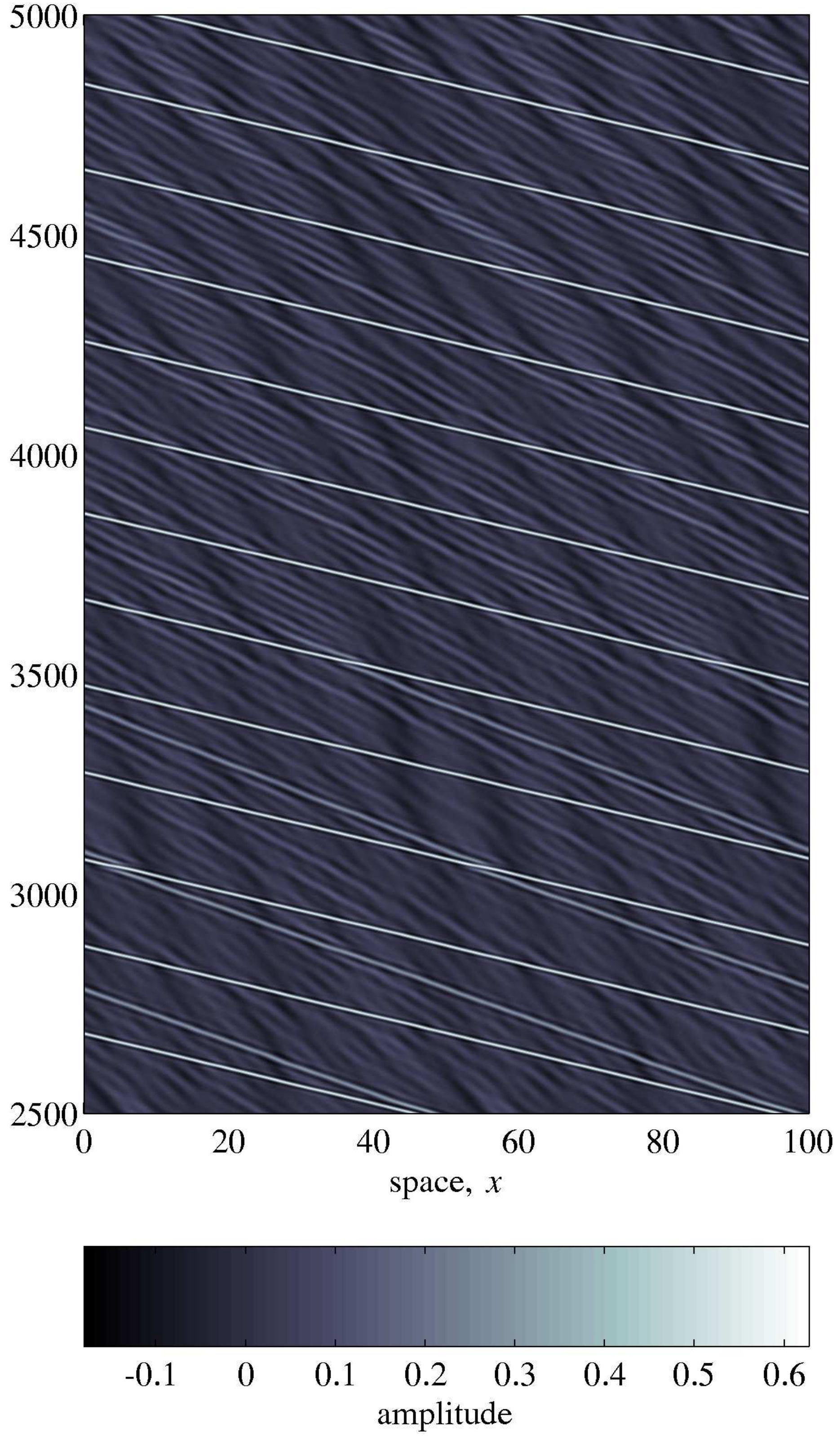

Fig. 25. Pseudocolor plot over two space periods in the case of $d_l = 1.2$, $b_l = 2.0$, $A = 0.57$ and $2500 < t < 5000$.

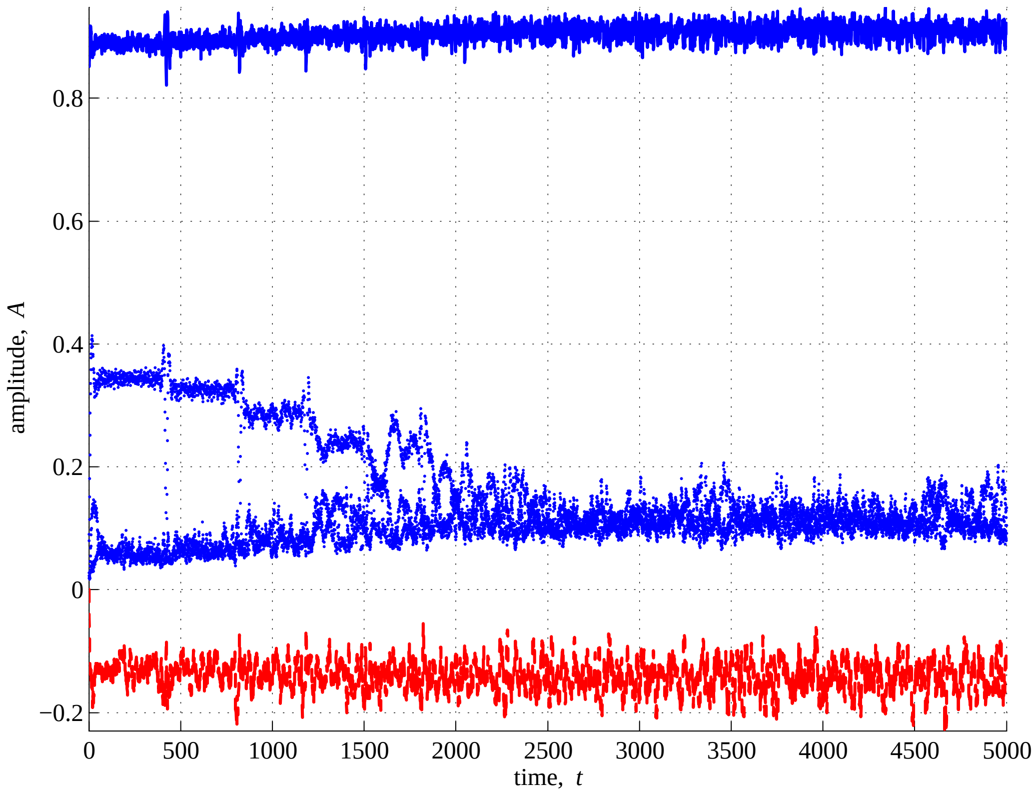

Fig. 26. Maxima of the solution, case $d_l = 1.2$, $b_l = 2.0$ and $A = 0.90$.

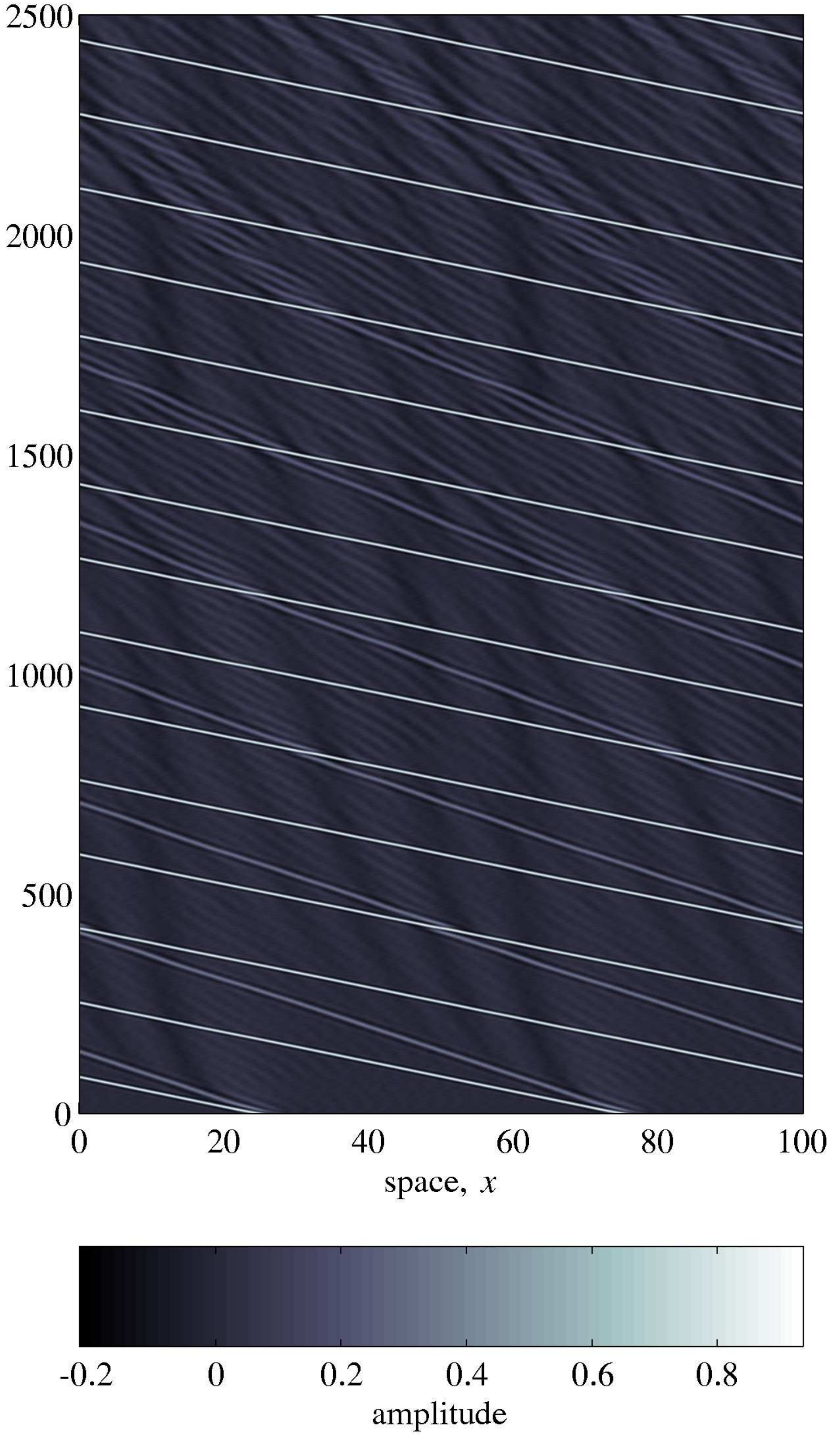

Fig. 27. Pseudocolor plot over two space periods in the case of $d_l = 1.2$, $b_l = 2.0$, $A = 0.90$ and $0 < t < 2500$.

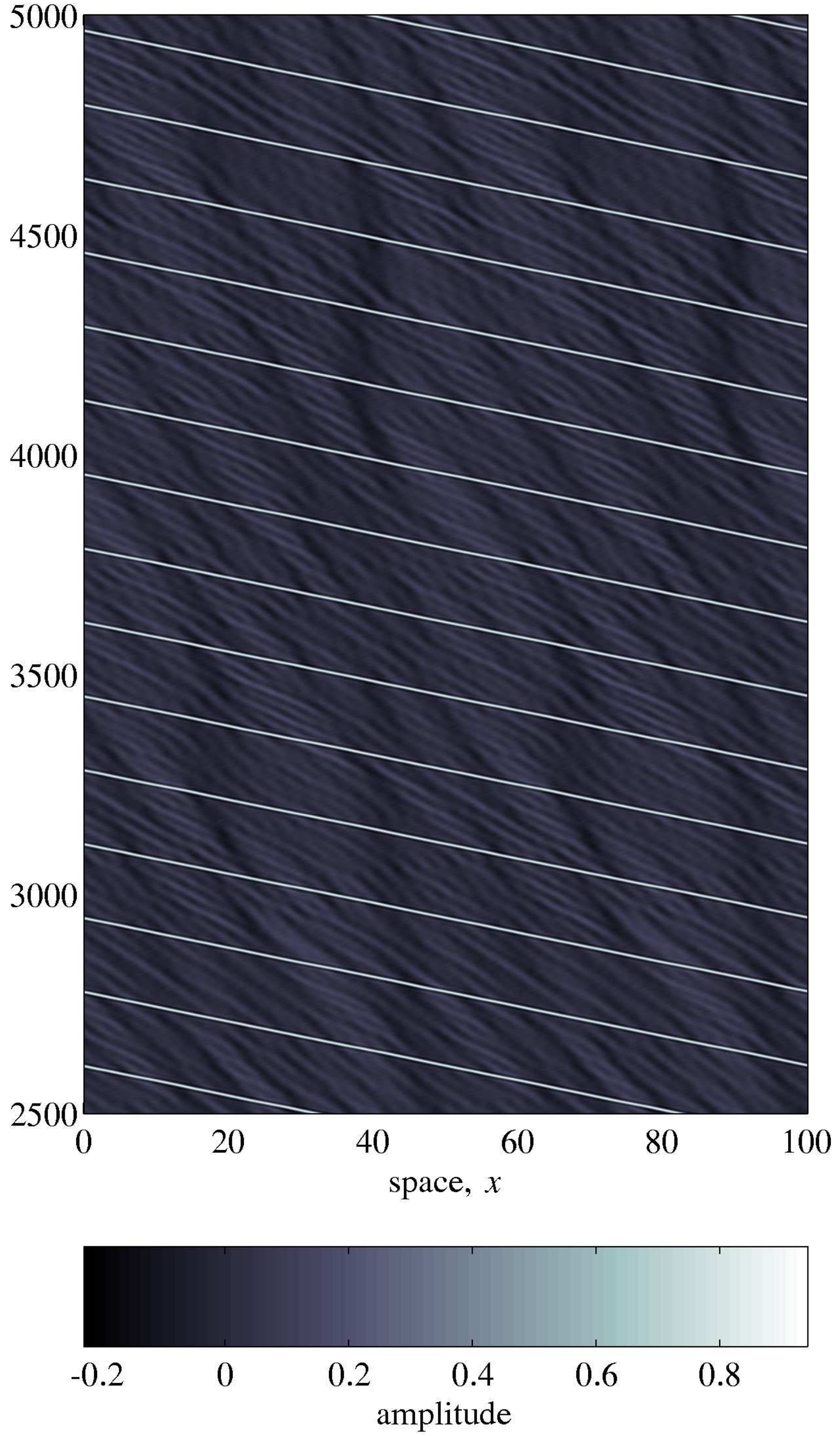

Fig. 28. Pseudocolor plot over two space periods in the case of $d_l = 1.2$, $b_l = 2.0$, $A = 0.90$ and $2500 < t < 5000$.

5 Conclusions

In this paper we studied the propagation of $a\,\mathrm{sech}^2\,bx$ type localised pulses, i.e., KdV solitons, in media characterised by higher order dispersive and higher order nonlinear effects. In [10] we used the same initial conditions in the case of normal dispersion — now the mixed dispersion case is under the consideration. In this sense the present paper can be considered as the second part of the paper [10] . Based on a large number of numerical experiments the following can be concluded:

- Two solution types were detected — solutions with irregular and regular behaviour.
- The first type, i.e., the irregular solution emerges in the case of small initial amplitude (small initial energy). This results in weak co-operation between dispersive and nonlinear effects and stable solitary wave(s) can not be formed.
- The second type, i.e., the regular solution can have three sub-types:
  (a) "plaited" solitons,
  (b) two solitary waves,
  (c) one solitary wave.
- Generally, the sub-type (a) is not stable — after a certain time interval it is changed to sub-type (b). Only in very few cases the "plaited" solitons lives until the end of simulation.
- In the case of sub-type (b) sequential nonelastic interactions take place and the whole process ends up in one solitary wave (sub-type (c)).
- The latter one was found in our numerical experiments to be stable, i.e., a single solitary wave can propagate with (nearly) constant speed and amplitude over long time intervals.
- The first type was found to be changed to the second type for $r_d \approx 1$. Therefore the main attention was paid to the domain in dispersion parameters plane where neither the third nor the fifth order dispersive effects are dominating.

By our opinion, the "plaited" solitons are the most interesting solution type. Porubov [5] describes the similar solitonic structures in the case of extended KdV equation and calls them two-humps and multi-humps. For example the two-humps localised structure is described to look like as a permanent interaction of two solitary waves (solitons): the higher surpasses the lower and becomes the lower and all the process is repeated. Our "plaited" solitons can be described in the same way. Porubov (see [5] pp. 22–25) found that such solitonic structures emerge due to the cubic nonlinear (of type $u^2 u_x$) and nonlinear dispersive (of type uu_{3x}) terms as a certain limit case. The model equation (2) does not include such terms. But again, the "plaited" solitons can be considered as a limit case of the two soliton solution when two closed amplitude solitary waves emerge from an initial localised wave.

There are still many open questions. For example how do two single solitary waves interact or how the single solitary wave and “plaited” solitons interact. Corresponding analysis is in progress and results will be published in forthcoming papers.

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