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2D spectral analysis of the KPI equation

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CONTENTS

1 Introduction - the Kadomtsev–Petviashvili equation

Kadomtsev–Petviashvili (KP) equation is partial differential equation composed of two spatial as well as a time dimension. The transverse perturbations are weak in this equation. It is named after Boris Kadomtsev and Vladimir Petviashvili who derived the equation in their paper. They were looking for a model to describe the evolution of long ion-acoustic waves of small amplitude propagating in plasmas under the effect of long transverse perturbations. The resulting KP equation is considered to be the two dimensional generalization of the Korteweg–de Vries (KdV) equation [1, 2].

Two different versions of the KP equation can be distinguished. In a normalized form they can both be described by

$$(u_t + 6uu_x + u_{xxx})_x + 3\sigma^2 u_{yy} = 0. \tag{1.1}$$

The case $\sigma^2 = -1$ is called the KPI equation and the case $\sigma^2 = 1$ is called the KPII equation. The former is used to describe waves in thin films with high surface tension as well as rogue waves and the latter can be used to model water waves with small surface tension [1, 3, 4].

As is the case for the KdV equation in one spatial dimension, the KP equation is also a universal integrable system in two spatial dimensions. Because of this, the KP equation is considered to be one of the most important equations in nonlinear wave theory and has thus been extensively studied by the mathematicians and physicists for the past 4 decades [1].

Since the KP equation makes up an integrable system, it is possible to describe it with an infinite number of conservation laws. The conserved quantities associated with the first few conservation laws for the KP equation are similar to the KdV equation [4]. Therefore

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u(x,y,t)\,dx\,dy = \text{const}, \tag{1.2}$$

and

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u^2(x,y,t)\,dx\,dy = \text{const}. \tag{1.3}$$

In discrete situations the integrals can be switched out with sums. In a certain special case [5, 6], which is also considered in this report, the following constraint

$$\int_{-\infty}^{\infty} u(x,y,t)\,dx = 0, \qquad \forall y, t, \tag{1.4}$$

holds for the equation.

2 Integrating the KP equation using the Pseudospectral Method

2.1 The purpose

The KP equation can have various forms depending on problems under consideration. The coefficient's values can be considered inessential as they can be modified by rescaling the appropriate variables [1]. The different values of the coefficients can also be viewed as representing the environment characteristics [7]. It is therefore essential to be able to calculate the numerical solution in case of any arbitrary set of coefficients. The KPI equation then takes the form of

$$(u_t + \alpha_1 u u_x + \alpha_2 u_{xxx})_x - \alpha_3 u_{yy} = 0 \tag{2.1}$$

equivalent to

$$u_{xt} + \alpha_1 \left(u_x^2 + u u_{xx}\right) + \alpha_2 u_{xxxx} - \alpha_3 u_{yy} = 0. \tag{2.2}$$

The coefficients α_1, α_2 and α_3 can be described as the nonlinear coefficient, the dispersion coefficient and the transverse perturbation coefficient, respectively. This report focuses on situations where $\alpha_1 = \alpha_3 = 1$, only the dispersion coefficient α_2 was changed.

Since the purpose of the work was to do numerical experiments, it was important to introduce discrete spatial axes as well as discrete steps in time. Therefore, the number of discrete points on either spatial axis was set as N_x and N_y (from here on $N_x = N_y = N$) on the x and y axes, respectively. The discrete time step just meant calculating the result on discrete moments in time such as $t = i\Delta t$, $i = 0, 1, 2, \ldots$. Because the Pseudospectral Method (PSM) uses the discrete Fourier transform (DFT) (and the fast Fourier transform (FFT) to calculate it), it is best to keep the number of spatial points equal to a power of 2. It is therefore imperative to choose the correct number of spatial points as increasing the number two times would mean the time to calculate the result would go up approximately 4 times.

The purpose of the work done for this report was to develop an algorithm based on the PSM for numerically solving 2D wave propagation problems; explain how the 2D spectral characteristics describe the solution; explain the effect of the equation coefficients as well different initial conditions to the solution.

2.2 Integrating the KP equation using the PSM

Since the KP equation is a partial differential equation, it is essential that the spatial derivatives be approximated using the current shape of the wave profile $u(x, y, t)$. The equation can then be solved using algorithms designed to solve ordinary differential equations. This can be done by using either local or global methods for approximating spatial derivatives. The PSM is a global method for approximating spatial derivatives. As mentioned before, it uses the FFT to calculate the DFT. The PSM expects periodic boundary conditions [8].

Making use of properties of the Fourier transform one can approximate spatial derivatives and integrals as follows [9]

$$\begin{aligned}
\frac{\partial^n u(x,y)}{\partial x^n} &= F_x^{-1}\left[\left(\frac{ik}{m}\right)^n F_x u\right], \\
\frac{\partial^n u(x,y)}{\partial y^n} &= F_y^{-1}\left[\left(\frac{ik}{m}\right)^n F_y u\right], \\
\int u(x,y)\,dx &= F_x^{-1}\left[\frac{F_x u}{\frac{ik}{m}}\right] + C(y).
\end{aligned} \tag{2.3}$$

Here F_x denotes the Fourier transform along the x axis and F_x^{-1} refers to the inverse Fourier transform along the same axis. The Fourier transform along the y axis is denoted similarly with

y instead of x in the subscript. It is important to note that m represents the spatial domain such that $-m\pi < x \leq m\pi$ and $-m\pi < y \leq m\pi$.

In order to solve the KP equation using algorithms designed to solve ordinary differential equations, it needs to be have the form

$$u_t = \Psi(u). \tag{2.4}$$

We can now use the equations (2.3) to give the KP equation (2.2) this form. Since the KP equations has a mixed derivative, we need to declare

$$\varphi = u_x, \tag{2.5}$$

and can then get the KP equations in the form

$$\varphi_t = -\alpha_1\left(\varphi^2 + u\varphi_x\right) - \alpha_2\varphi_{xxx} + \alpha_3 u_{yy}. \tag{2.6}$$

It is important to note that this equation has both variables u and φ. The equation (2.6) alongside

$$\begin{aligned}
\varphi_x &= F_x^{-1}\left[\frac{ik}{m}F_x\varphi\right],\\
\varphi_{xxx} &= F_x^{-1}\left[-\frac{ik^3}{m^3}F_x\varphi\right],\\
u &= F_x^{-1}\left[\frac{F_x\varphi}{\frac{ik}{m}}\right] + C(y),\\
u_{yy} &= F_y^{-1}\left[-\frac{k^2}{m^2}F_y u\right]
\end{aligned} \tag{2.7}$$

make up the system we are looking for which can also be expressed as

$$\begin{cases}
\varphi_t = -\alpha_1\left(\varphi^2 + F_x^{-1}\left[\frac{F_x\varphi}{\frac{ik}{m}} + C(y)\right]F_x^{-1}\left[\frac{ik}{m}F_x\varphi\right]\right) - \alpha_2 F_x^{-1}\left[-\frac{ik^3}{m^3}F_x\varphi\right] + \\
\qquad\qquad + \alpha_3 F_y^{-1}\left[-\frac{k^2}{m^2}F_y u\right],\\
u = F_x^{-1}\left[\frac{F_x\varphi}{\frac{ik}{m}}\right] + C(y).
\end{cases} \tag{2.8}$$

2.3 Discrete spectral analysis

In this subsection, spectral amplitudes (or the amplitude spectra) $a_k(y,t)$ as well as the phase spectra $\phi_k(y,t)$ are considered as discrete spectral characteristics. Spectral amplitudes $a_k(y,t)$ are defines as

$$\begin{aligned}
a_0(y,t) &= \frac{U(0,y,t)}{N} && j = 1,2,\ldots,N,\\
a_k(y,t) &= \frac{2|U(k,y,t)|}{N} && j = 1,2,\ldots,N,\quad k = 1,2,\ldots,\frac{N}{2}-1,\\
a_k(y,t) &= \frac{|U(k,y,t)|}{N} && j = 1,2,\ldots,N,\quad k = \frac{N}{2},
\end{aligned} \tag{2.9}$$

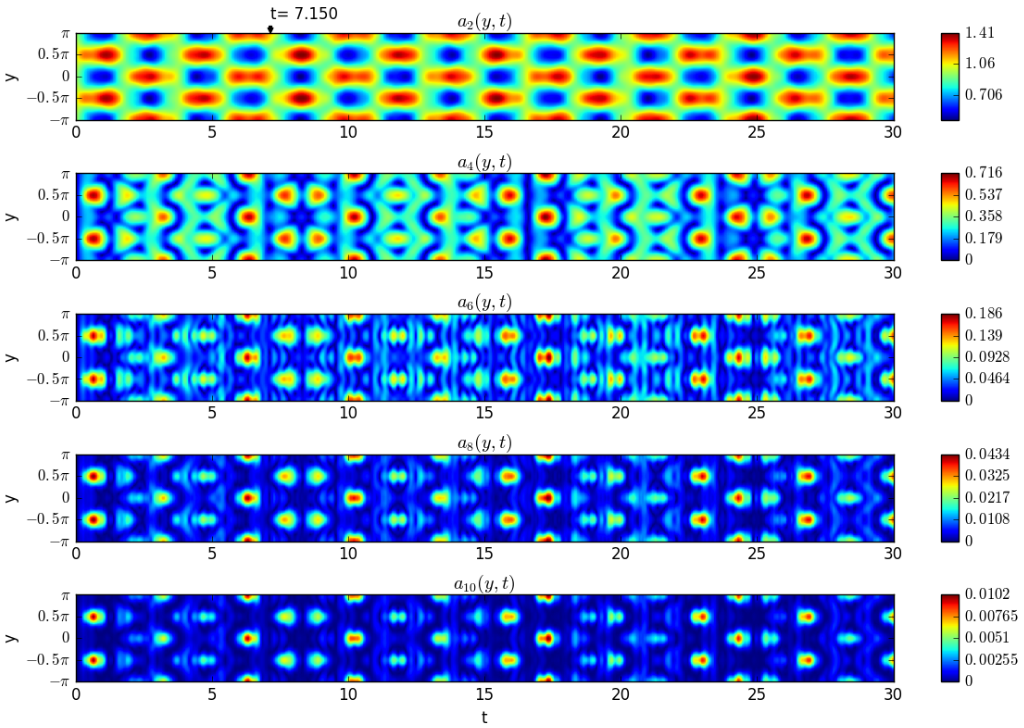

Figure 2.1: 2D spectral amplitudes. Only even spectral amplitudes are shown here since the odd spectral amplitudes were extremely small (10^{-12}) in the case of this specific initial condition

where $U(k,y,t) = F_x(u(x,y,t))$ and phase spectrum is defines as $\varphi_k(y,t) = \text{angle}U(k,y,t)$ (the argument of the complex number $U(k,y,t)$). Here the values $k = 0,1,\ldots,\frac{N}{2}$ are of interest [8].

Discrete spectral analysis (DSA) has been used do describe the behavior of 1D waves [8]. In the present paper the method is generalized for 2D waves.

If DSA is applied for 1D waves all the spectral amplitudes can be shown within a single axis with a different line referring to each spectral amplitude at different moments in time. However, in a situation with two spatial dimensions, such an image would refer to just a single discrete value of 1 spatial axis. Therefore, there should have to be N such images. In order to combat this problem, a new way of looking at spectral analysis was developed. In this depiction, a 3D pseudocolor[1] plot is formed for each spectral amplitude in questions. Since in most cases, only the first few spectral amplitudes are of interest (the higher spectral amplitudes are usually smaller), this results in only a few different images. An example of such a depiction is shown in Fig. 2.1. In this figure the horizontal axis refers to the time t while the vertical axis refers to the spatial axis y, the spectral amplitudes are that of the Fourier' transform of the x axis.

This kind of image shows us how the different spectral amplitudes behave in time. The most

[1] In a pseudocolor plot each value is given a distinct color – this is similar to maps that indicate height with colors

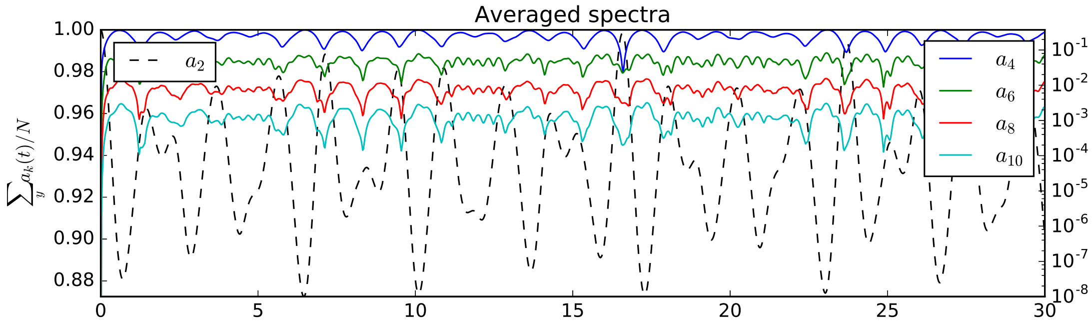

Figure 2.2: Spectral amplitudes averaged over y

immediate result from integration would be a set of 2D wave profiles at different times. In order to show these, one would have to use a 3D surface at different times. This, however, is often difficult to follow and does not show all the information about spatio-temporal behavior of waves. That is why spectral amplitudes are so useful in this case.

2.3.1 Recurrence

Recurrence was first introduced by Zabusky and Kruskal[10]. In general recurrence means that at a certain point in time (let us call this time t_R) a situation extremely similar to the initial condition is formed [8].

It is possible to use spectral amplitudes to track recurrence. When $t \to t_R$, $a_k(y,t) \to a_k(y,0)$, $\forall k, y$. If recurrence takes place at each $t_{R_j} \approx j t_R$, $j = 2, 3, \ldots$, the solution is called quasi-periodic.

From Fig. 2.1 it would seem that at $t \approx 7.15$ (shown with a little arrow in Fig. 2.1) the 2nd spectral amplitude looks similar to what it was at the beginning. To get the precise time of recurrence, the individual spectral amplitudes in Fig. 2.1 are averaged over y resulting in Fig. 2.2 which is much easier to read. Doing so allows (in case of a sinusaoidal initial condition) using local maxima of the first[2] spectral amplitude to find times where recurrence could occur. Figures very similar to Fig. 2.2 have previously been used in case of KdV [8, 11, 12]. In order to show the recurrence state even more clearly, a combination of spectra, averaged spectra and wave profiles is used. An example at $t = 7.15$ can be seen in Fig. 2.3. The spectra show how close the spectral amplitudes at t_R are to the spectral amplitudes at $t = 0$; the local maximum of the 2nd spectral amplitude can be seen from the averaged spectra; the distribution of the spectral amplitudes across y at $t = t_R$ can be seen more clearly on the lower left part and the wave profiles show how close the wave profile itself is to the initial condition. It is important to note that in figures such as Figs. 2.2 and 2.3, the summed spectra as well as the lower left part of the figure are both shown on two different y axes where the 1st (or the 2nd in cases like the current one where there are 2 periods within the domain) is on a separate (linear) axis (left side) and the rest of the spectral amplitudes are on a logarithmic axis (right side).

In case of Fig. 2.3 it is clear that the 2nd spectral amplitude has a local maximum at $t_R = 7.15$, however it is also clear that neither the spectra nor the wave profiles are exactly alike. This

[2] Or 2nd if the initial condition has 2 periods of the sinusoid

is to be expected. Another issue to point out is the fact that the wave profile at an arbitrary recurrence time t_R is not always in phase with the initial condition. That is why the wave profile at $t = t_R$ is shifted along the x axis to best match the initial condition. We are able to do this because the PSM operates in periodic boundary conditions. It will become apparant later (in subsection 4.1.2) that sometimes an additional shift with respect to the y axis will also be needed.

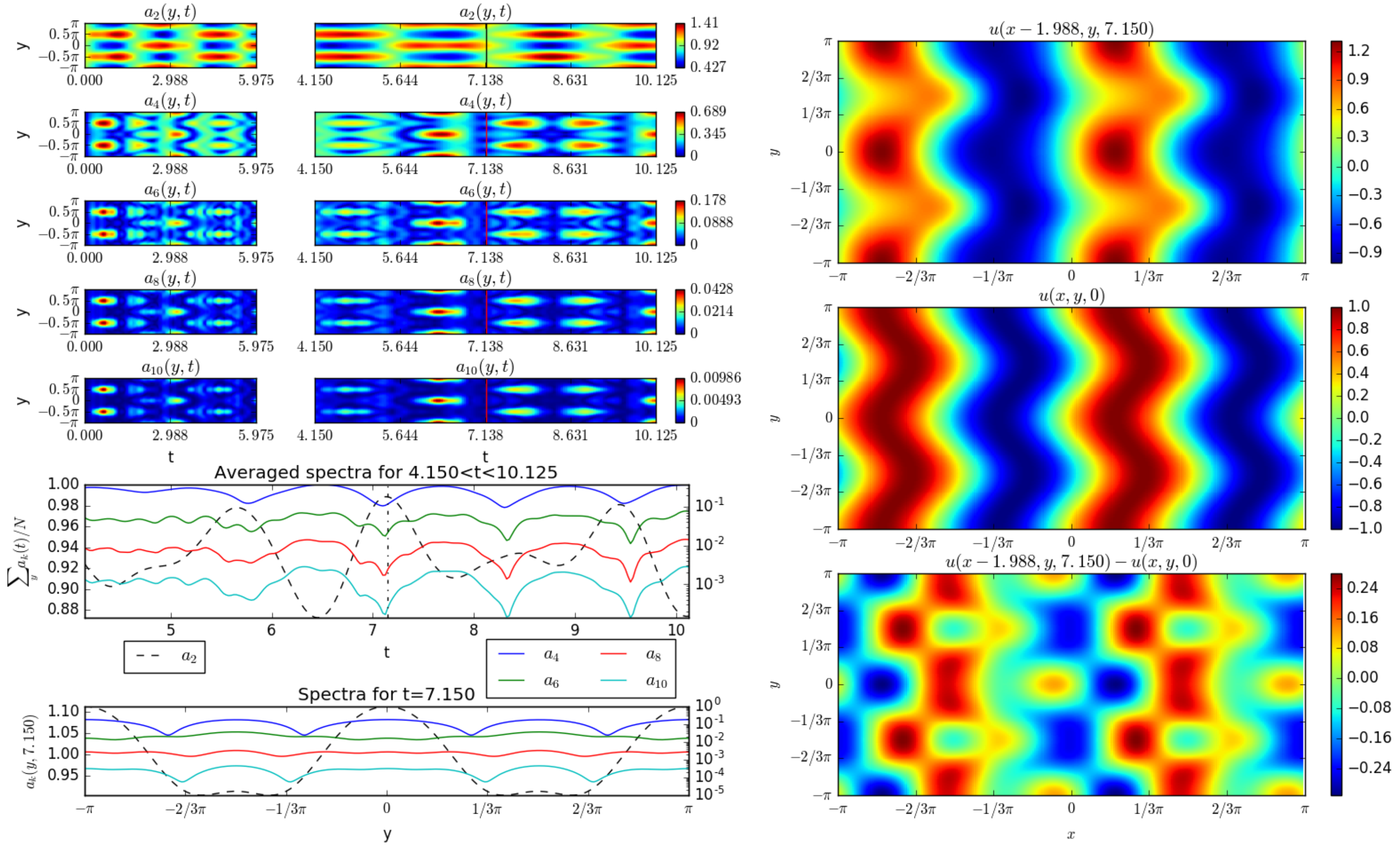

Figure 2.3: Recurrence at $t = 7.15$: full spectrum (upper left, time indicated by black/red line), averaged spectrum (middle left, time indicated by dotted line), spectra variance along y (lower left), wave profile at specified time (upper right), initial condition (middle right) and theri difference (lower right)

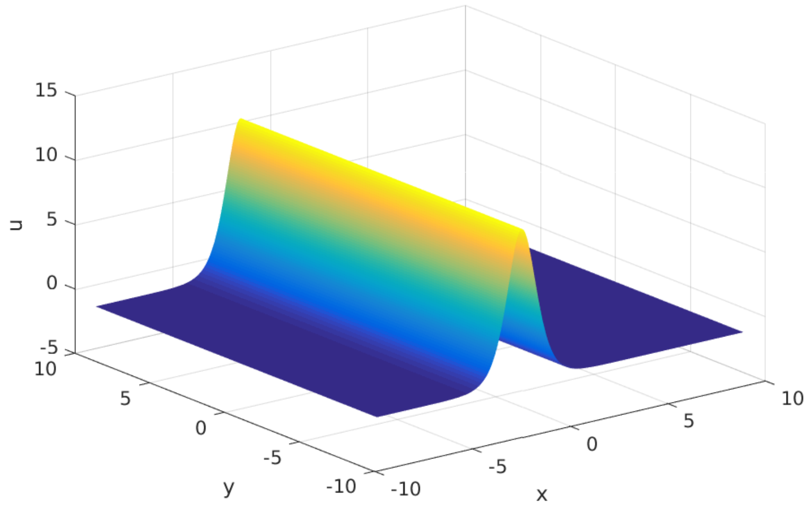

Figure 3.1: Initial condition (3.2) in case of parameter $x_0 = -2$ and 6π-periodic domain

3 Test cases and initial conditions

A number of different initial conditions were used over the course of the work described here. A few were used to determine that the approach taken is valid and another set was used to see interesting results.

3.1 Test cases

In order to compare the results to that of [13] as well as an exact solution, two initial conditions presented there were used. The first was essentially the exact solution of KdV equation

$$u(x, y, t) = \frac{12}{\cosh^2(x - x_0 - 4t)} \tag{3.1}$$

at $t = 0$ (see Fig. 3.1)

$$u(x, y, 0) = \frac{12}{\cosh^2(x - x_0)} \tag{3.2}$$

and the second was a slight variation of it (see Fig. 3.2)

$$u(x, y, 0) = \frac{12}{\cosh^2\left(x - x_0 + \beta\cos(\delta y)\right)}. \tag{3.3}$$

In both initial conditions x_0 is an arbitrary constant and in the second one β and δ are arbitrary constants as well. It is clear that in case of initial condition (3.2) u does not depend on the variable y. In this case the y dependent derivative $u_{yy} = 0$ and that part can be ignored, in which case the KP equation acts like the KdV equation.

Because of the constraint (1.4), the exact solution used was not (3.2) but rather a soliton solution which complies with the mentioned constraint was used. For this the solution in the form

$$\begin{aligned} u(x,y,t) &= u_\infty + \frac{12}{\cosh^2(x-x_0-ct)} \\ c &= u_\infty + 4 \end{aligned} \tag{3.4}$$

where u_∞ is an arbitrary constant that changes the zero level elevation of the solution and the speed c depends on that elevation was looked at. In order to find the initial condition that complies with (1.4), it needs to comply with it at $t=0$, i.e

$$\int_{-m\pi}^{m\pi} \left(u_\infty + \frac{12}{\cosh^2(x-x_0)} \right) dx = \left[u_\infty x + 12\tanh(x-x_0) \right] \Big|_{-m\pi}^{m\pi} = 0. \tag{3.5}$$

In case of $x_0 = 0$ and 6π periodic domain this gives

$$\begin{aligned} &2m\pi u_\infty + 24\tanh(m\pi) = 0 \\ &u_\infty = -\frac{12\tanh(m\pi)}{m\pi} \Rightarrow c = 4 - \frac{12\tanh(m\pi)}{m\pi}. \end{aligned} \tag{3.6}$$

In the end we get the solution in the form

$$u(x,y,t) = \frac{12}{\cosh^2\left(x - \left(4 - \frac{12\tanh(m\pi)}{m\pi}\right)t\right)} - \frac{12\tanh(m\pi)}{m\pi}. \tag{3.7}$$

Using exact solution (3.7) at $t=0$ as the initial condition allowed the calculated solution to be compared to the exact solution. Additionally it was shown how close the conserved value associated with the second conservation law is to the initial value. Such comparisons are shown in Fig. 3.3.

As the next test case, the initial condition (3.3) was used in case of parameters $x_0 = -25$, $\beta = 0.4$, $\sigma = 0.2$ and a 20π periodic domain. This is the same case that had previously been published in [13]. Both results looked extremely similar at $t=6$, the solution found within this work can be seen in Fig. 3.4.

Additionally, the KdV 2-soliton solution

$$u(x,y,0) = \frac{9}{\cosh^2(x-x_0)} \tag{3.8}$$

was also used to verify that the scheme works. The variable x_0 is arbitrary. This initial condition was not used so extensively, however. As in the previous case, the zero-level elevation of the solution would be dropped before integration in order to comply with (1.4).

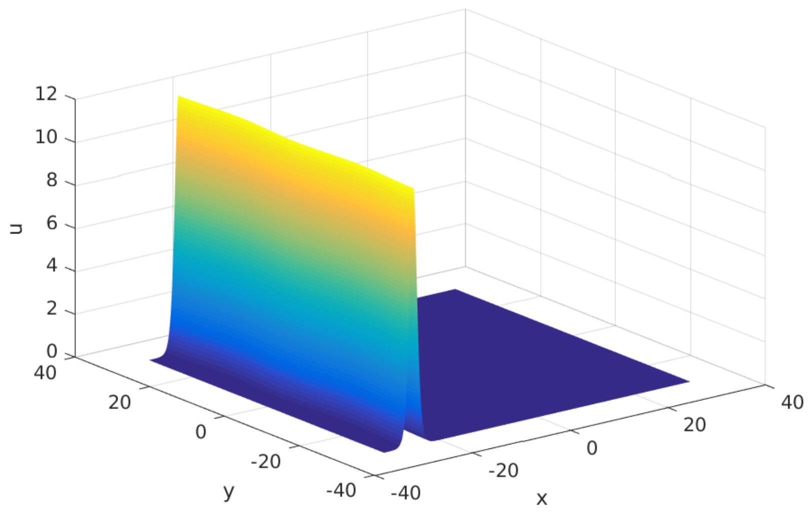

Figure 3.2: Initial condition (3.3) in case of parameters $x_0 = -25$, $\beta = 0.4$, $\delta = 0.2$ and 20π-periodic domain

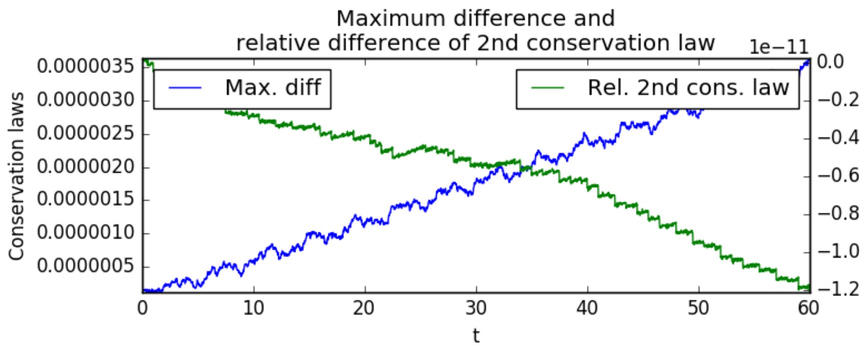

Figure 3.3: The difference between the calculated solution and the exact solution (3.7). The maximal difference with respect to the exact solution and the difference from the constant related to the second conservation law (1.3)

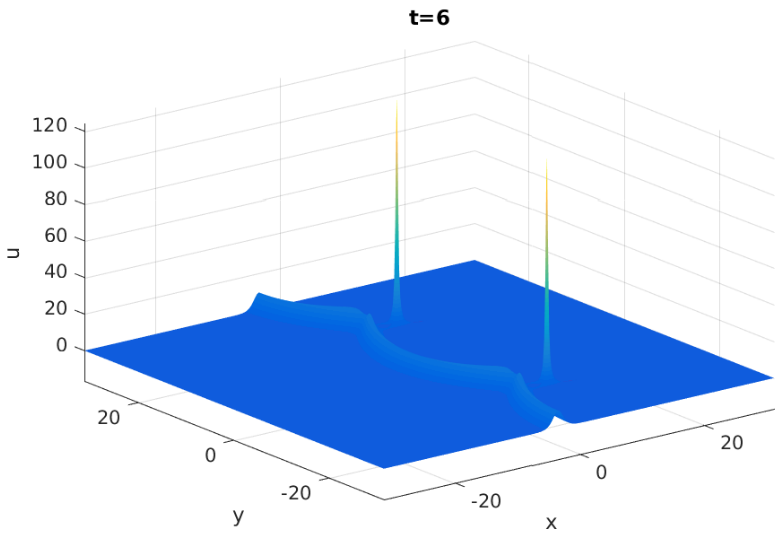

Figure 3.4: Wave profile at $t = 6$ in case of initial condition (3.3) and parameters $x_0 = -25$, $\beta = 0.4$ and $\delta = 0.2$ and 20π periodic domain

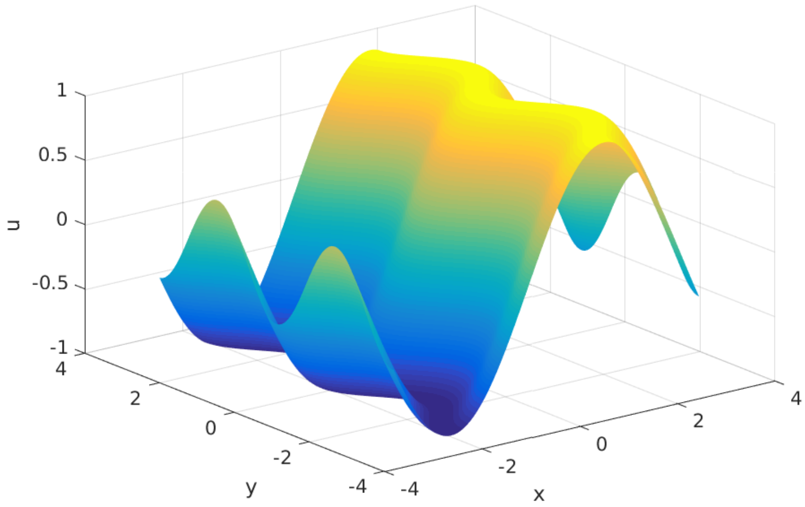

Figure 3.5: Initial condition (3.9) in case of parameters $a_x = 1$, $\beta = 0.4$ and $a_y = 2$

3.2 Most used initial conditions

Most of the work described here revolves around another type of initial condition, however. This initial condition is a sort of bendy sinusoid (see Fig. 3.5)

$$u(x, y, 0) = \sin(a_x x + \beta \cos(a_y y)). \tag{3.9}$$

The parameters a_x, β and a_y are arbitrary parameters, the first and last of which describe the amount of periods within the x and y axes within a 2π long space, respectively.

The initial condition described here was used most extensively, mainly because it satisfied the condition (1.4) at arbitrary time $t \geq 0$.

3.3 What could have gone better

During testing of the approach used, another initial condition was used. This initial condition was used to test if it can be compared to the exact lump soliton solution

$$u(x, y, t) = 4\frac{-(x + ay + 3(a^2 - b^2)t)^2 + b^2(y + 6at)^2 + 1/b^2}{\left[(x + ay + 3(a^2 - b^2)t)^2 + b^2(y + 6at)^2 + 1/b^2\right]^2}. \tag{3.10}$$

The initial condition used was the above exact solution at $t = 0$ (see Fig. 3.6)

$$u(x, y, 0) = 4\frac{-(x + ay)^2 + b^2 y^2 + 1/b^2}{\left[(x + ay)^2 + b^2 y^2 + 1/b^2\right]^2}. \tag{3.11}$$

The difference with respect to the exact solution and the calculated solution at $t = 0.01$ can be seen in Fig. 3.7. The solutions seem very similar at this time, but going forward with integration increases the difference between the two. This is because the PSM expects

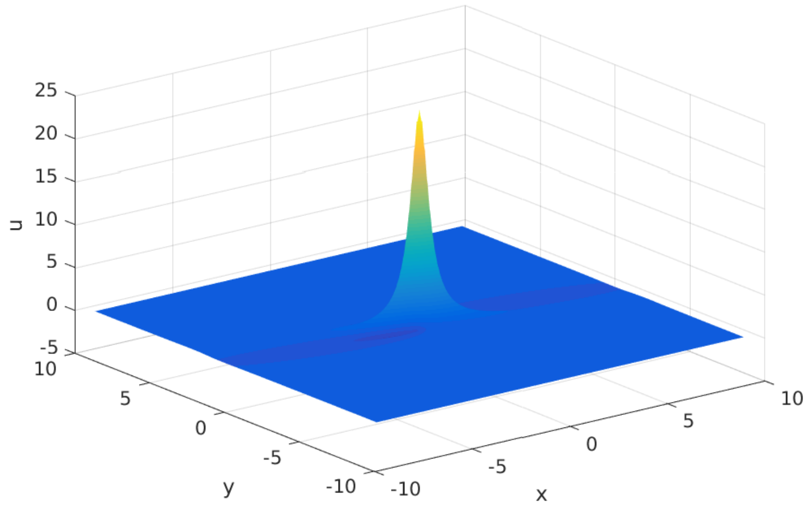

Figure 3.6: Initial condition (3.11) in case of parameters $a = b = 2.5$ and 6π periodic domain

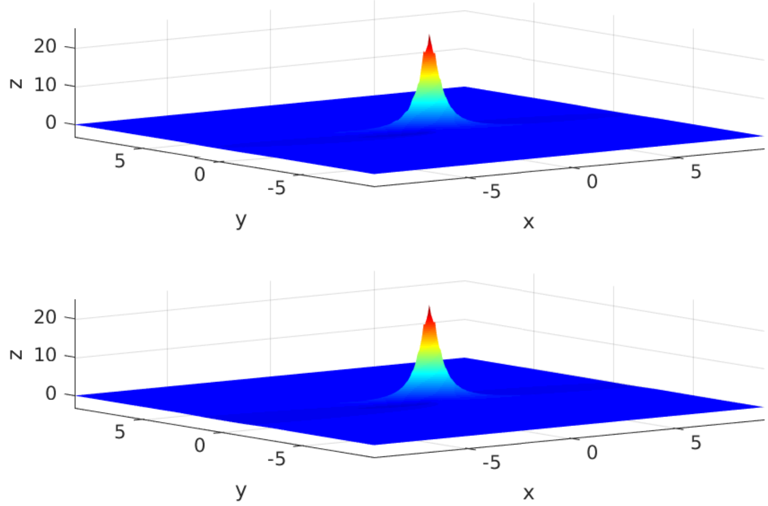

Figure 3.7: Calculated solution at $t = 0.01$ and initial condition (3.11) with parameters $a = b = 2.5$ (top) and exact solution (3.10) at $t = 0.01$ (bottom)

periodic boundary conditions, which the initial condition (nor the exact solution) do not satisfy. Furthermore, the initial condition does not satisfy the constraint (1.4) either. Therefore this initial condition was not used in the present study.

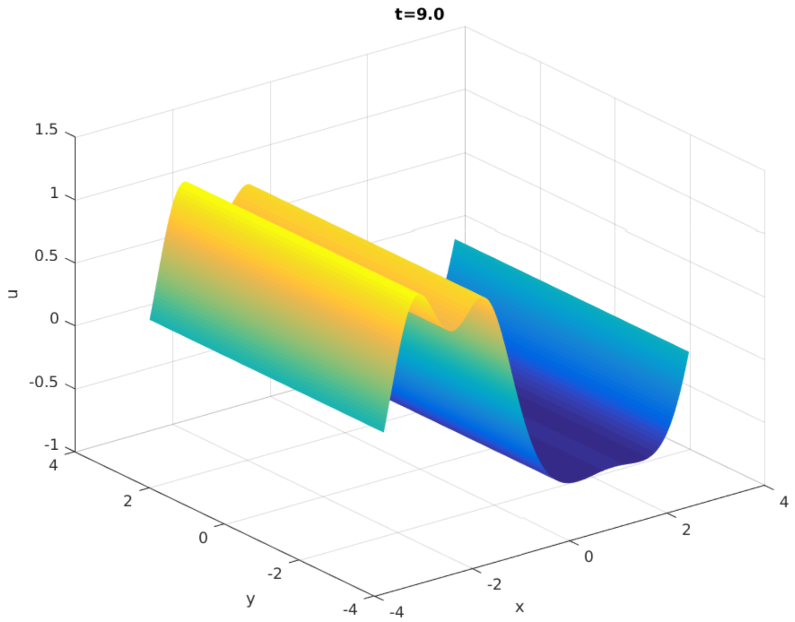

Figure 4.1: 3-soliton sequence in case of the initial condition (3.9) with parameters $a_x = 1$, $\beta = 0$ at $t = 9$

4 Analysis of solutions

4.1 Initial condition (3.9)

In this subsection we apply the initial condition (3.9). If we use the initial condition with the parameter $\beta = 0$ we get the harmonic initial condition previously used for the KdV equation [14–18]. More precisely, if we use $a_x = 1$ and $\beta = 0$, plus set the dispersion parameter $\alpha_2 = 0.1$, we get a 3-soliton sequence as depicted in Fig. 4.1 similarly to the KdV equation [14].

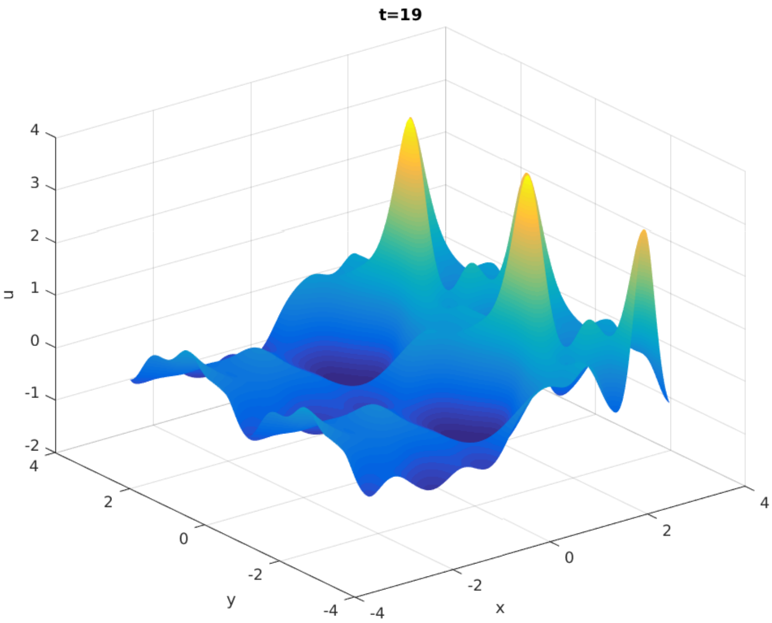

Figure 4.2: Solitary waves with parameters $a_x = 1$, $\beta = 0.4$, $a_y = 2$ and $d_l = 1.4$ at $t = 19$

If we set $\beta \neq 0$, we get solitary waves that are localized with respect to both the x and the y axes (as in Fig. 4.2). In most cases, the spectral amplitudes of each specific solution are shown. Otherwise a video would be needed to convey the time wise evolution[3].

4.1.1 Dispersion parameter and its effect on the solution

In this subsection we will be looking at the initial condition with parameters $a_x = 1$, $\beta = 0.4$ and $a_y = 2$. Different values of the dispersion parameter will be tested to see what kind of effect it has on the solution. In order to do that we introduce the logarithmic dispersion parameter d_l:

$$\alpha_2 = 10^{-d_l}. \tag{4.1}$$

Within this work, a solution was found for $d_l = 1, 1.1, ..., 1.8$. The amplitude spectra of the appropriate solutions are shown in Figs. 4.3-4.11. It is clear that the bigger the logarithmic dispersion parameter (and thus smaller the dispersion parameter) the less frequent the first spectral amplitude is dominating over the others (the (pseudo)period is lengthening such as in the KdV [16]). The precise value to be shifted is found such that the resulting profile most closely matches the initial condition. Therefore the solitary waves get thinner as the logarithmic dispersion parameter grows similarly to the KdV [15, 16]. Furthermore, it can be observed that the significance of higher spectral amplitudes increases when the logarithmic dispersion parameter increases as well.

The maximum height of the highest solitary wave also increases when the parameter d_l increases. This can be observed in Fig. 4.12.

[3]This method was also used within the work done here. Look for an appendix

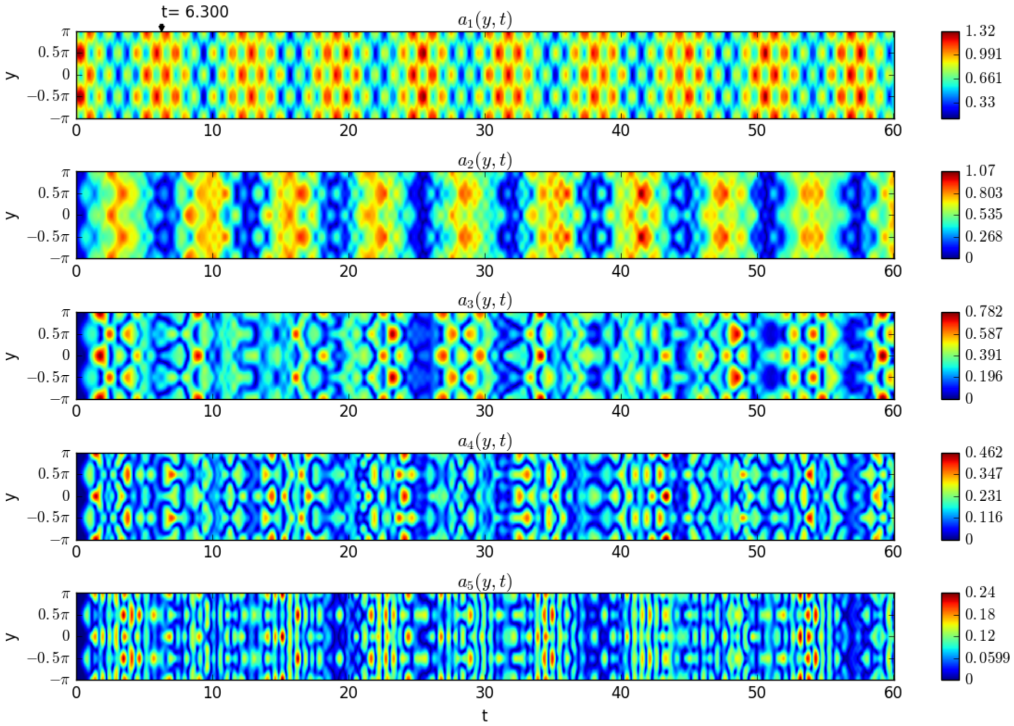

Figure 4.3: Spectral amplitude in case of the initial condition (3.9), parameters $a_x = 1$, $\beta = 0.4$, $a_y = 2$ and logaritmic dispersion parameter $d_l = 1$, recurrence times shown with a small arrow

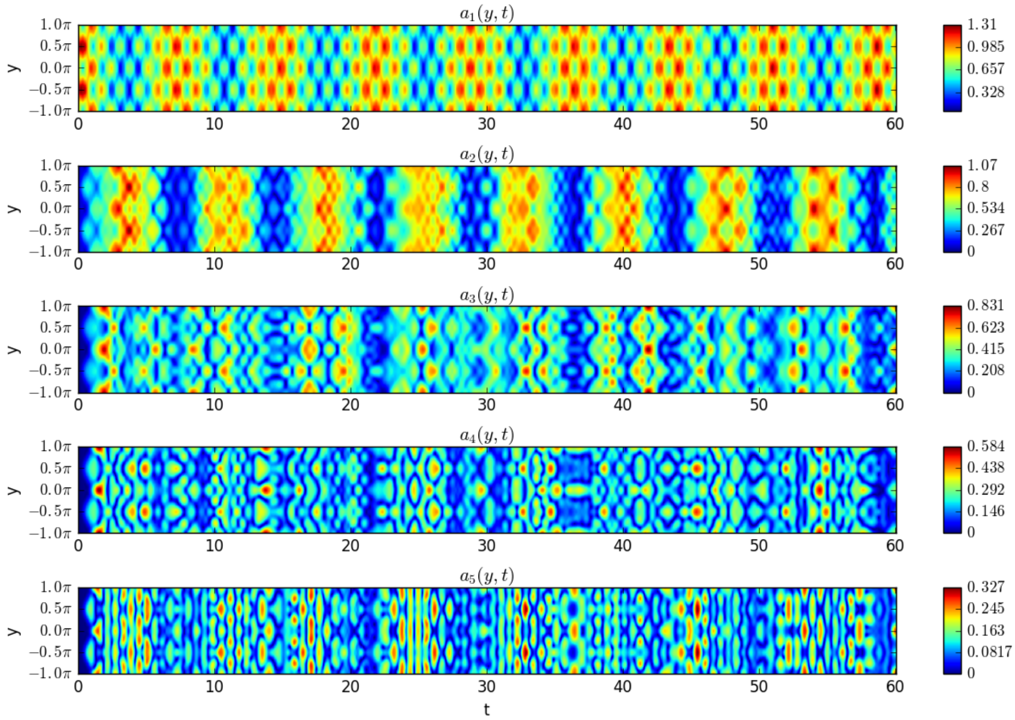

Figure 4.4: Spectral amplitude in case of the initial condition (3.9), parameters $a_x = 1$, $\beta = 0.4$, $a_y = 2$ and logaritmic dispersion parameter $d_l = 1.1$

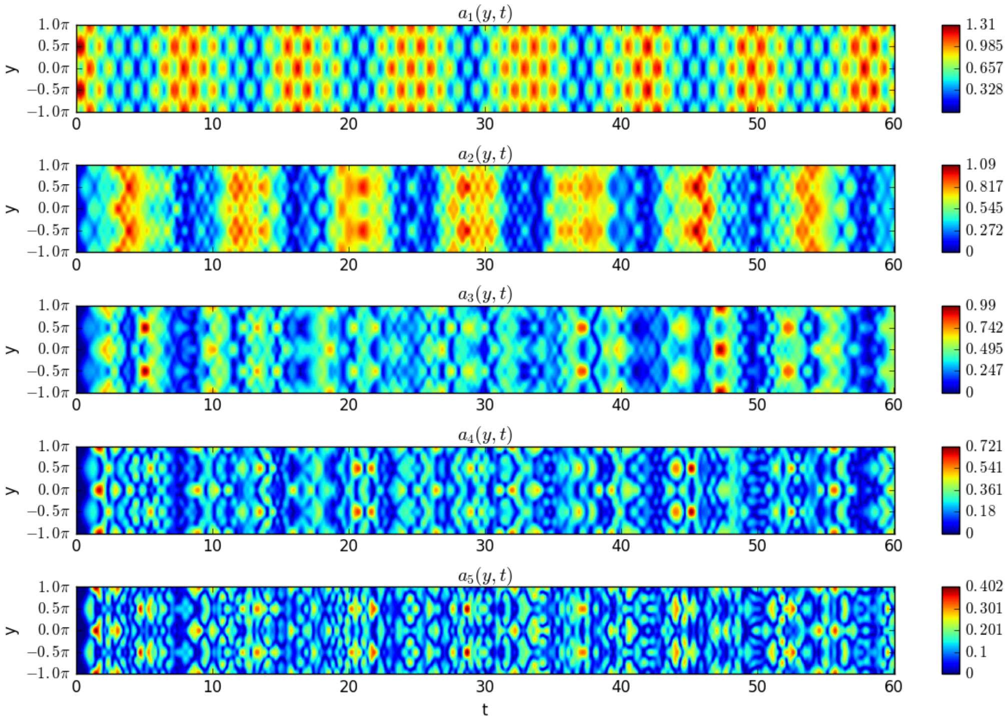

Figure 4.5: Spectral amplitude in case of the initial condition (3.9), parameters $a_x = 1$, $\beta = 0.4$, $a_y = 2$ and logaritmic dispersion parameter $d_l = 1.2$

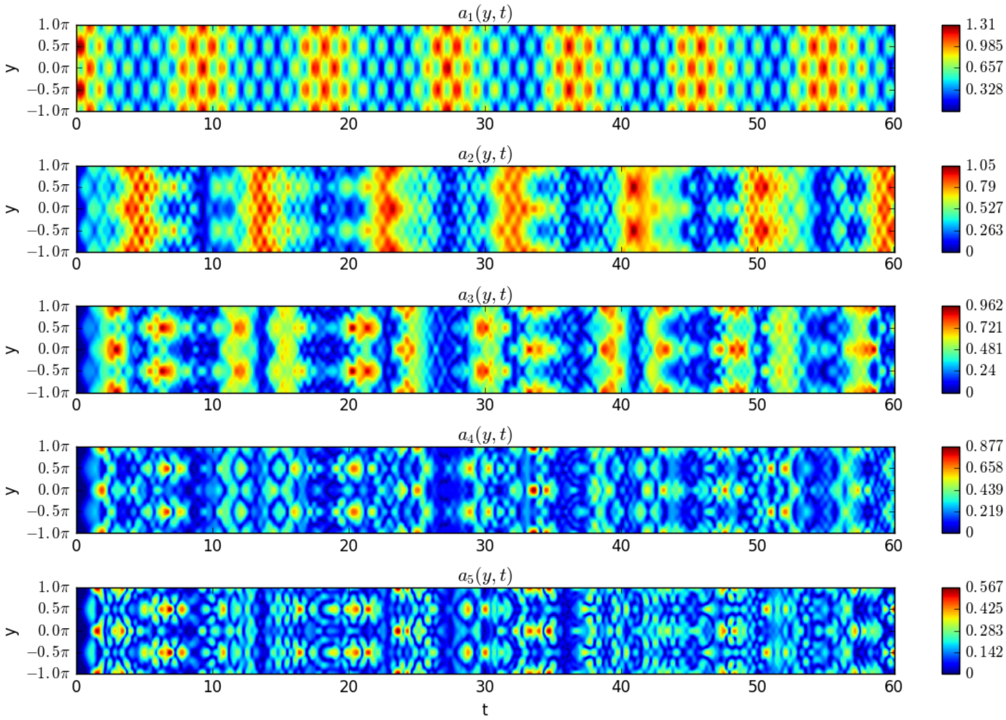

Figure 4.6: Spectral amplitude in case of the initial condition (3.9), parameters $a_x = 1$, $\beta = 0.4$, $a_y = 2$ and logaritmic dispersion parameter $d_l = 1.3$

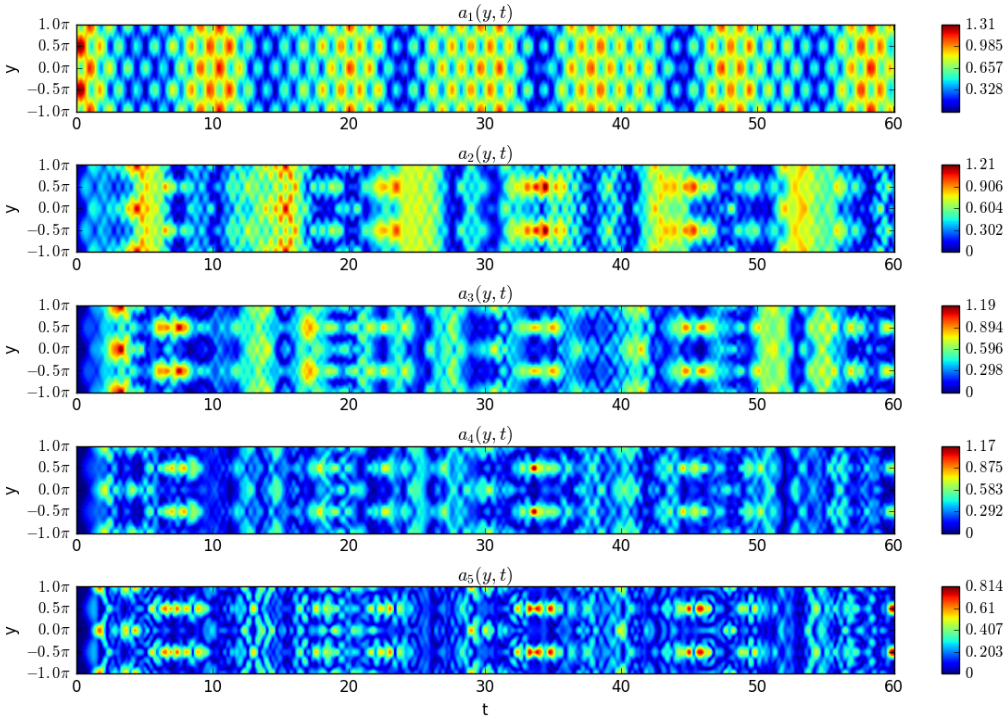

Figure 4.7: Spectral amplitude in case of the initial condition (3.9), parameters $a_x = 1$, $\beta = 0.4$, $a_y = 2$ and logaritmic dispersion parameter $d_l = 1.4$

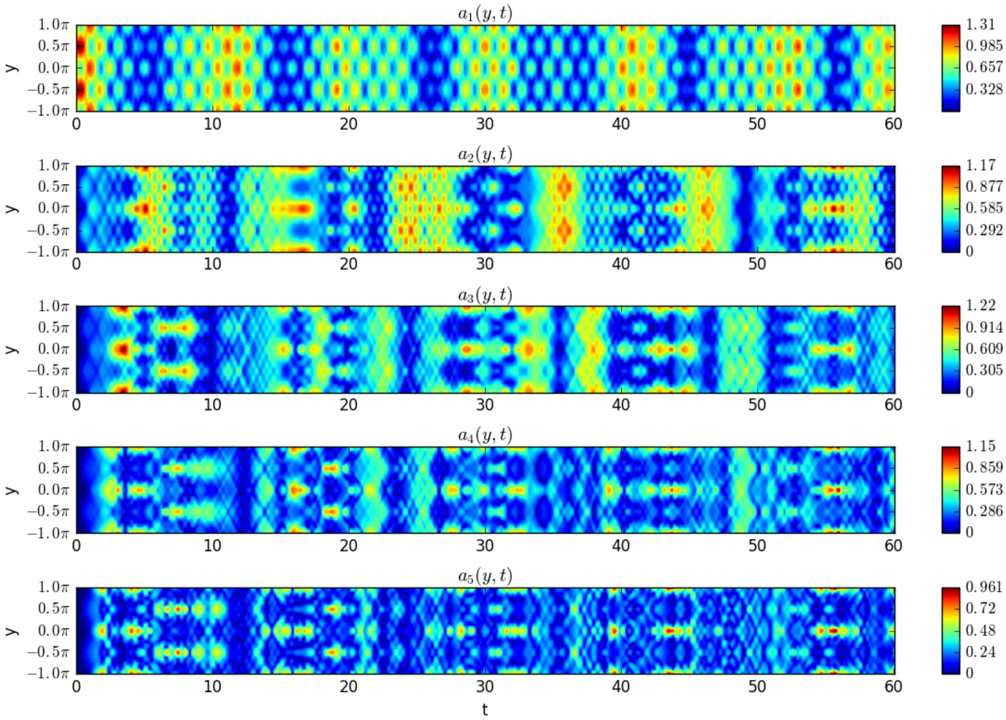

Figure 4.8: Spectral amplitude in case of the initial condition (3.9), parameters $a_x = 1$, $\beta = 0.4$, $a_y = 2$ and logaritmic dispersion parameter $d_l = 1.5$

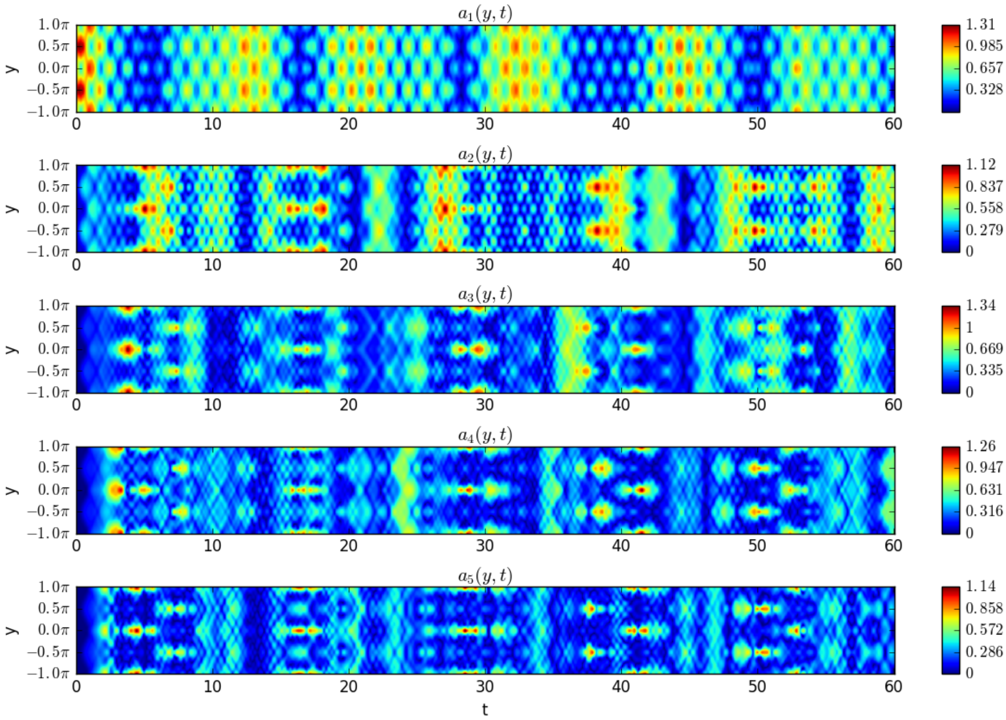

Figure 4.9: Spectral amplitude in case of the initial condition (3.9), parameters $a_x = 1$, $\beta = 0.4$, $a_y = 2$ and logaritmic dispersion parameter $d_l = 1.6$

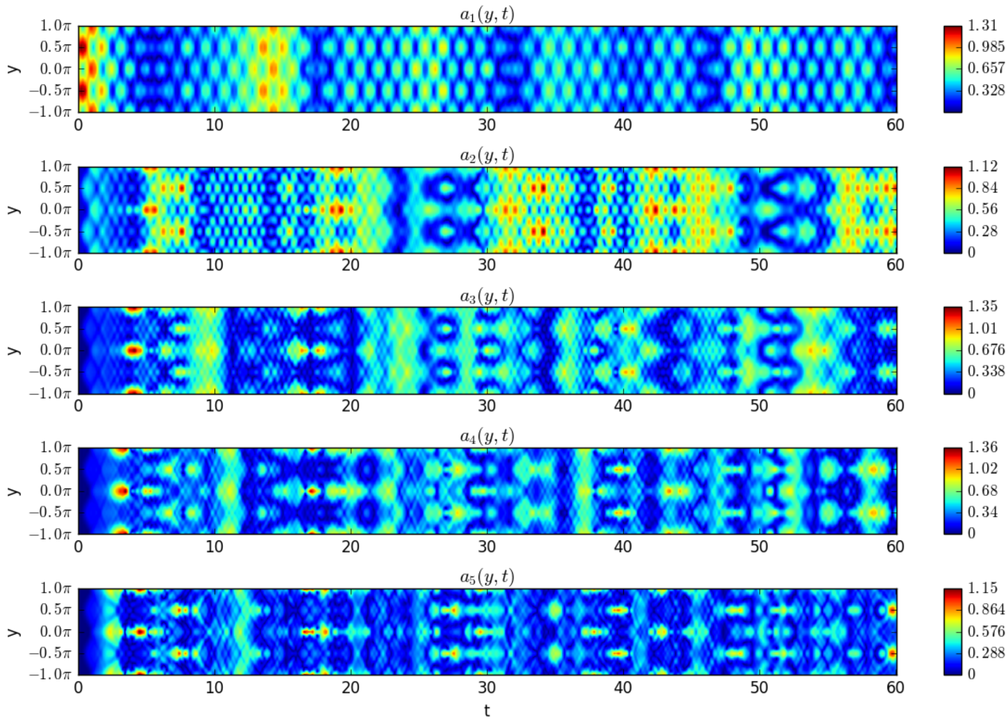

Figure 4.10: Spectral amplitude in case of the initial condition (3.9), parameters $a_x = 1$, $\beta = 0.4$, $a_y = 2$ and logaritmic dispersion parameter $d_l = 1.7$

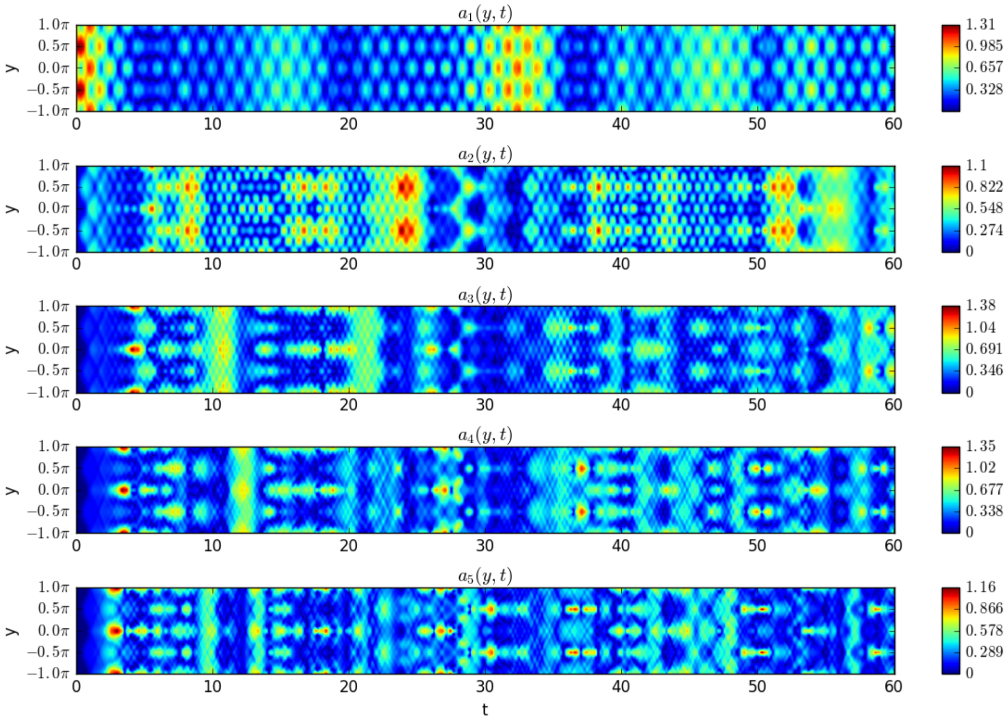

Figure 4.11: Spectral amplitude in case of the initial condition (3.9), parameters $a_x = 1$, $\beta = 0.4$, $a_y = 2$ and logaritmic dispersion parameter $d_l = 1.8$

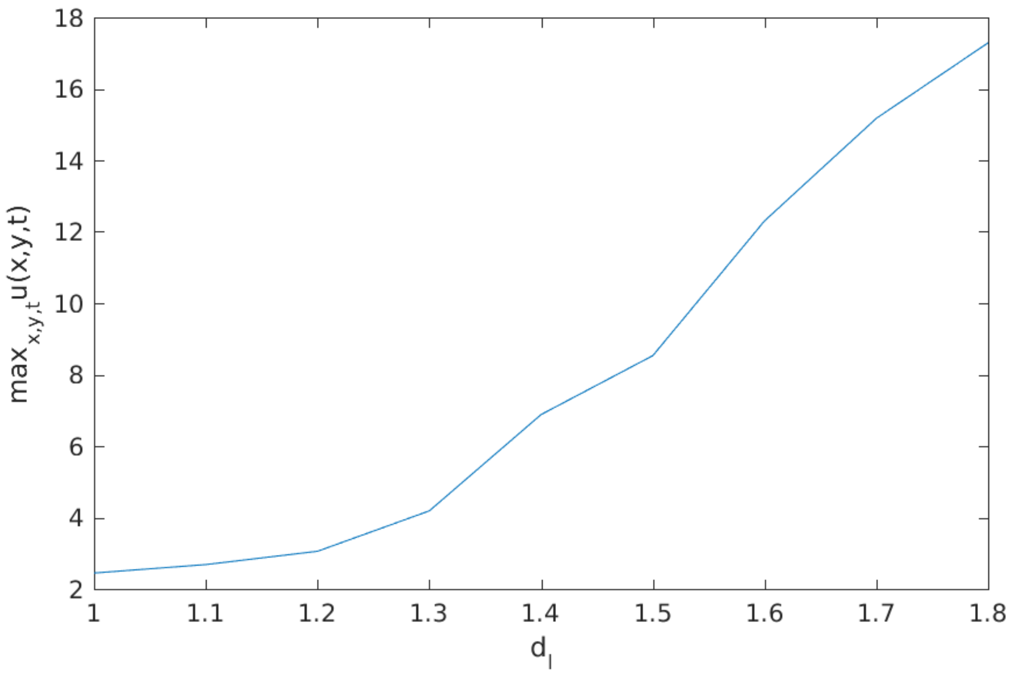

Figure 4.12: Maximum value of the solution as dependent from the logarithmic dispersion parameter d_l

4.1.2 Recurrence

As previously mentioned, recurrence states can exist within a KP solution. One such recurrence can be seen when using parameters $a_x = 2$, $\beta = 0.4$ and $a_y = 2$ and setting the dispersion parameter $\alpha_2 = 0.1$. Such an example was actually shown already in Fig. 2.3. The spectral amplitudes of the above mentioned solution are shown in Fig. 4.13. We looked at how to find recurrences in section 2.3.1.

Sometimes the spectral amplitudes can seem extremely similar, but it is not possible to see the recurrent state just be shifting the wave profile along the x axis. In these cases, there is a relatively big difference between the initial condition and wave profile at time t_R and a $\pi/2$ shift along the y axis is in order. The value $\pi/2$ comes from the fact that the transverse perubation is periodic and has 2 periods within the domain[4]. As mentioned before, such shifting is not a problem because the PSM operates within periodic boundary conditions with respect to both spatial axes. Having shifted the wave profile along the y axis, the difference between the initial condition and that of wave profile at $t_R = 16.6$ becomes rather small as seen in Fig. 4.14.

Such recurrences can, of course, take place in case of other initial conditions as well as dispersion parameter values. One such example is in case of dispersion parameter $\alpha_2 = 1$ and

[4]We will see that there are only 2 different phases the solutions take in subsection 4.1.4

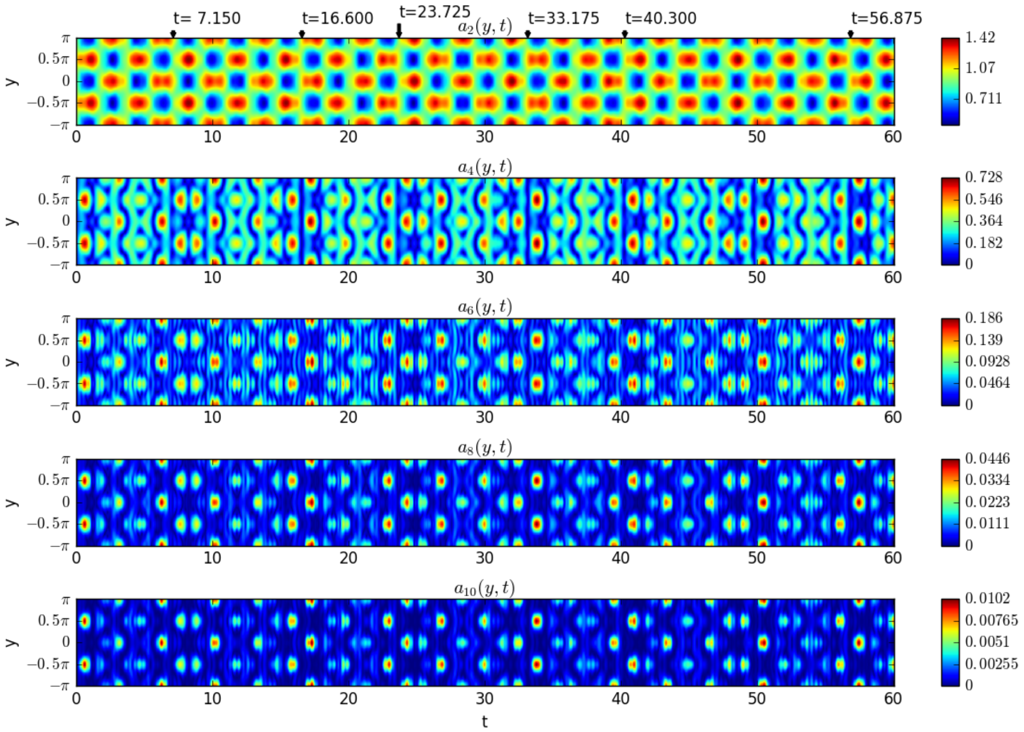

Figure 4.13: Spectral amplitudes in case of parameters $a_x = 2$, $\beta = 0.4$, $a_y = 2$ and $\alpha_2 = 0.1$, recurrence times shown with a small arrow

initial condition with parameters $a_x = 2$, $\beta = 0.4$ and $a_y = 2$, the amplitude spectrum of which is shown in Fig. 4.19.

It is clear that in this case a recurrence state is at $t_R = 1.6$ (see Fig. 4.20). Additionally, it is evident that the differences are small in this case as well. From Fig. 4.19 it would also seem that at $t = 6.375$ a period of evolution ends and a new and similar one begins for this solution.

A number of similar recurrences can be observed for a multitude of different initial condition parameters and dispersion parameters. Some of these are described in tabel 4.1.

In case of $d_l > 1$ no proper recurrence could be found. This is because in cases where the logarithmic dispersion parameter has a higher value, the higher order spectral amplitudes are big enough (Figs. 4.3 - 4.11) to have a significant effect on the wave profile. It therefore becomes extremely rare for the different spectral amplitudes to all correspond to the initial condition at the same time. One of the more promising examples of attmpting to find a recurrence for the case $\alpha_2 = 1.1$ is shown in Fig. 4.38. Similar results have been obtained in case of the KdV equation as well [18].

It must be noted that in the case of two spatial axes, finding recurrence states becomes more difficult than it is in the one dimensional case. It is not always easy to notice recurrence times from the amplitud spectrum and even if one manages to notice them, the solution at that time might differ from the initial condition quite significantly. The reason for this is that energy can

Table 4.1: Recurrences for solutions for different initial conditions and dispersion parameter values

Initial condition parameters	Dispersion parameter	t_R	Recurrence Fig.	Spectra Fig.
$a_x = 2$, $\beta = 0.4$, $a_y = 2$	$\alpha_2 = 0.1$	7.15	2.3	4.13
		16.6	4.14	
		23.725	4.15	
		33.175	4.16	
		40.3	4.17	
		56.875	4.18	
$a_x = 2$, $\beta = 0.4$, $a_y = 2$	$\alpha_2 = 1$	1.6	4.20	4.19
		3.175	4.21	
		4.75	4.22	
		6.325	4.23	
		9.525	4.24	
		11.1	4.25	
		12.7	4.26	
		14.275	4.27	
		15.85	4.28	
		17.425	4.29	
		18.975	4.30	
		20.575	4.31	
		22.225	4.32	
		23.8	4.33	
		25.375	4.34	
		26.95	4.35	
		28.525	4.36	
$a_x = 1$, $\beta = 0.4$, $a_y = 2$	$\alpha_2 = 0.1$	25.5	4.37	4.3
$a_x = 2$, $\beta = 0.2$, $a_y = 2$	$\alpha_2 = 0.1$	10.9	4.40	4.39
		18.125	4.41	
		38.6	4.42	
		45.825	4.43	
		56.7	4.44	
$x_x = 2$, $\beta = 0.3$, $a_y = 2$	$\alpha_2 = 0.1$	7.2	4.46	4.45
		16.7	4.47	
		27.575	4.48	
		34.725	4.49	
		51.4	4.50	

be distributed not only between spectral amplitudes within the same y value, but also across different values of y. All in all, this means that a spectral amplitude can (locally) have a higher value at a later time than it did in the initial condition. This was not possible in the 1D case.

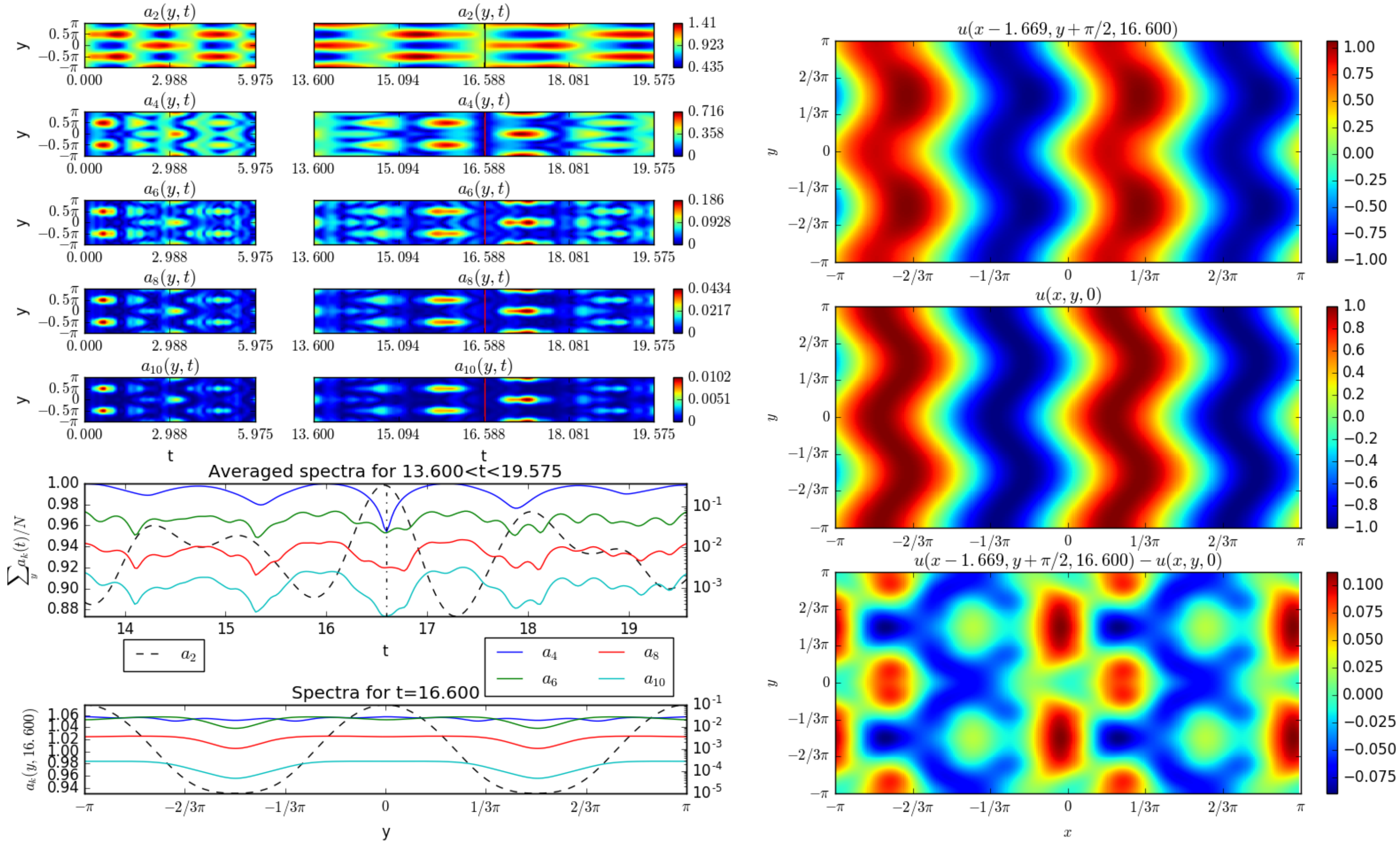

Figure 4.14: Recurrence at $t = 16.6$: full spectrum (upper left, time indicated by black/red line), averaged spectrum (middle left, time indicated by dotted line), spectra variance along y (lower left), wave profile at specified time (upper right), initial condition (middle right) and theri difference (lower right)

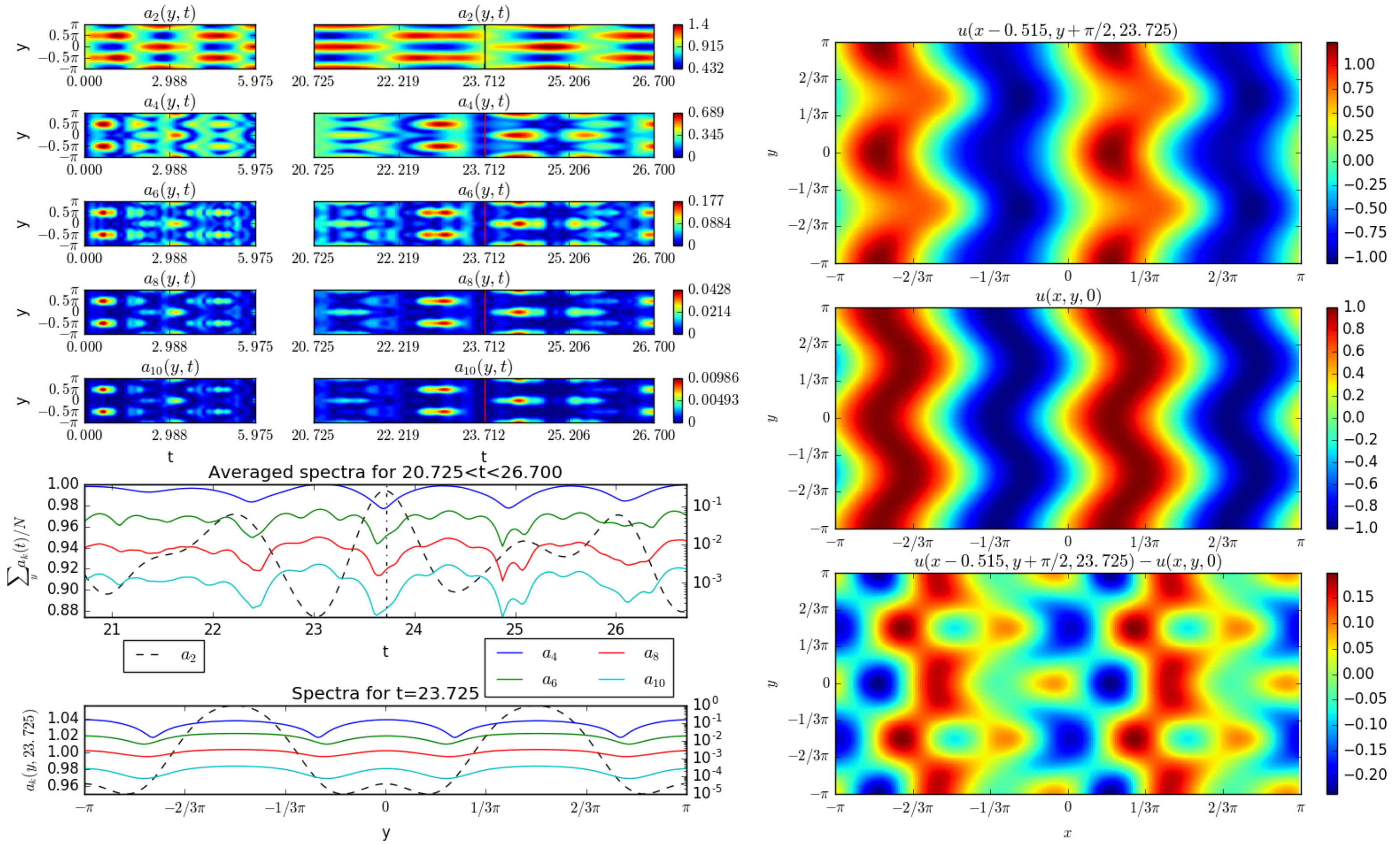

Figure 4.15: Recurrence in case of (3.9) and parameters $a_x = 2$, $\beta = 0.4$, $a_y = 2$ and dispersion parameter $\alpha_2 = 0.1$ at $t = 23.725$: full spectrum (upper left, time indicated by black/red line), averaged spectrum (middle left, time indicated by dotted line), spectra variance along y (lower left), wave profile at specified time (upper right), initial condition (middle right) and theri difference (lower right)

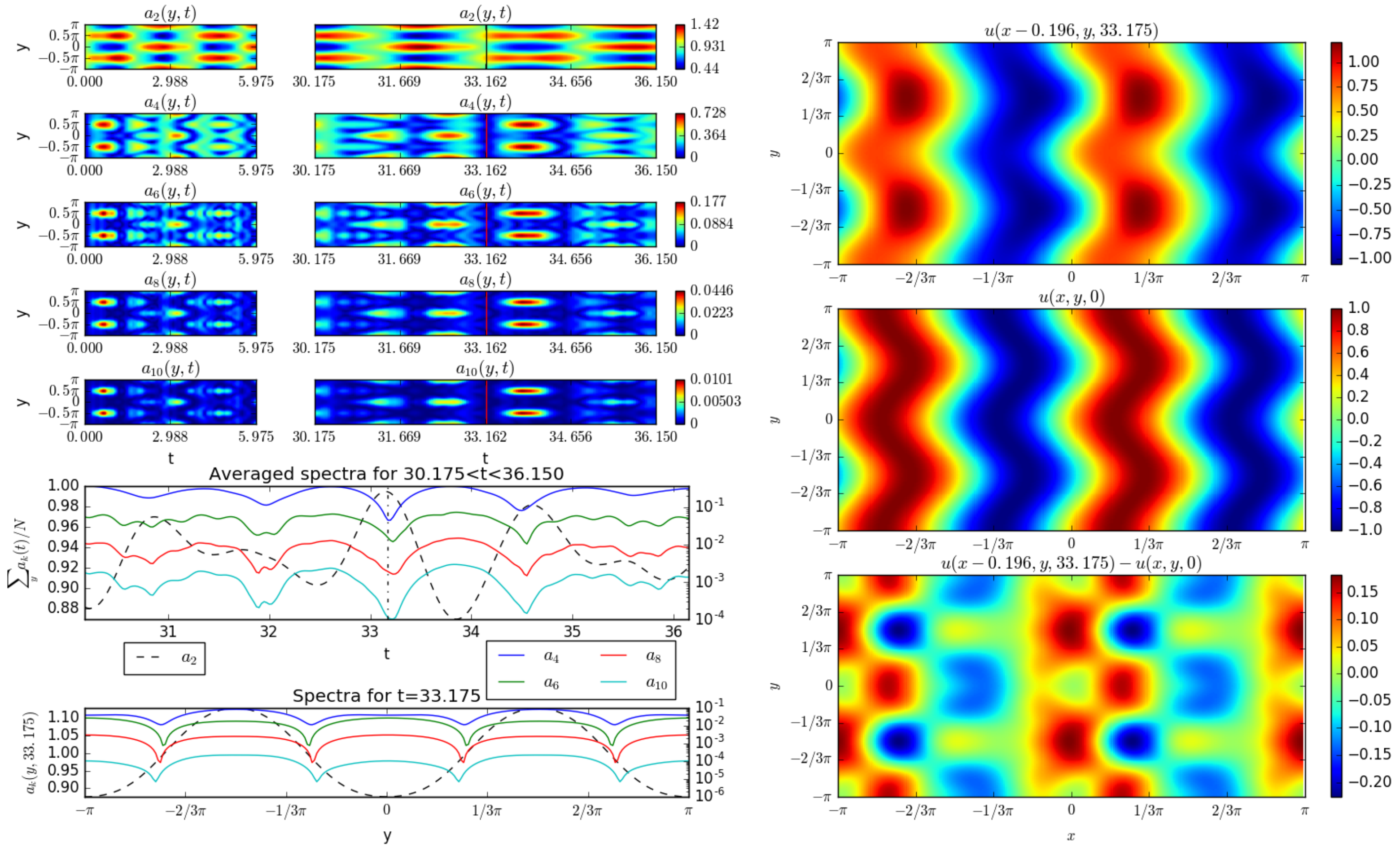

Figure 4.16: Recurrence in case of (3.9) and parameters $a_x = 2$, $\beta = 0.4$, $a_y = 2$ and dispersion parameter $\alpha_2 = 0.1$ at $t = 33.175$: full spectrum (upper left, time indicated by black/red line), averaged spectrum (middle left, time indicated by dotted line), spectra variance along y (lower left), wave profile at specified time (upper right), initial condition (middle right) and theri difference (lower right)

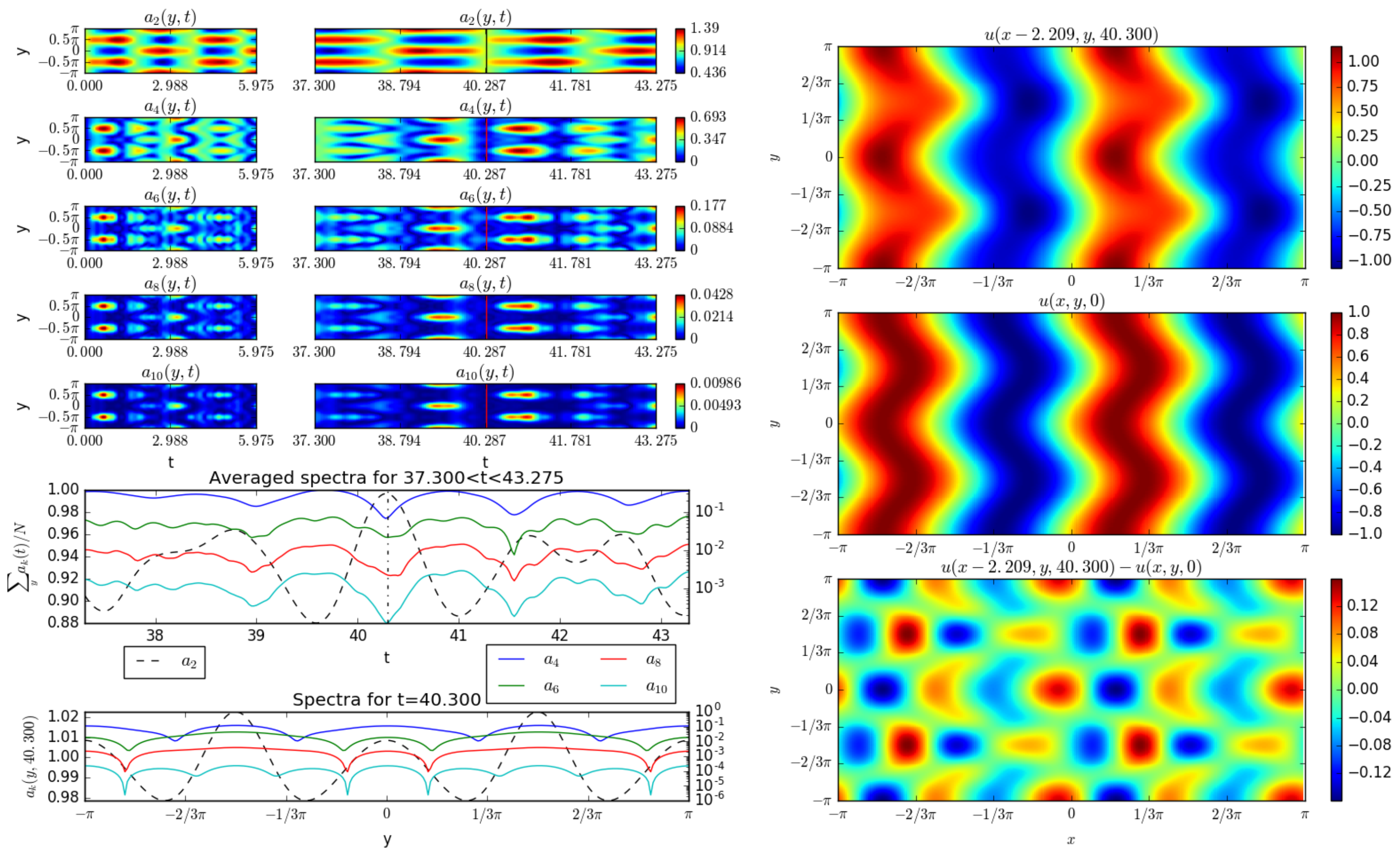

Figure 4.17: Recurrence in case of (3.9) and parameters $a_x = 2$, $\beta = 0.4$, $a_y = 2$ and dispersion parameter $\alpha_2 = 0.1$ at $t = 40.3$: full spectrum (upper left, time indicated by black/red line), averaged spectrum (middle left, time indicated by dotted line), spectra variance along y (lower left), wave profile at specified time (upper right), initial condition (middle right) and theri difference (lower right)

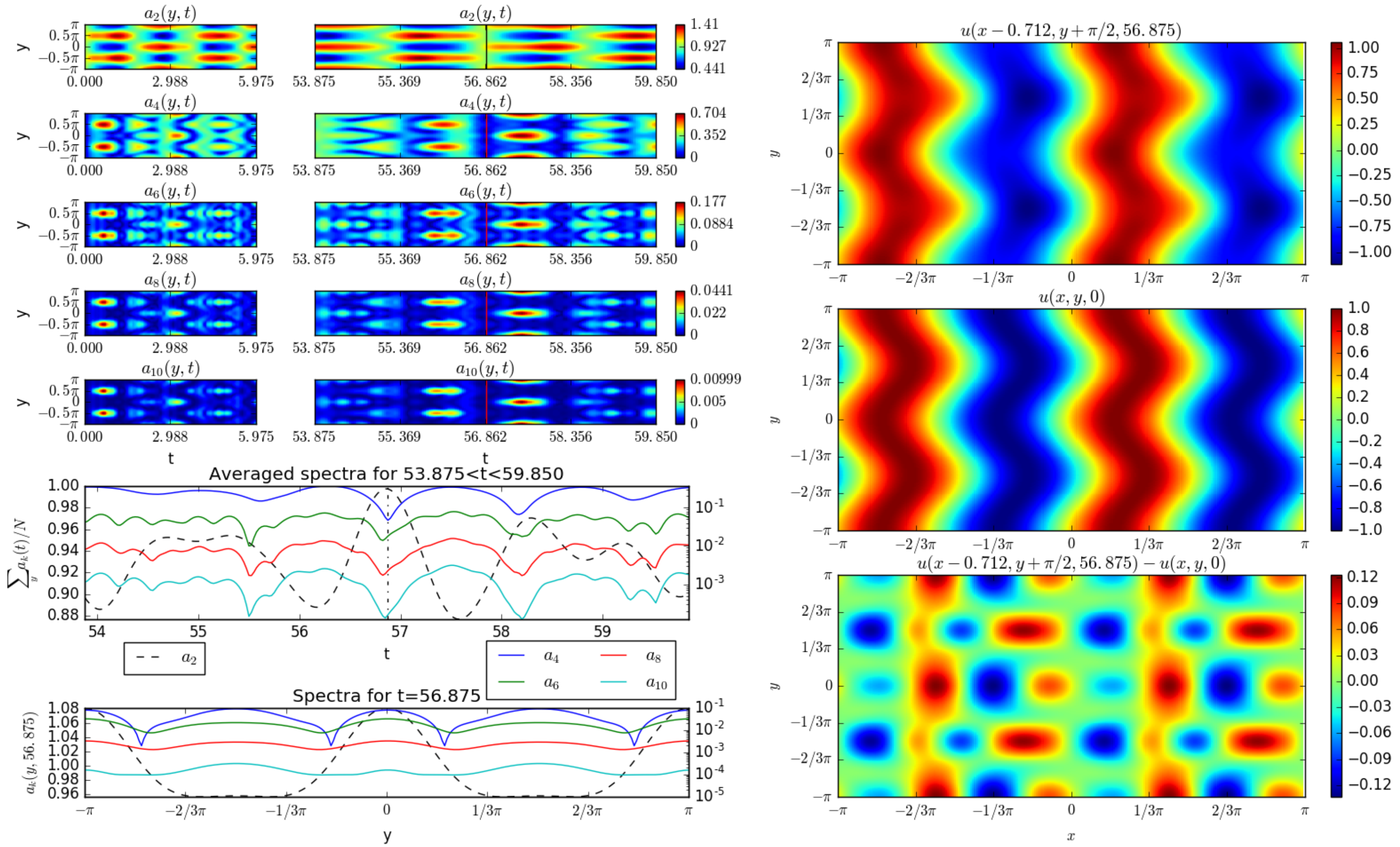

Figure 4.18: Recurrence in case of (3.9) and parameters $a_x = 2$, $\beta = 0.4$, $a_y = 2$ and dispersion parameter $\alpha_2 = 0.1$ at $t = 56.875$: full spectrum (upper left, time indicated by black/red line), averaged spectrum (middle left, time indicated by dotted line), spectra variance along y (lower left), wave profile at specified time (upper right), initial condition (middle right) and theri difference (lower right)

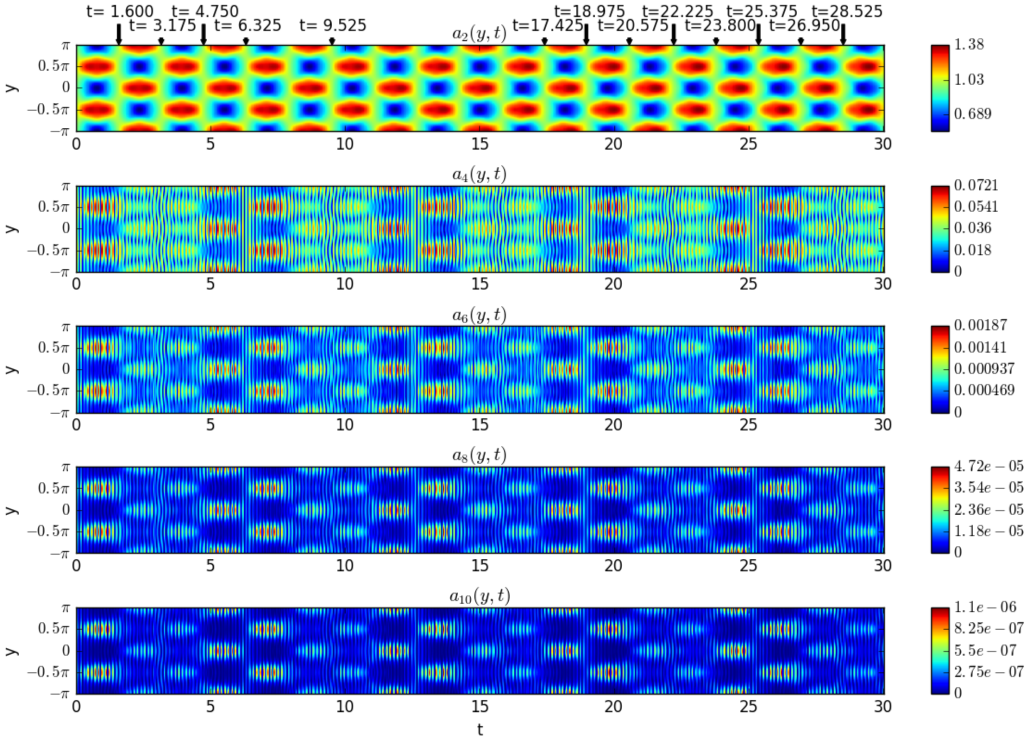

Figure 4.19: Spectral amplitudes in case of parameters $a_x = 2$, $\beta = 0.4$, $a_y = 2$ and $\alpha_2 = 1$, recurrence times shown with a small arrow

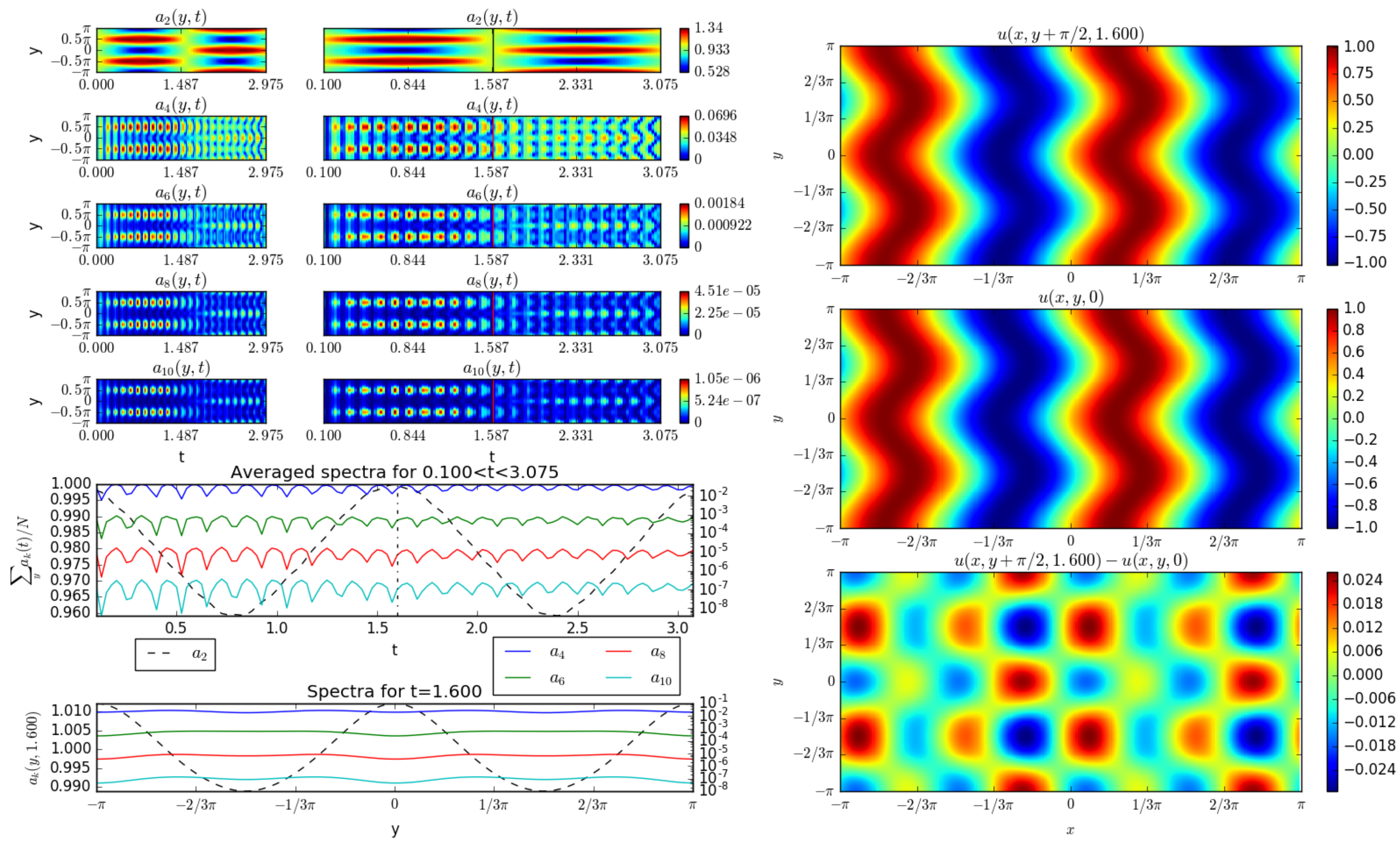

Figure 4.20: Recurrence in case of (3.9) and parameters $a_x = 2$, $\beta = 0.4$, $a_y = 2$ and dispersion parameter $\alpha_2 = 1$ at $t = 1.6$: full spectrum (upper left, time indicated by black/red line), averaged spectrum (middle left, time indicated by dotted line), spectra variance along y (lower left), wave profile at specified time (upper right), initial condition (middle right) and theri difference (lower right)

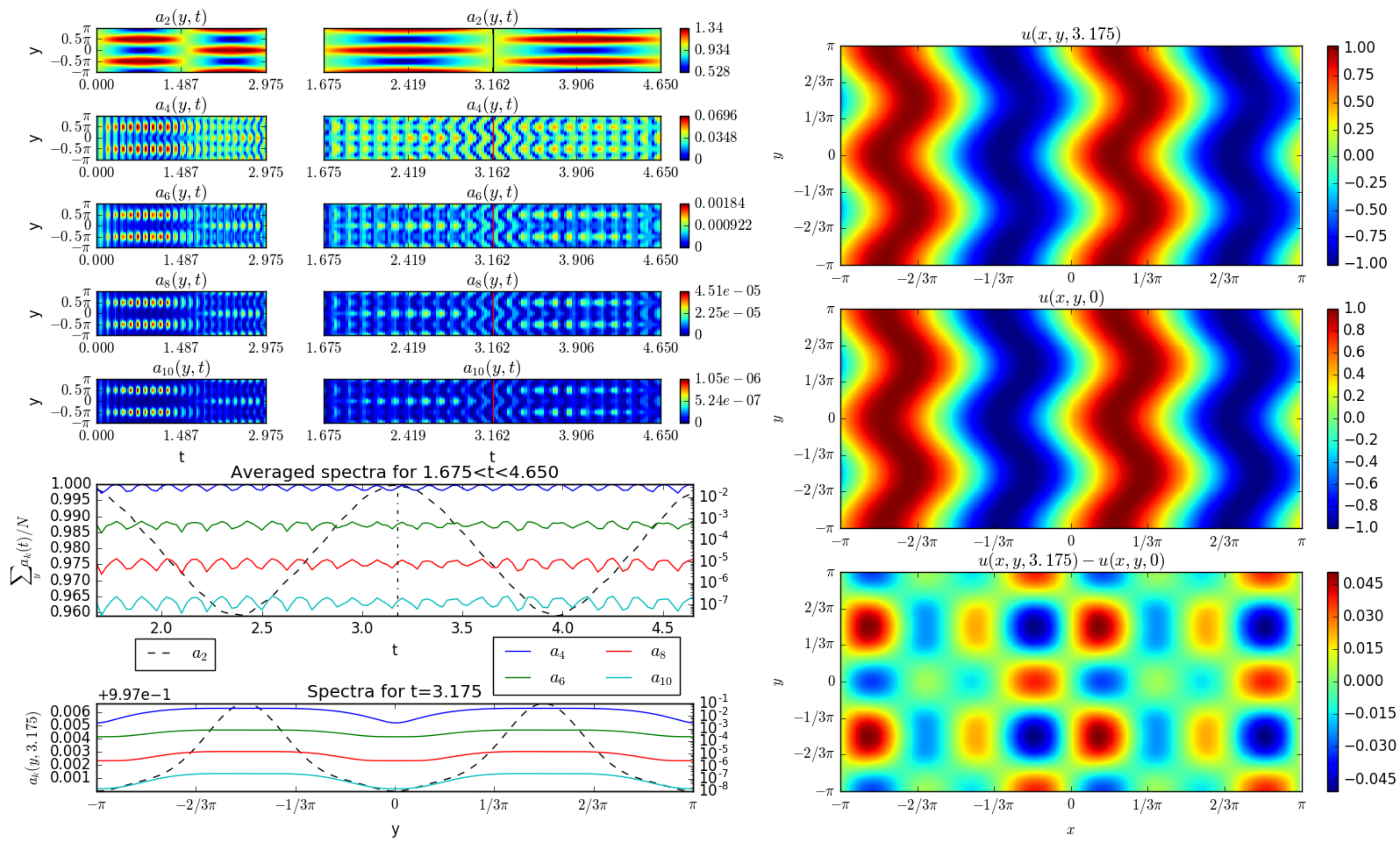

Figure 4.21: Recurrence in case of (3.9) and parameters $a_x = 2$, $\beta = 0.4$, $a_y = 2$ and dispersion parameter $\alpha_2 = 1$ at $t = 3.175$: full spectrum (upper left, time indicated by black/red line), averaged spectrum (middle left, time indicated by dotted line), spectra variance along y (lower left), wave profile at specified time (upper right), initial condition (middle right) and theri difference (lower right)

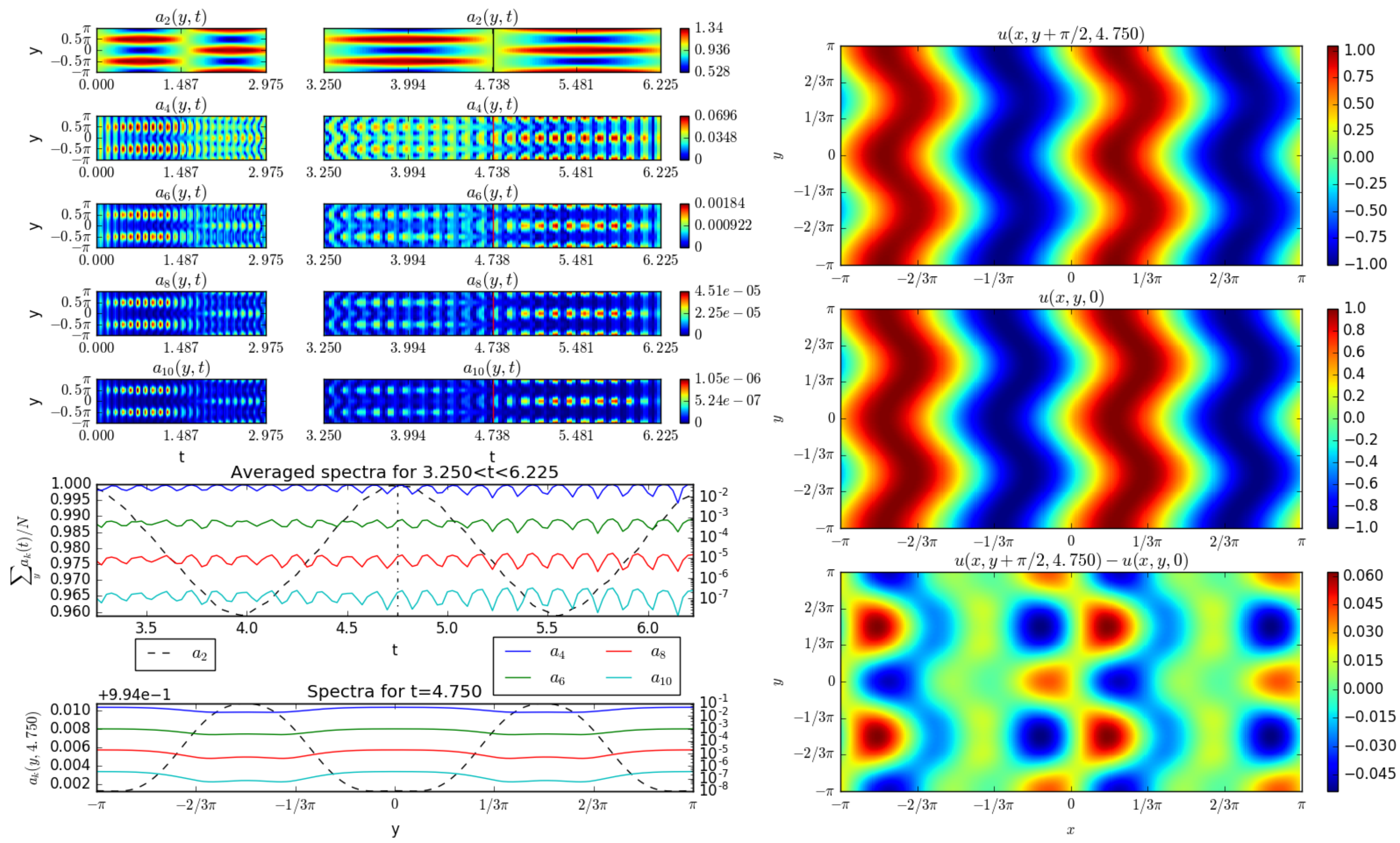

Figure 4.22: Recurrence in case of (3.9) and parameters $a_x = 2$, $\beta = 0.4$, $a_y = 2$ and dispersion parameter $\alpha_2 = 1$ at $t = 4.75$: full spectrum (upper left, time indicated by black/red line), averaged spectrum (middle left, time indicated by dotted line), spectra variance along y (lower left), wave profile at specified time (upper right), initial condition (middle right) and theri difference (lower right)

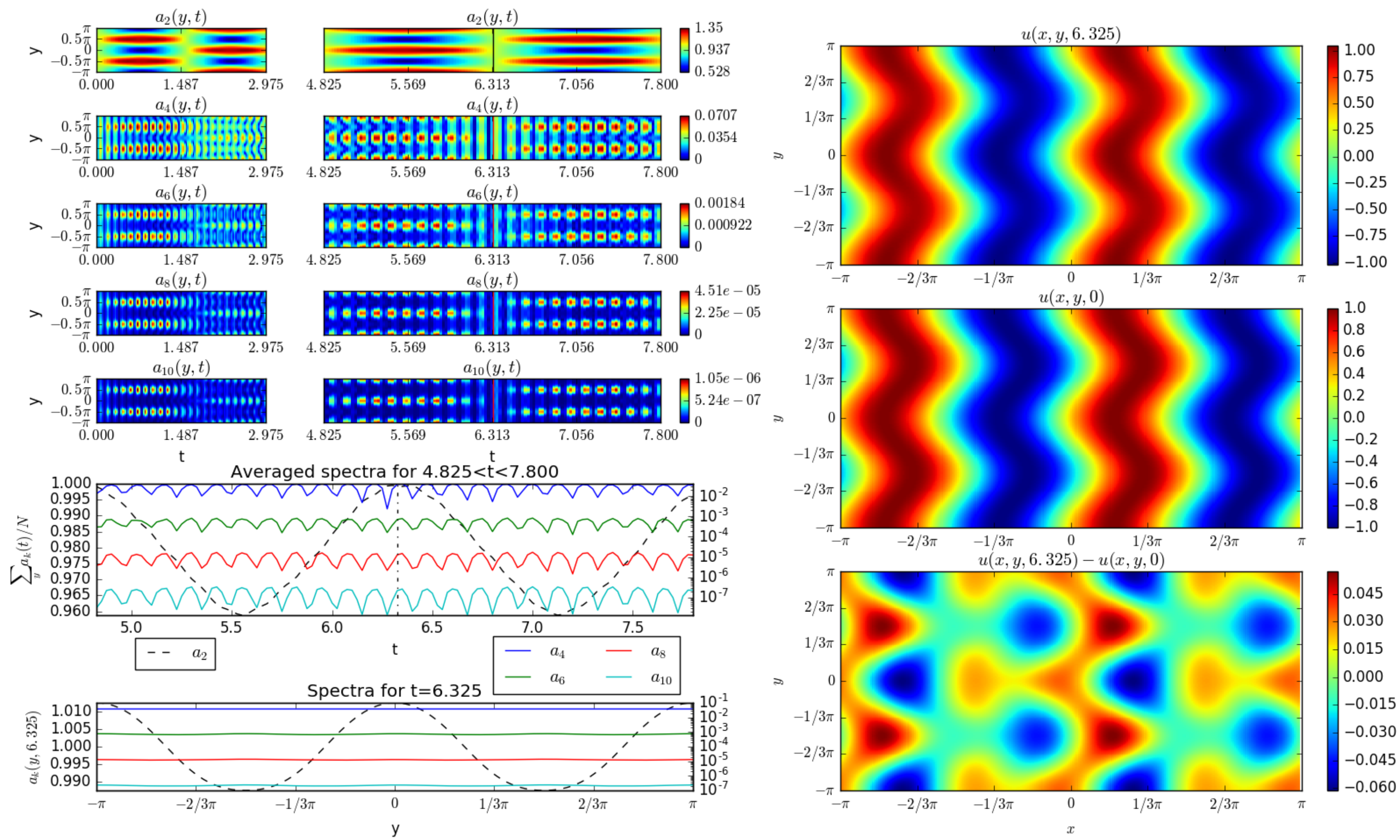

Figure 4.23: Recurrence in case of (3.9) and parameters $a_x = 2$, $\beta = 0.4$, $a_y = 2$ and dispersion parameter $\alpha_2 = 1$ at $t = 6.325$: full spectrum (upper left, time indicated by black/red line), averaged spectrum (middle left, time indicated by dotted line), spectra variance along y (lower left), wave profile at specified time (upper right), initial condition (middle right) and theri difference (lower right)

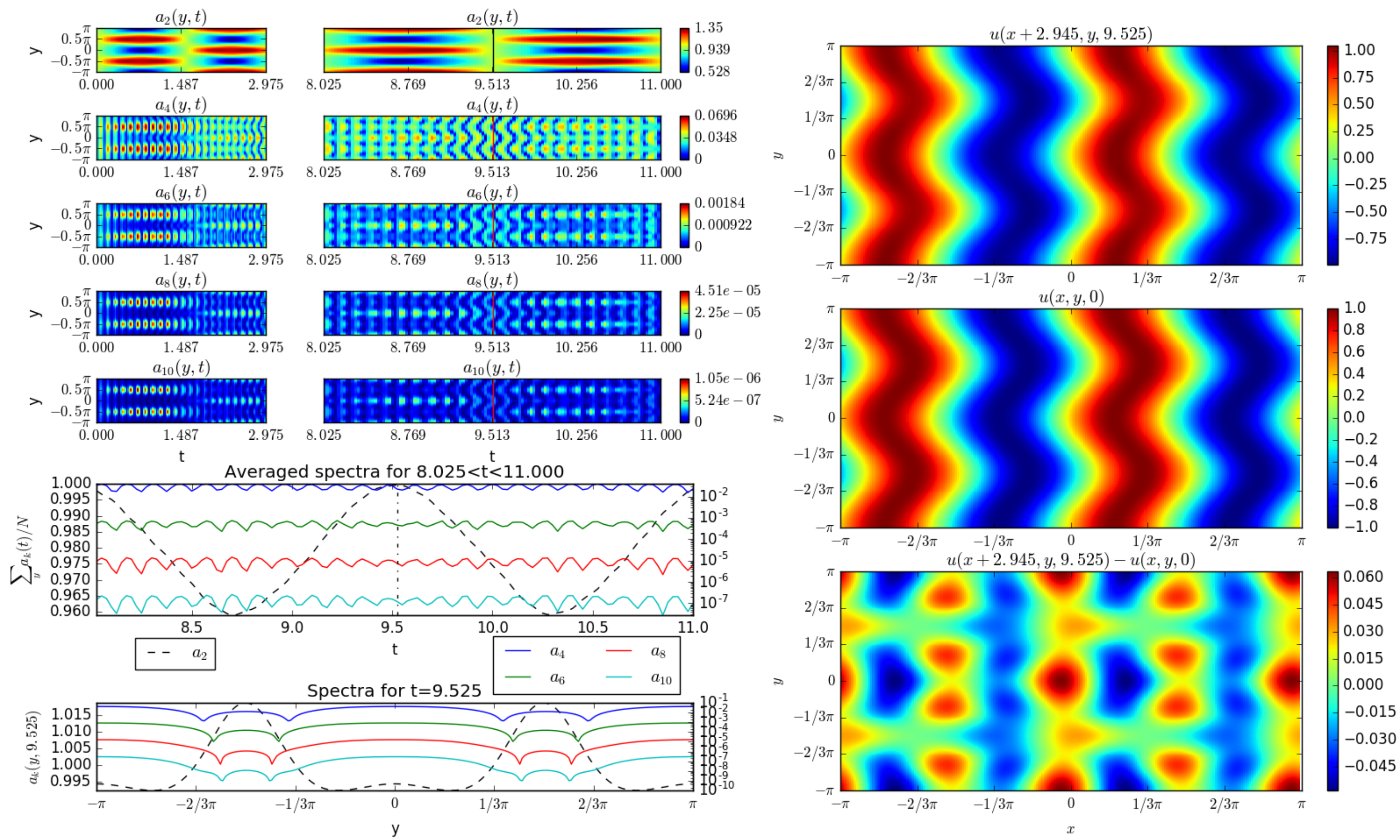

Figure 4.24: Recurrence in case of (3.9) and parameters $a_x = 2$, $\beta = 0.4$, $a_y = 2$ and dispersion parameter $\alpha_2 = 1$ at $t = 9.525$: full spectrum (upper left, time indicated by black/red line), averaged spectrum (middle left, time indicated by dotted line), spectra variance along y (lower left), wave profile at specified time (upper right), initial condition (middle right) and theri difference (lower right)

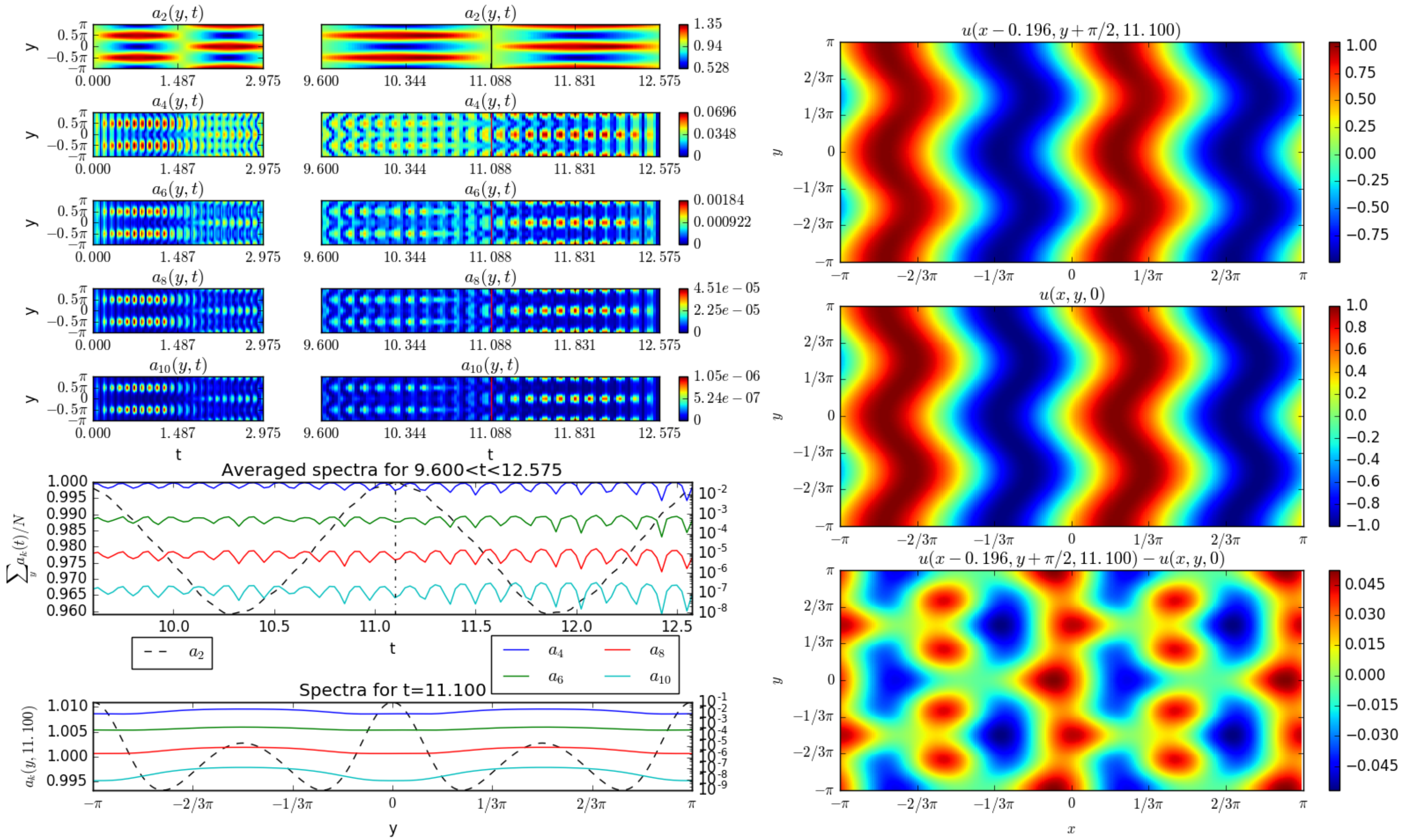

Figure 4.25: Recurrence in case of (3.9) and parameters $a_x = 2$, $\beta = 0.4$, $a_y = 2$ and dispersion parameter $\alpha_2 = 1$ at $t = 11.1$: full spectrum (upper left, time indicated by black/red line), averaged spectrum (middle left, time indicated by dotted line), spectra variance along y (lower left), wave profile at specified time (upper right), initial condition (middle right) and theri difference (lower right)

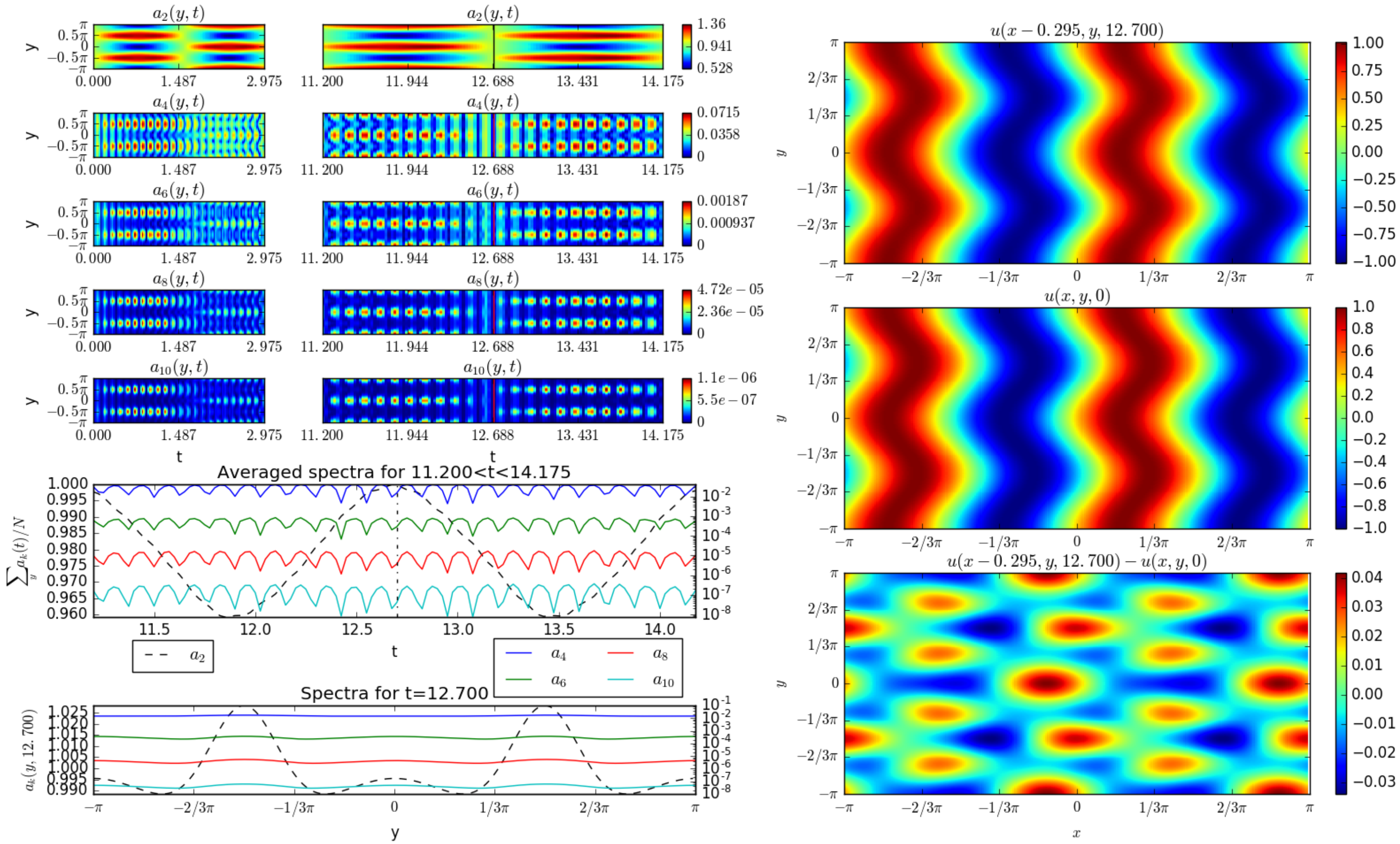

Figure 4.26: Recurrence in case of (3.9) and parameters $a_x = 2$, $\beta = 0.4$, $a_y = 2$ and dispersion parameter $\alpha_2 = 1$ at $t = 12.7$: full spectrum (upper left, time indicated by black/red line), averaged spectrum (middle left, time indicated by dotted line), spectra variance along y (lower left), wave profile at specified time (upper right), initial condition (middle right) and theri difference (lower right)

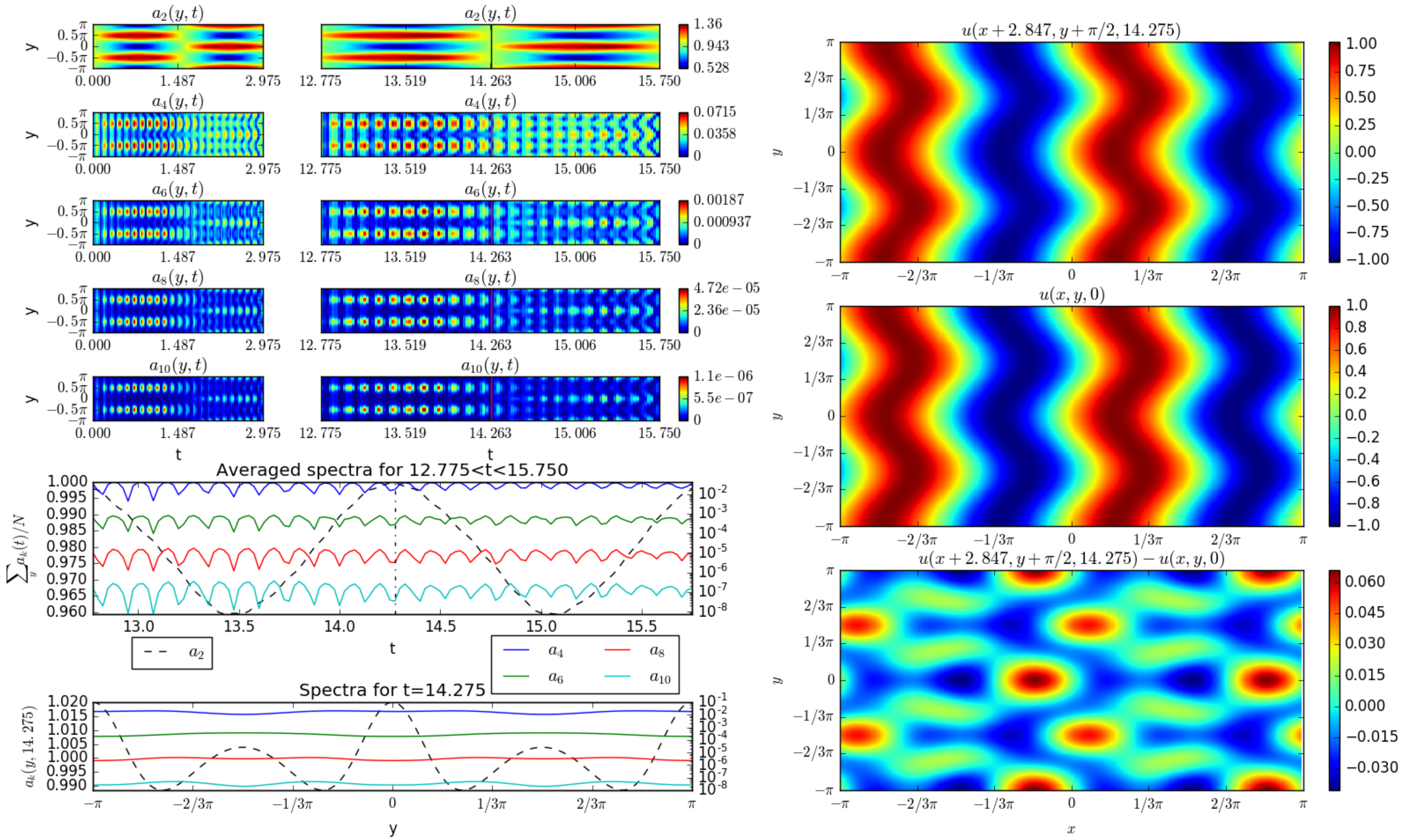

Figure 4.27: Recurrence in case of (3.9) and parameters $a_x = 2$, $\beta = 0.4$, $a_y = 2$ and dispersion parameter $\alpha_2 = 1$ at $t = 14.275$: full spectrum (upper left, time indicated by black/red line), averaged spectrum (middle left, time indicated by dotted line), spectra variance along y (lower left), wave profile at specified time (upper right), initial condition (middle right) and theri difference (lower right)

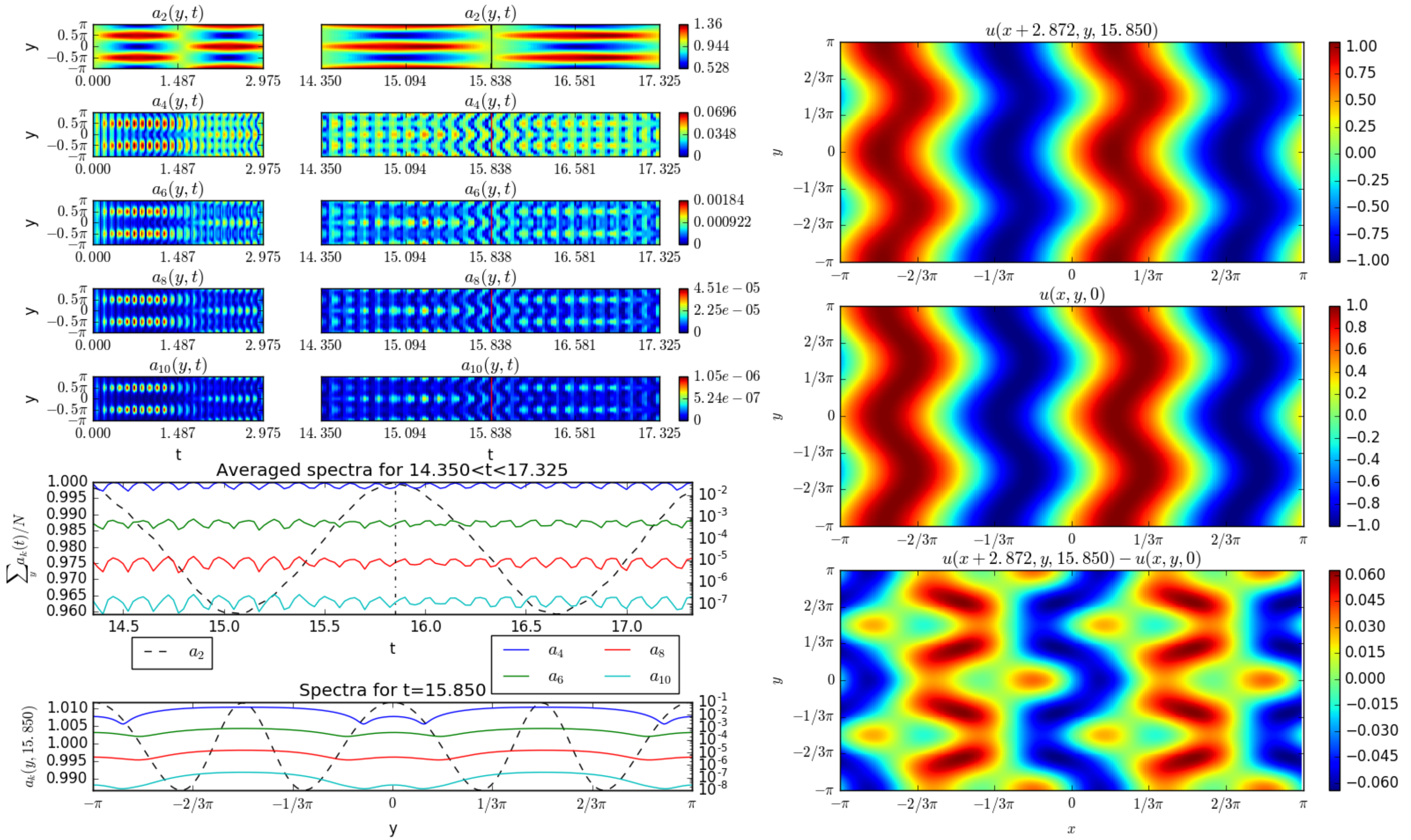

Figure 4.28: Recurrence in case of (3.9) and parameters $a_x = 2$, $\beta = 0.4$, $a_y = 2$ and dispersion parameter $\alpha_2 = 1$ at $t = 15.85$: full spectrum (upper left, time indicated by black/red line), averaged spectrum (middle left, time indicated by dotted line), spectra variance along y (lower left), wave profile at specified time (upper right), initial condition (middle right) and theri difference (lower right)

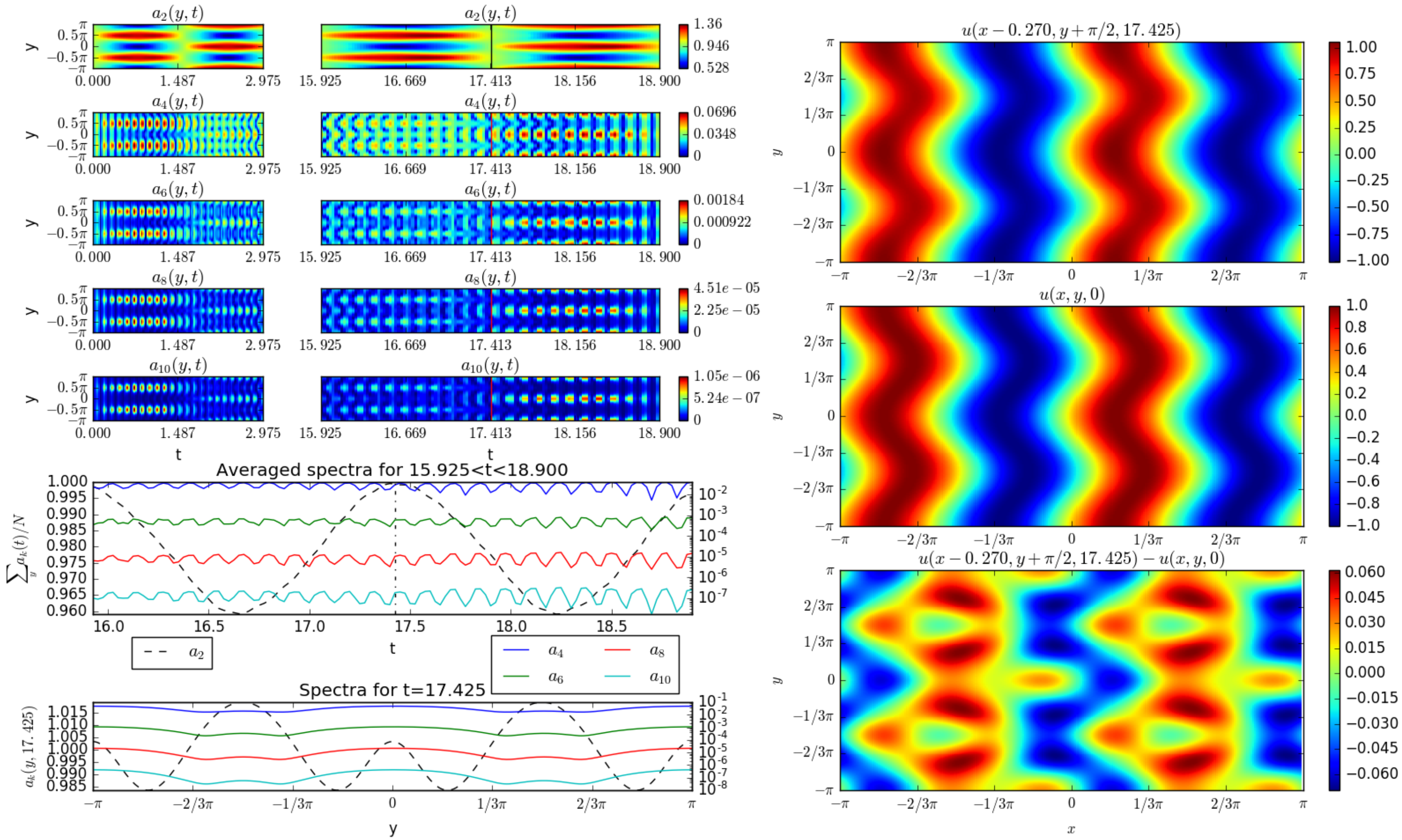

Figure 4.29: Recurrence in case of (3.9) and parameters $a_x = 2$, $\beta = 0.4$, $a_y = 2$ and dispersion parameter $\alpha_2 = 1$ at $t = 17.425$: full spectrum (upper left, time indicated by black/red line), averaged spectrum (middle left, time indicated by dotted line), spectra variance along y (lower left), wave profile at specified time (upper right), initial condition (middle right) and theri difference (lower right)

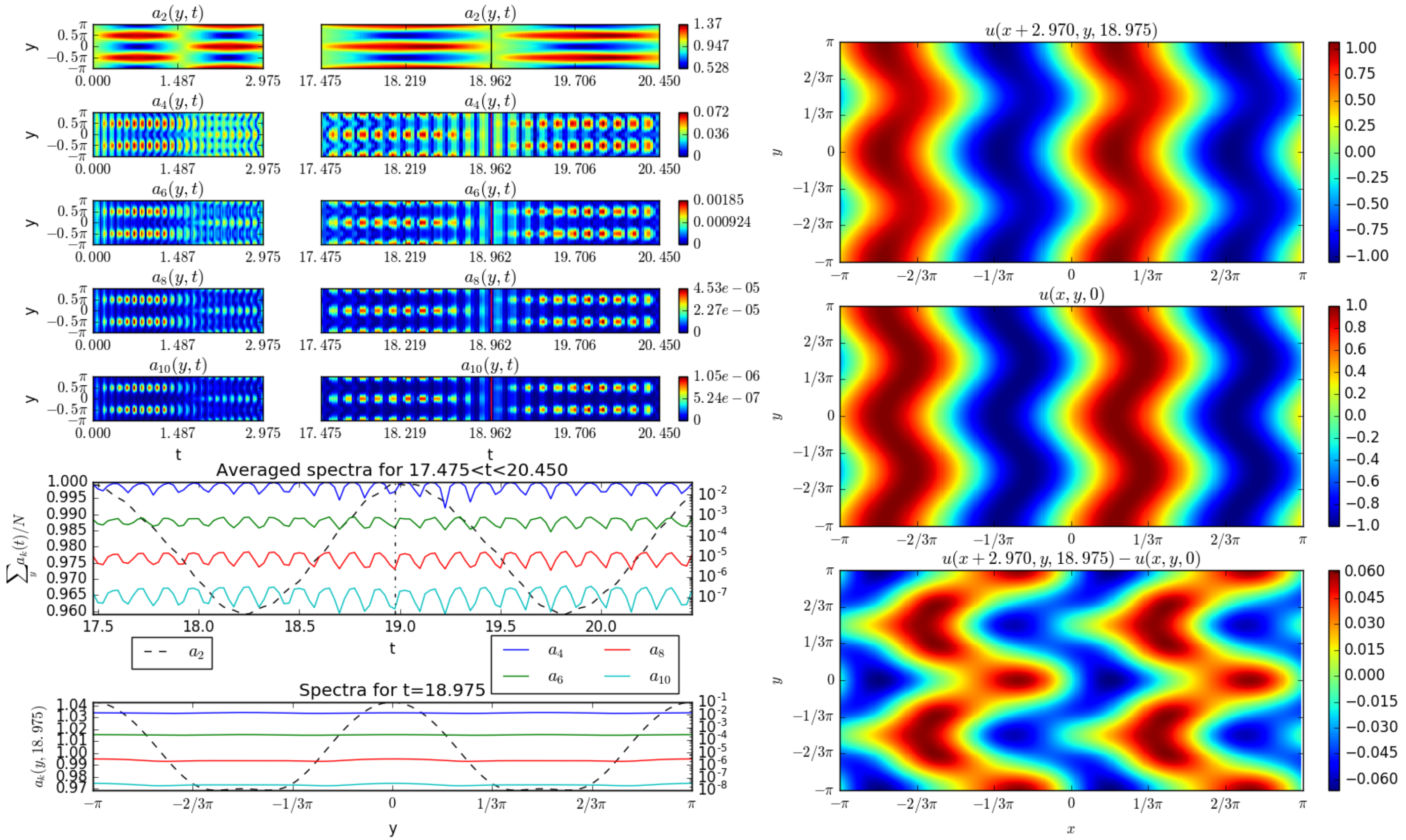

Figure 4.30: Recurrence in case of (3.9) and parameters $a_x = 2$, $\beta = 0.4$, $a_y = 2$ and dispersion parameter $\alpha_2 = 1$ at $t = 18.975$: full spectrum (upper left, time indicated by black/red line), averaged spectrum (middle left, time indicated by dotted line), spectra variance along y (lower left), wave profile at specified time (upper right), initial condition (middle right) and theri difference (lower right)

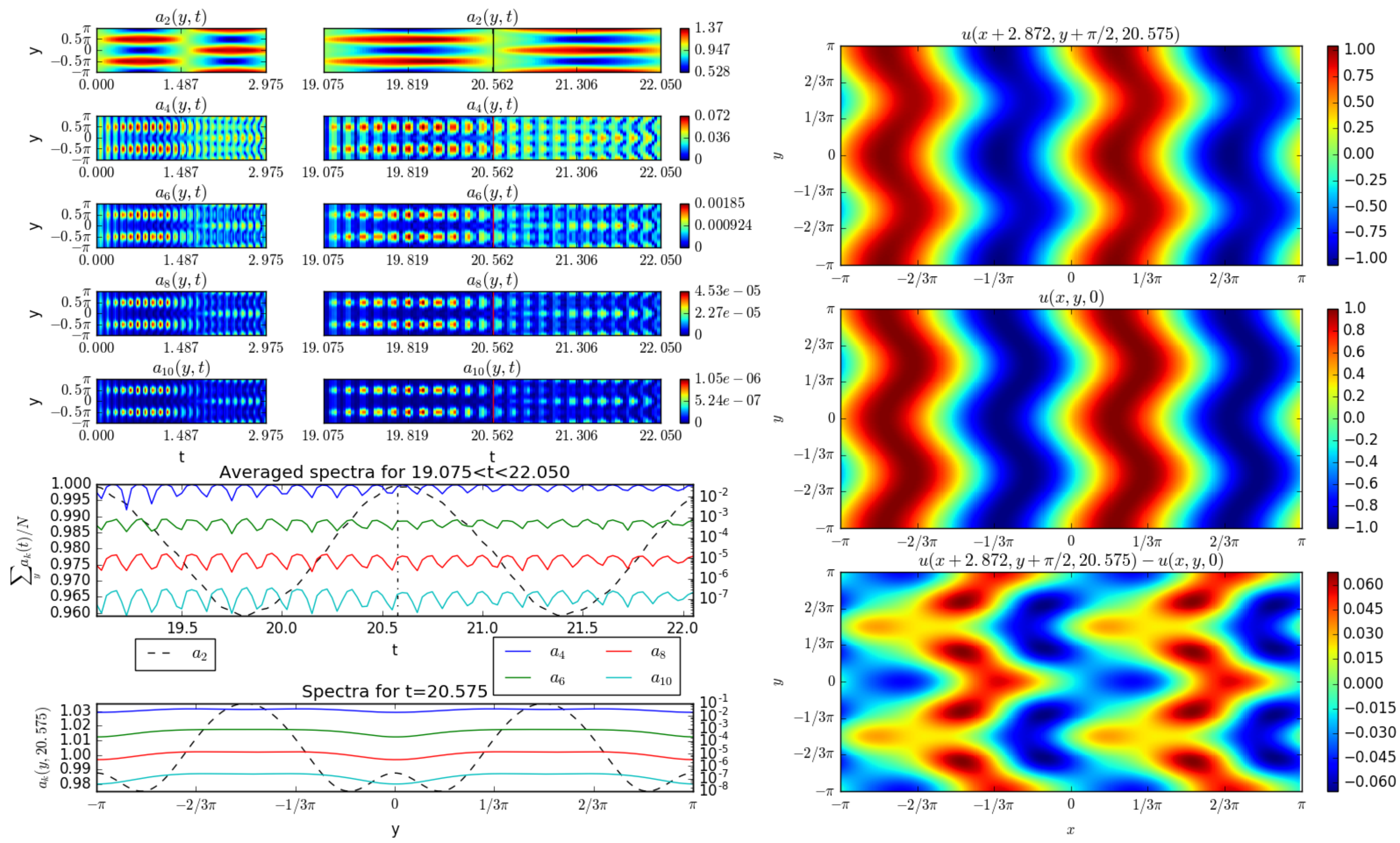

Figure 4.31: Recurrence in case of (3.9) and parameters $a_x = 2$, $\beta = 0.4$, $a_y = 2$ and dispersion parameter $\alpha_2 = 1$ at $t = 20.575$: full spectrum (upper left, time indicated by black/red line), averaged spectrum (middle left, time indicated by dotted line), spectra variance along y (lower left), wave profile at specified time (upper right), initial condition (middle right) and theri difference (lower right)

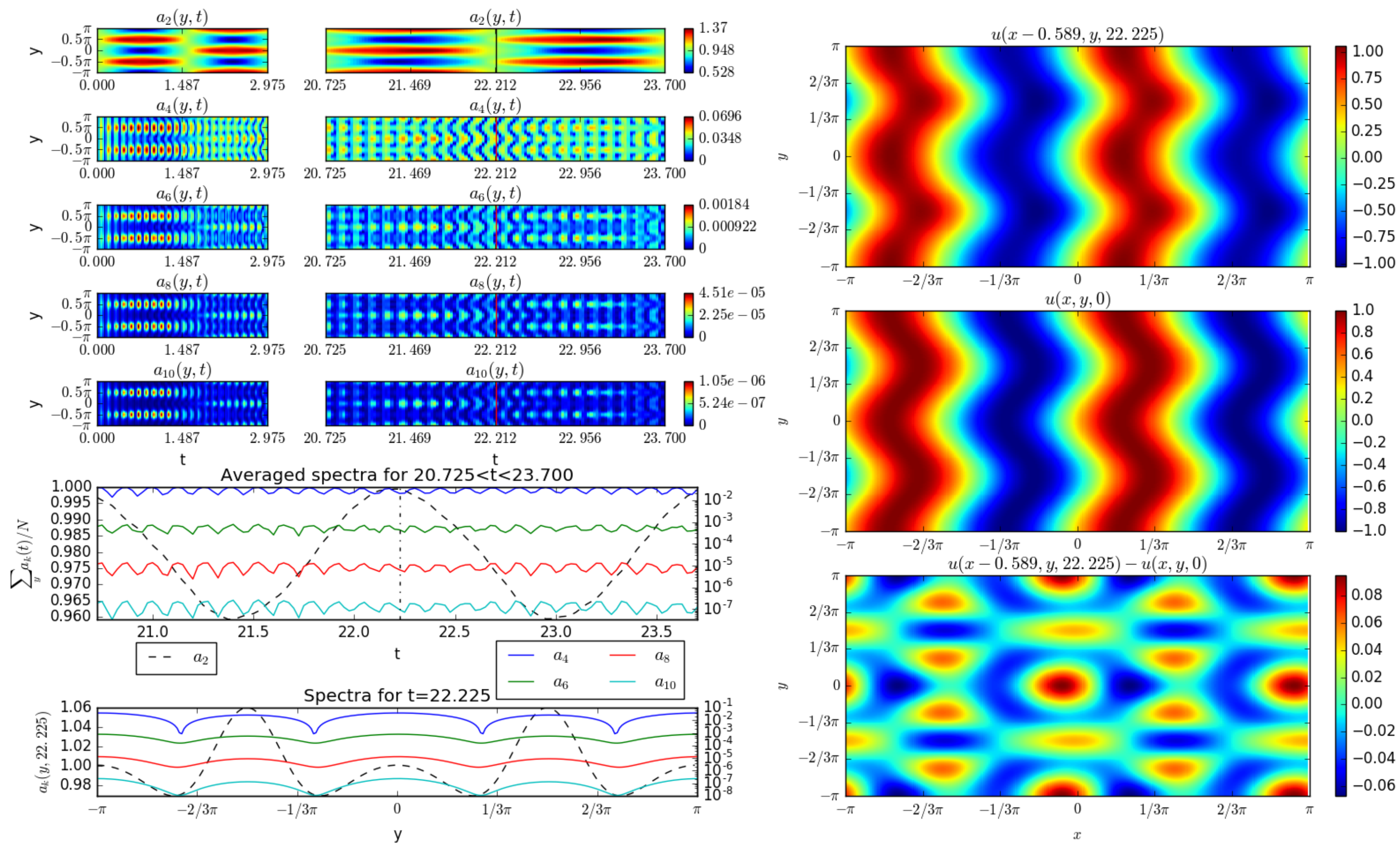

Figure 4.32: Recurrence in case of (3.9) and parameters $a_x = 2$, $\beta = 0.4$, $a_y = 2$ and dispersion parameter $\alpha_2 = 1$ at $t = 22.225$: full spectrum (upper left, time indicated by black/red line), averaged spectrum (middle left, time indicated by dotted line), spectra variance along y (lower left), wave profile at specified time (upper right), initial condition (middle right) and theri difference (lower right)

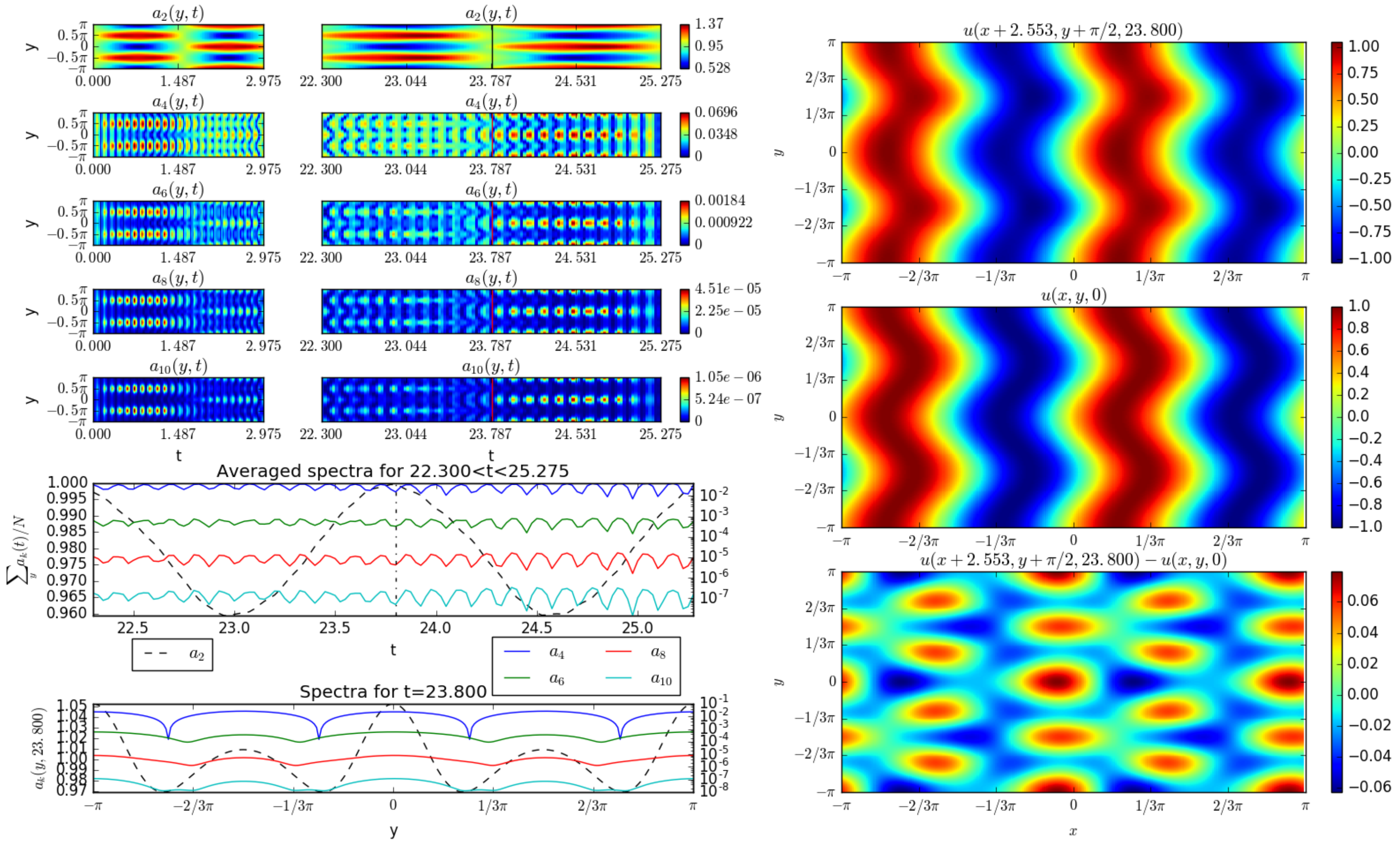

Figure 4.33: Recurrence in case of (3.9) and parameters $a_x = 2$, $\beta = 0.4$, $a_y = 2$ and dispersion parameter $\alpha_2 = 1$ at $t = 23.8$: full spectrum (upper left, time indicated by black/red line), averaged spectrum (middle left, time indicated by dotted line), spectra variance along y (lower left), wave profile at specified time (upper right), initial condition (middle right) and theri difference (lower right)

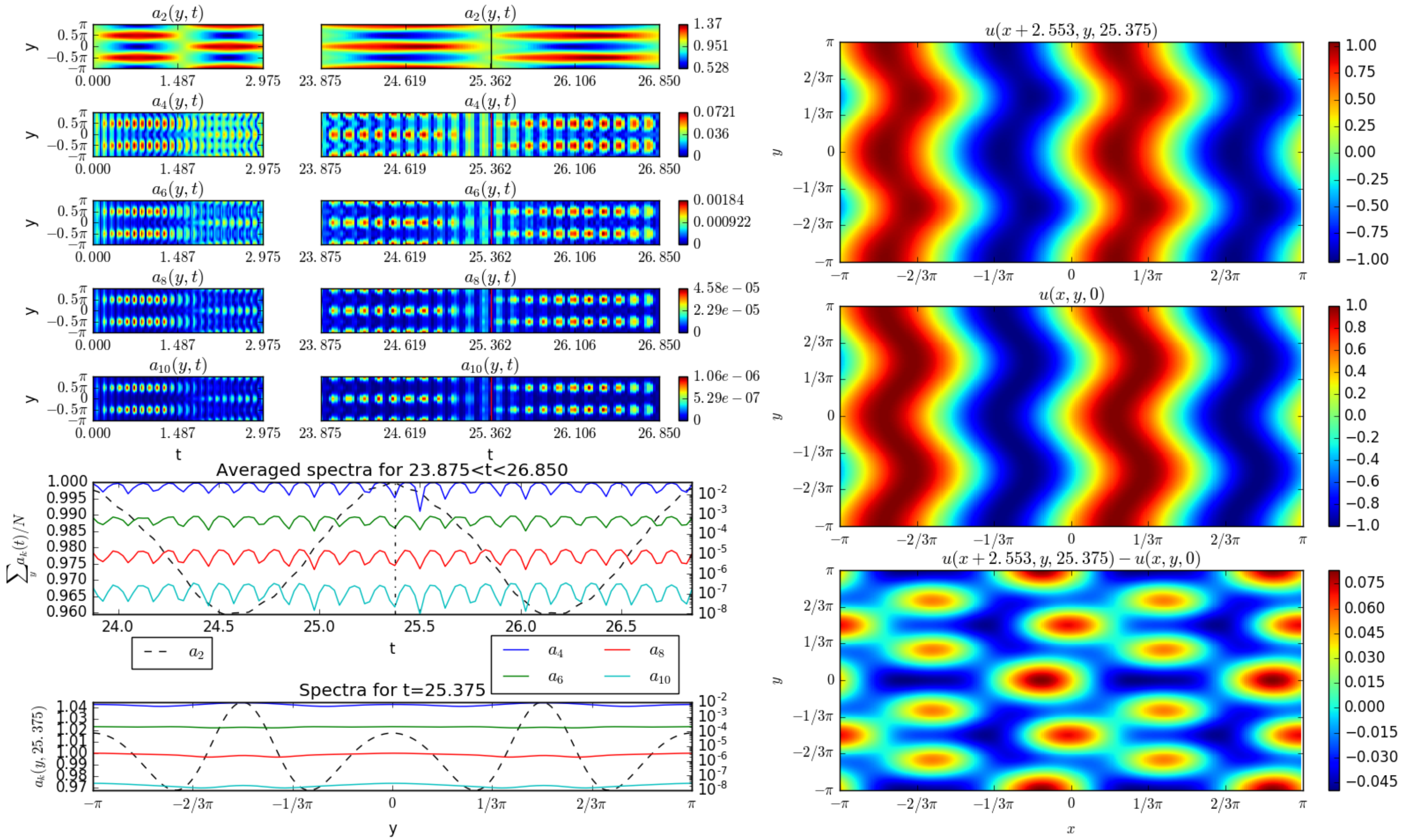

Figure 4.34: Recurrence in case of (3.9) and parameters $a_x = 2$, $\beta = 0.4$, $a_y = 2$ and dispersion parameter $\alpha_2 = 1$ at $t = 25.375$: full spectrum (upper left, time indicated by black/red line), averaged spectrum (middle left, time indicated by dotted line), spectra variance along y (lower left), wave profile at specified time (upper right), initial condition (middle right) and theri difference (lower right)

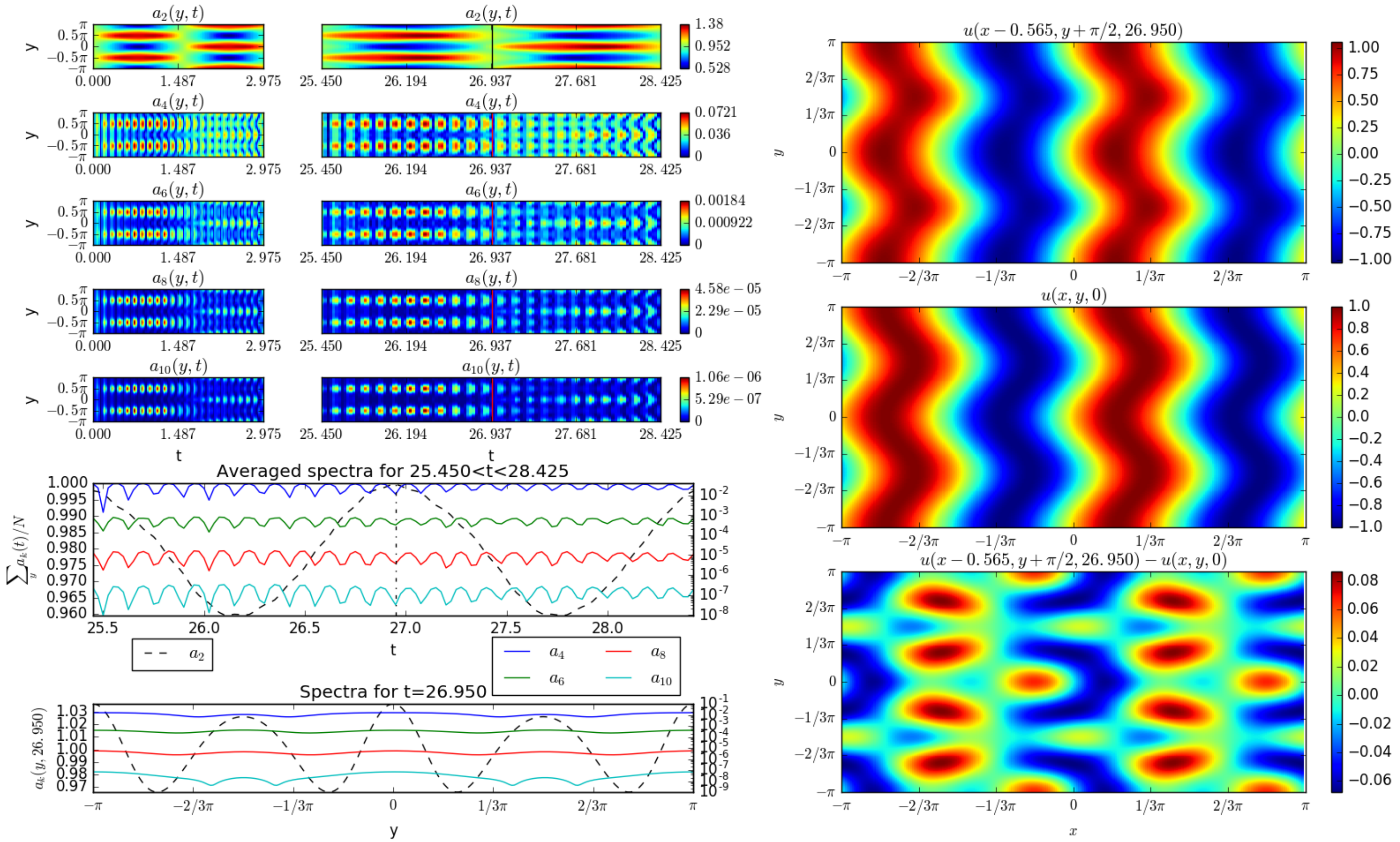

Figure 4.35: Recurrence in case of (3.9) and parameters $a_x = 2$, $\beta = 0.4$, $a_y = 2$ and dispersion parameter $\alpha_2 = 1$ at $t = 26.95$: full spectrum (upper left, time indicated by black/red line), averaged spectrum (middle left, time indicated by dotted line), spectra variance along y (lower left), wave profile at specified time (upper right), initial condition (middle right) and theri difference (lower right)

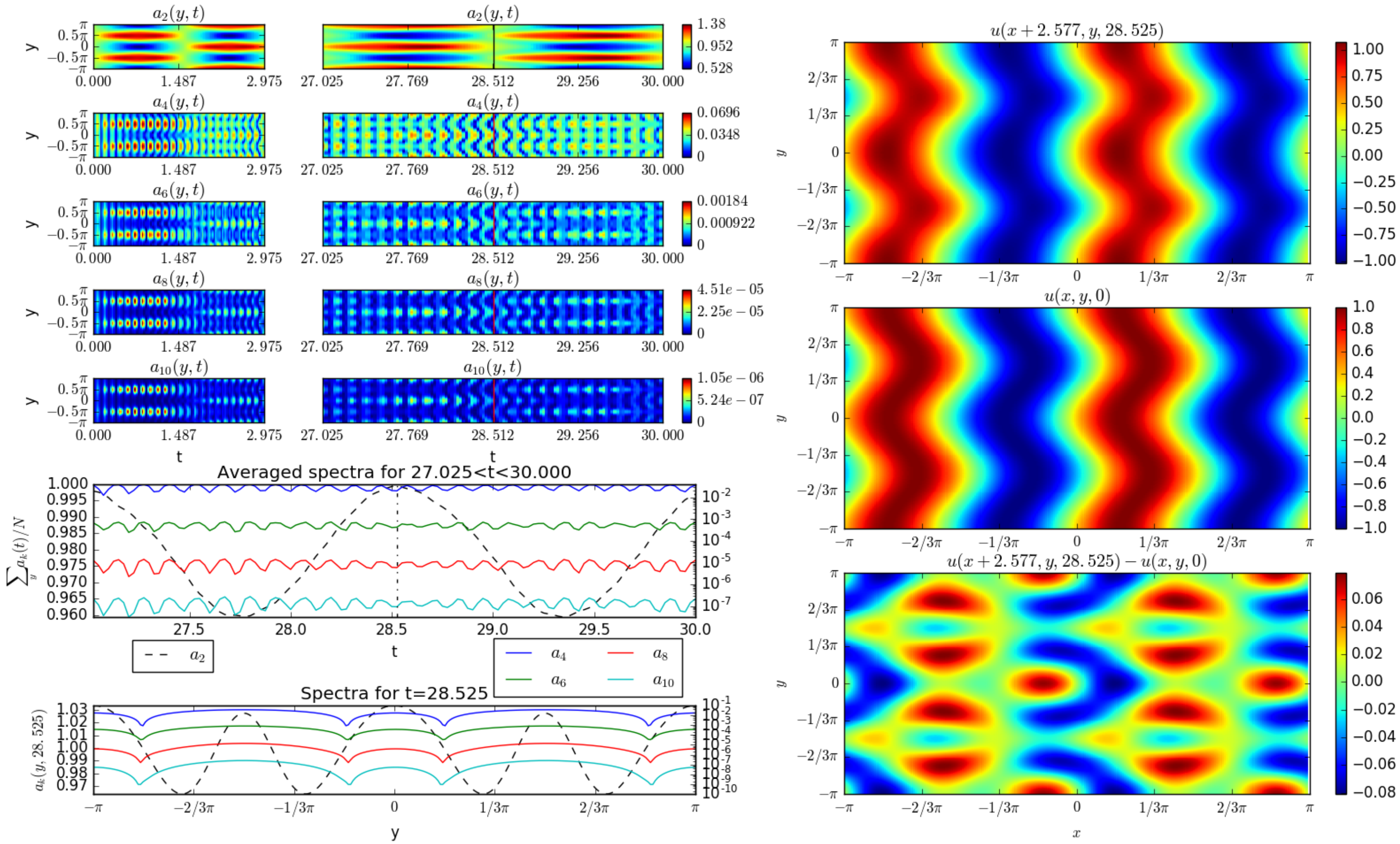

Figure 4.36: Recurrence in case of (3.9) and parameters $a_x = 2$, $\beta = 0.4$, $a_y = 2$ and dispersion parameter $\alpha_2 = 1$ at $t = 28.525$: full spectrum (upper left, time indicated by black/red line), averaged spectrum (middle left, time indicated by dotted line), spectra variance along y (lower left), wave profile at specified time (upper right), initial condition (middle right) and theri difference (lower right)

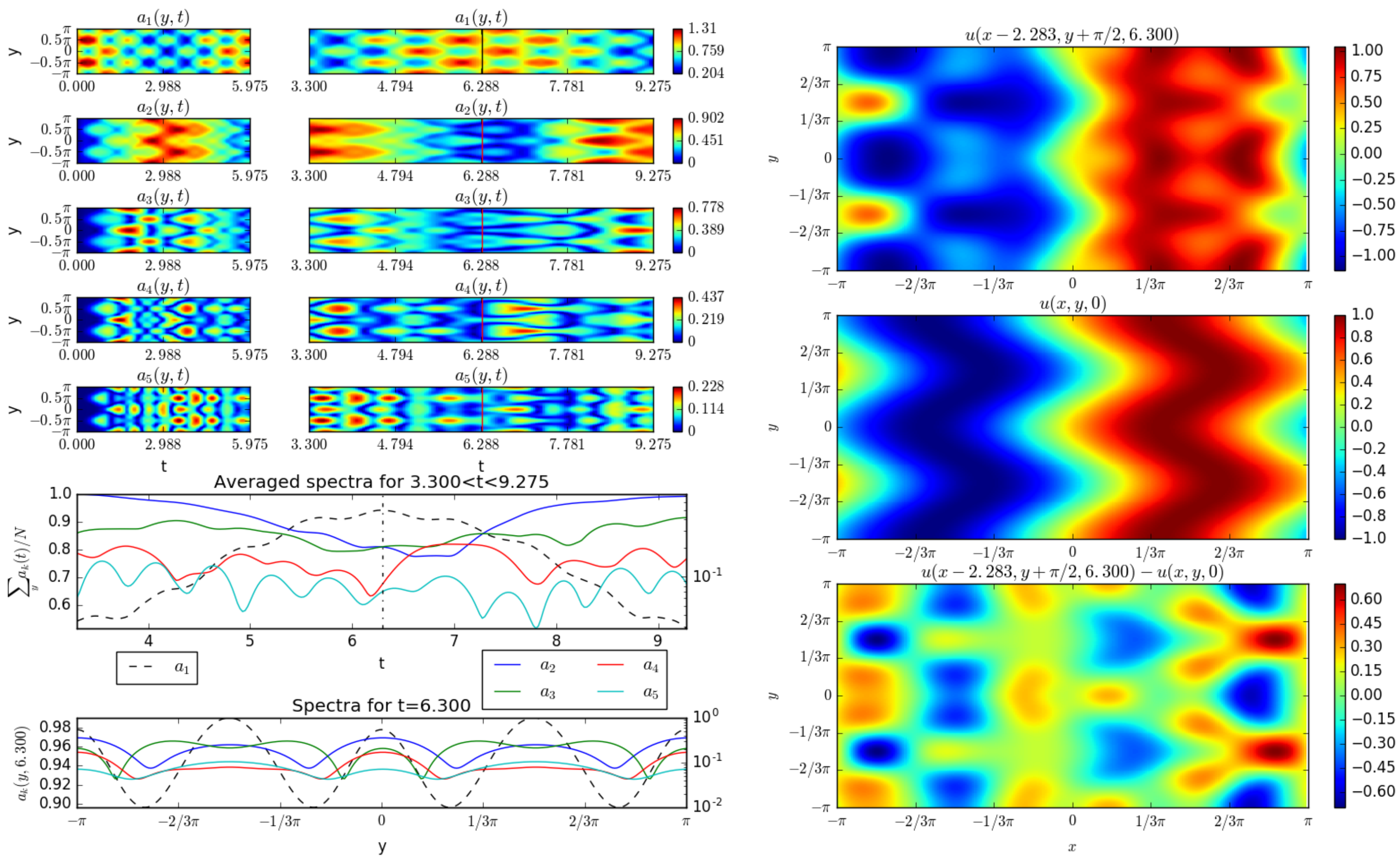

Figure 4.37: Recurrence in case of (3.9) and parameters $a_x = 1$, $\beta = 0.4$, $a_y = 2$ and dispersion parameter $\alpha_2 = 0.1$ at $t = 6.3$: full spectrum (upper left, time indicated by black/red line), averaged spectrum (middle left, time indicated by dotted line), spectra variance along y (lower left), wave profile at specified time (upper right), initial condition (middle right) and theri difference (lower right)

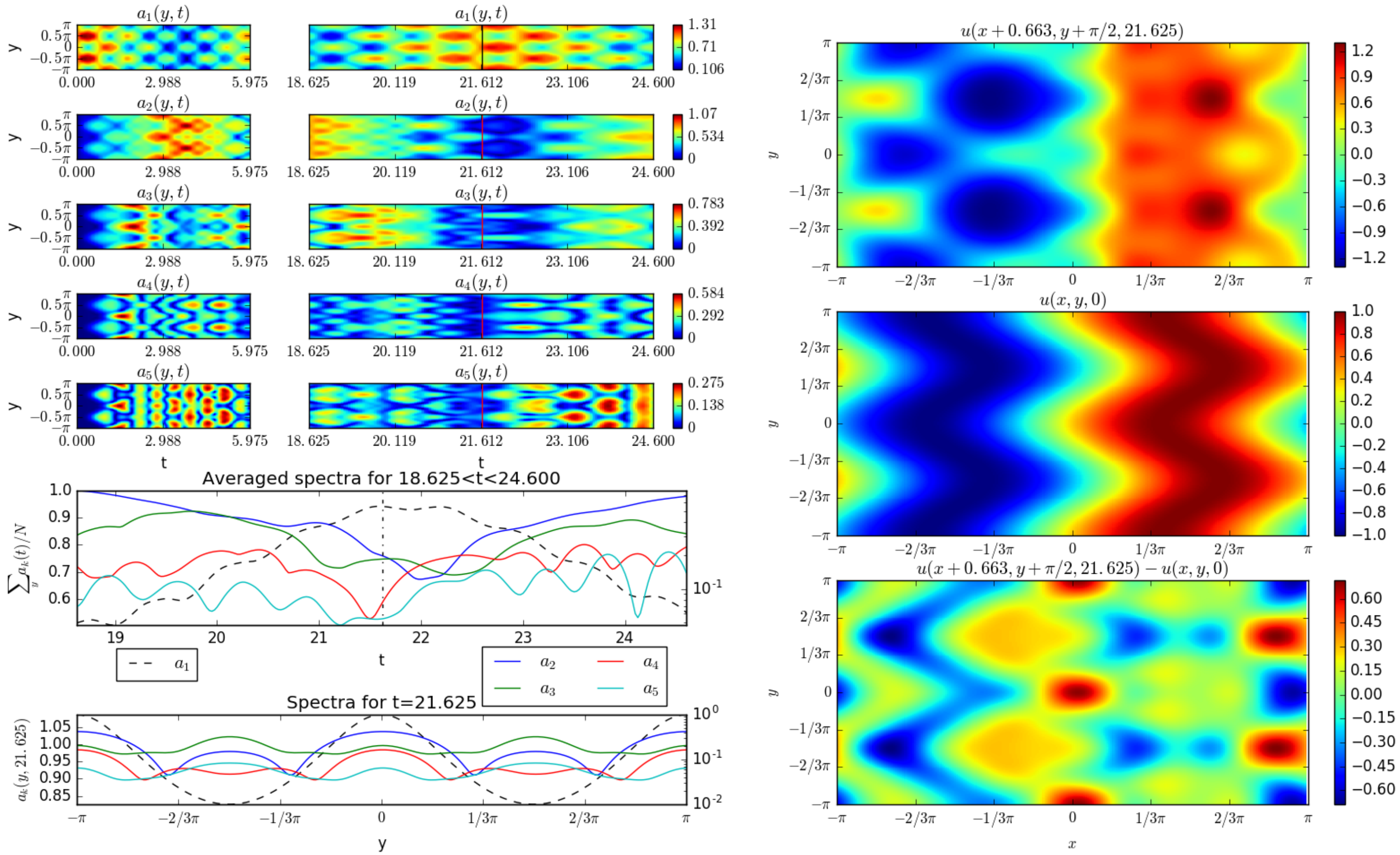

Figure 4.38: Looking for recurrence in case of (3.9) and parameters $a_x = 1$, $\beta = 0.4$, $a_y = 2$ and dispersion parameter $\alpha_2 = 10^{1.1}$ at $t = 21.625$: full spectrum (upper left, time indicated by black/red line), averaged spectrum (middle left, time indicated by dotted line), spectra variance along y (lower left), wave profile at specified time (upper right), initial condition (middle right) and theri difference (lower right)

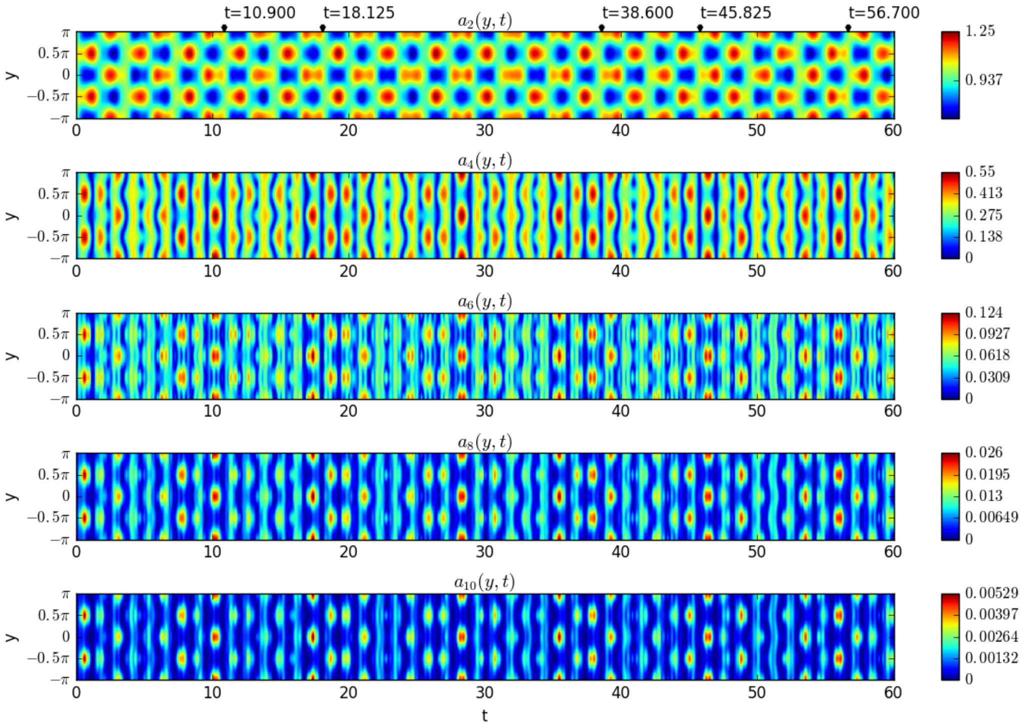

Figure 4.39: Spectral amplitude in case of the initial condition (3.9), parameters $a_x = 2$, $\beta = 0.2$, $a_y = 2$ and dispersion parameter $\alpha_2 = 0.1$, recurrence times shown with a small arrow

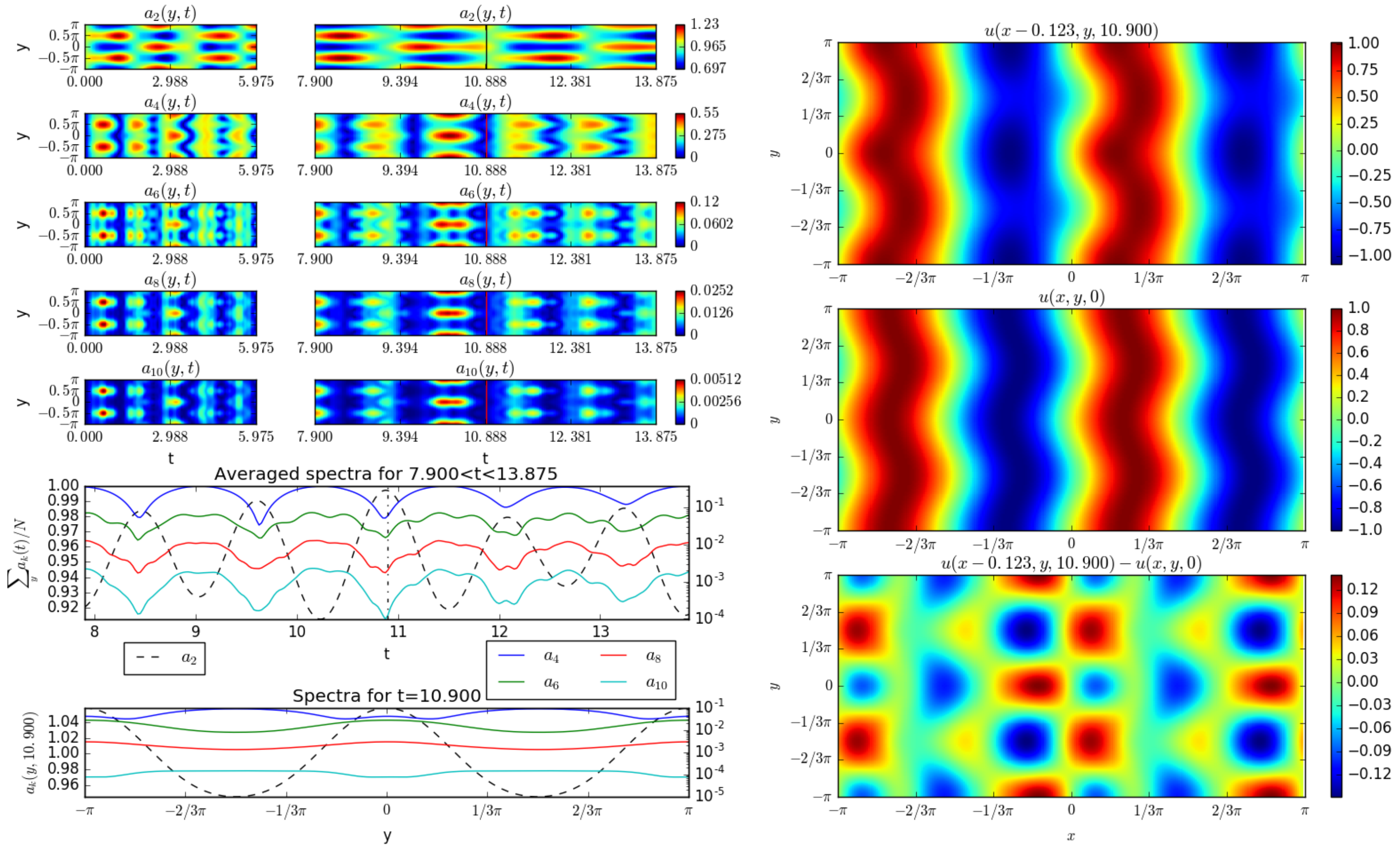

Figure 4.40: Recurrence in case of (3.9) and parameters $a_x = 2$, $\beta = 0.2$, $a_y = 2$ and dispersion parameter $\alpha_2 = 0.1$ at $t = 10.9$: full spectrum (upper left, time indicated by black/red line), averaged spectrum (middle left, time indicated by dotted line), spectra variance along y (lower left), wave profile at specified time (upper right), initial condition (middle right) and theri difference (lower right)

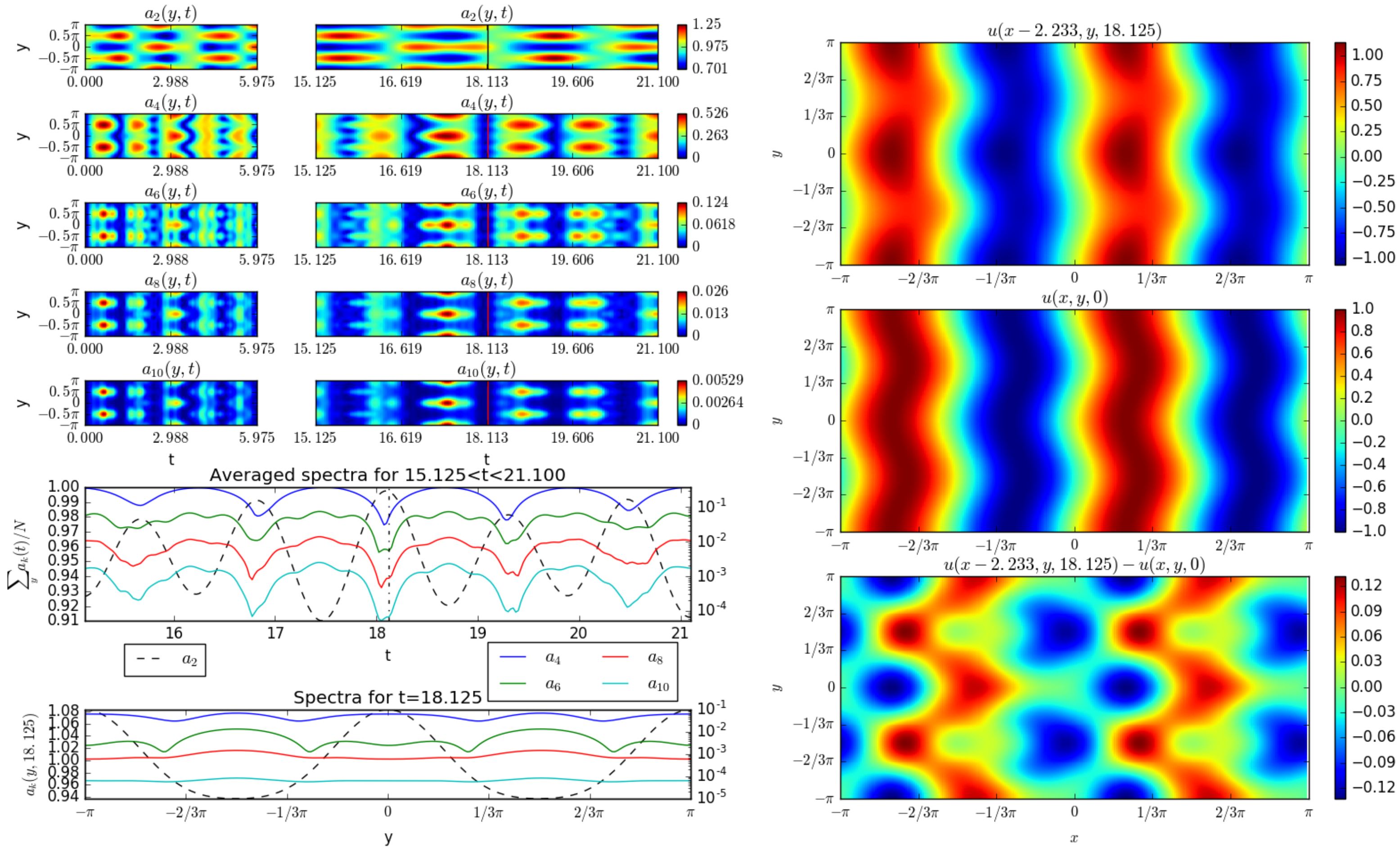

Figure 4.41: Recurrence in case of (3.9) and parameters $a_x = 2$, $\beta = 0.2$, $a_y = 2$ and dispersion parameter $\alpha_2 = 0.1$ at $t = 18.125$: full spectrum (upper left, time indicated by black/red line), averaged spectrum (middle left, time indicated by dotted line), spectra variance along y (lower left), wave profile at specified time (upper right), initial condition (middle right) and theri difference (lower right)

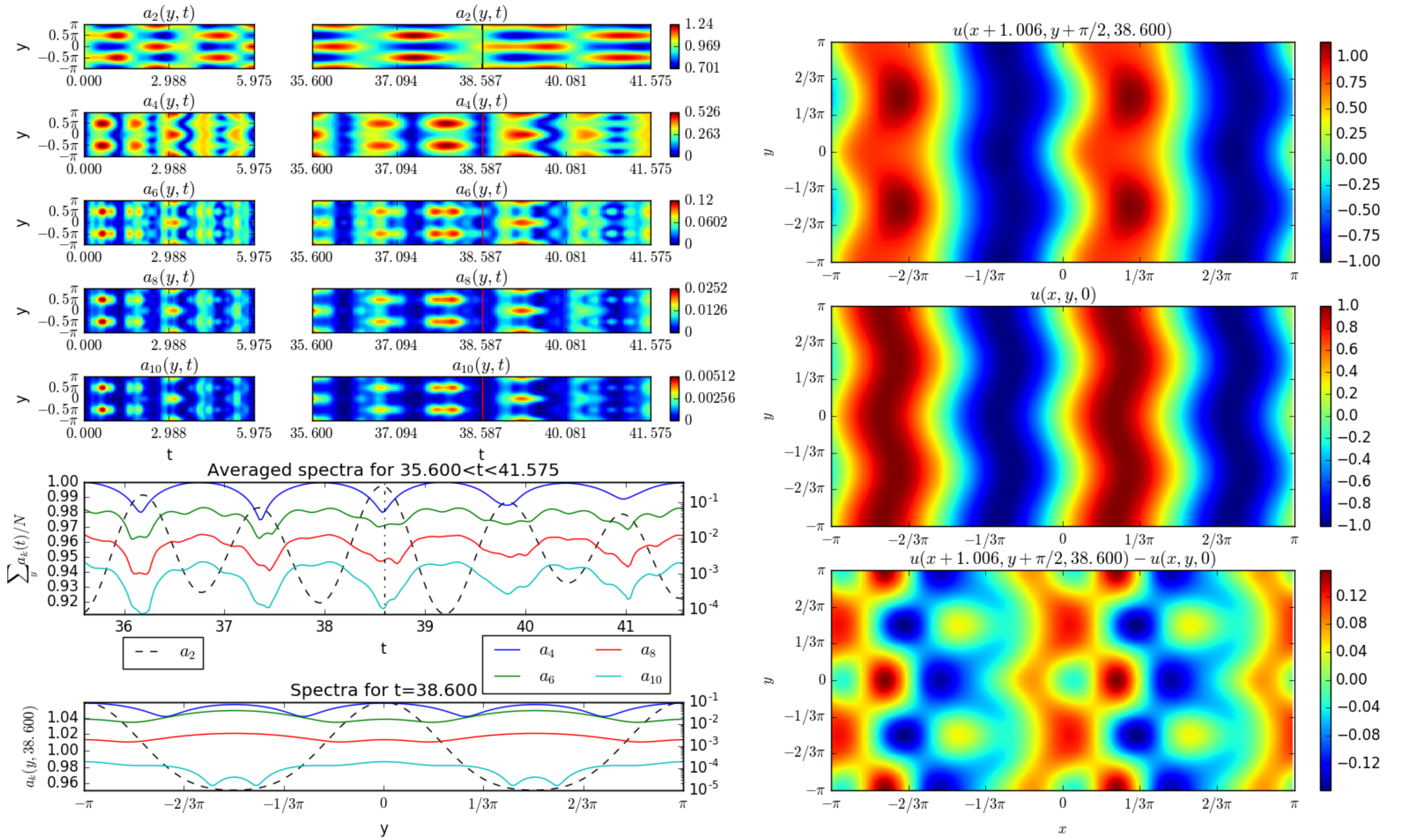

Figure 4.42: Recurrence in case of (3.9) and parameters $a_x = 2$, $\beta = 0.2$, $a_y = 2$ and dispersion parameter $\alpha_2 = 0.1$ at $t = 38.6$: full spectrum (upper left, time indicated by black/red line), averaged spectrum (middle left, time indicated by dotted line), spectra variance along y (lower left), wave profile at specified time (upper right), initial condition (middle right) and theri difference (lower right)

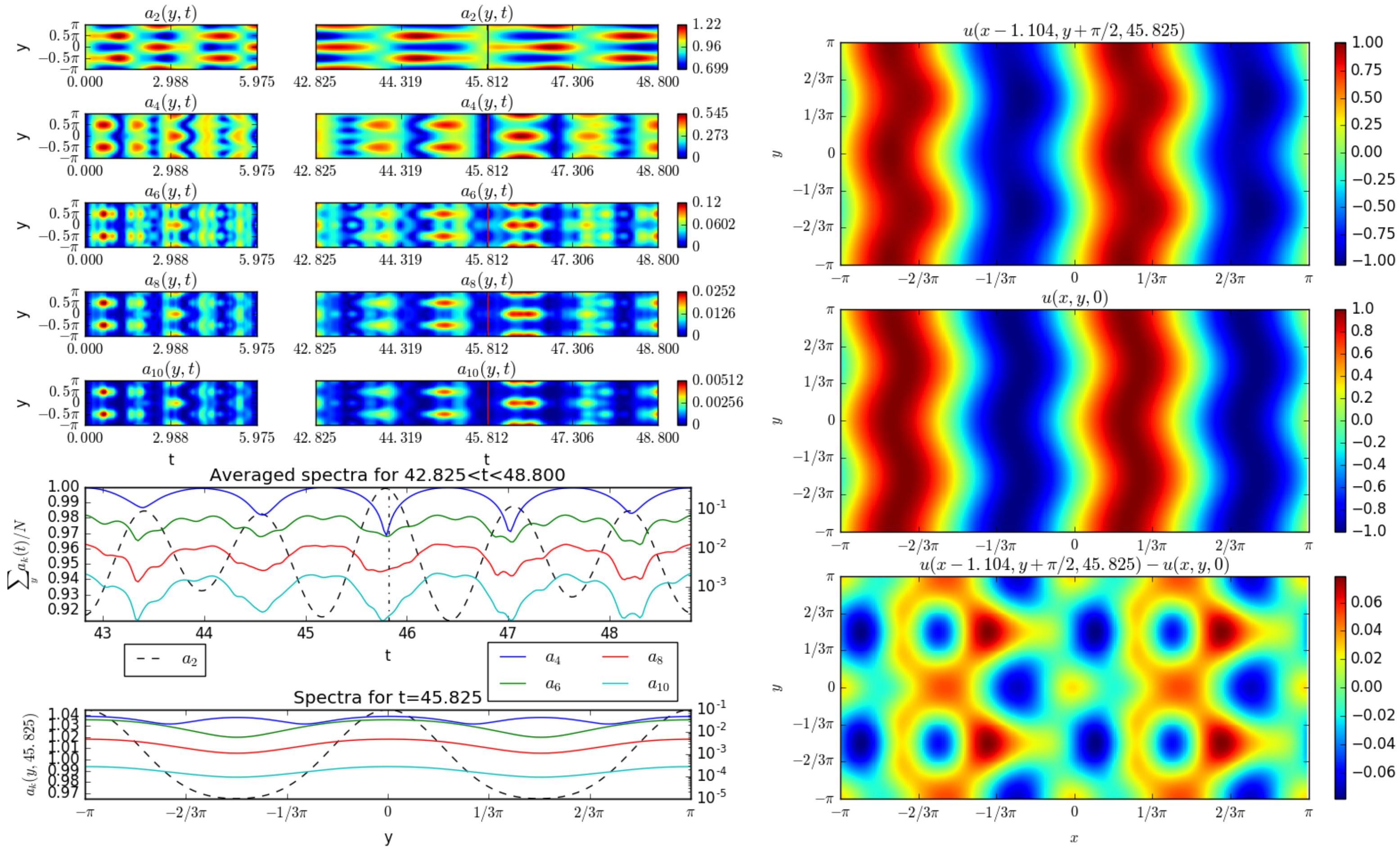

Figure 4.43: Recurrence in case of (3.9) and parameters $a_x = 2$, $\beta = 0.2$, $a_y = 2$ and dispersion parameter $\alpha_2 = 0.1$ at $t = 45.825$: full spectrum (upper left, time indicated by black/red line), averaged spectrum (middle left, time indicated by dotted line), spectra variance along y (lower left), wave profile at specified time (upper right), initial condition (middle right) and theri difference (lower right)

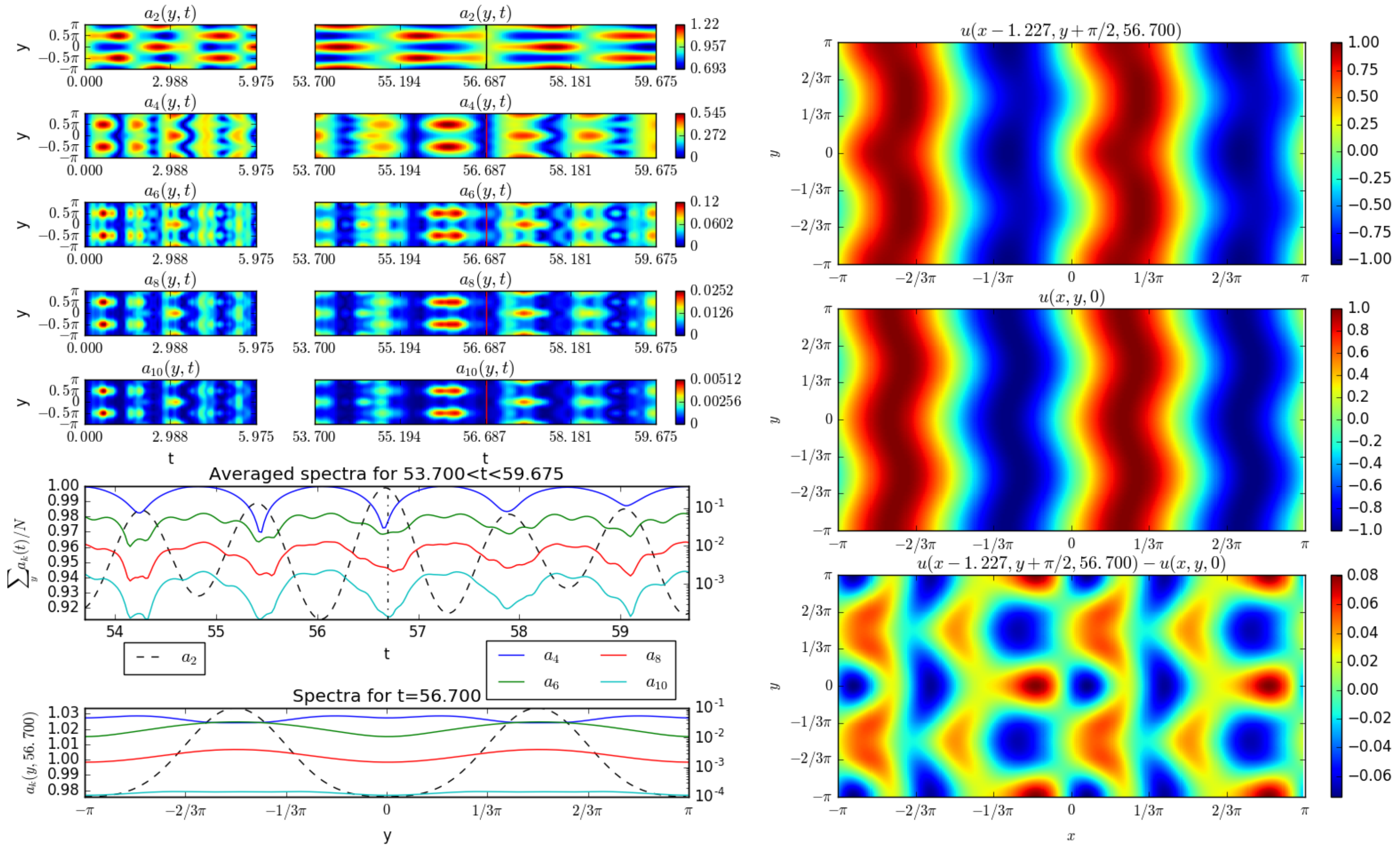

Figure 4.44: Recurrence in case of (3.9) and parameters $a_x = 2$, $\beta = 0.2$, $a_y = 2$ and dispersion parameter $\alpha_2 = 0.1$ at $t = 56.7$: full spectrum (upper left, time indicated by black/red line), averaged spectrum (middle left, time indicated by dotted line), spectra variance along y (lower left), wave profile at specified time (upper right), initial condition (middle right) and theri difference (lower right)

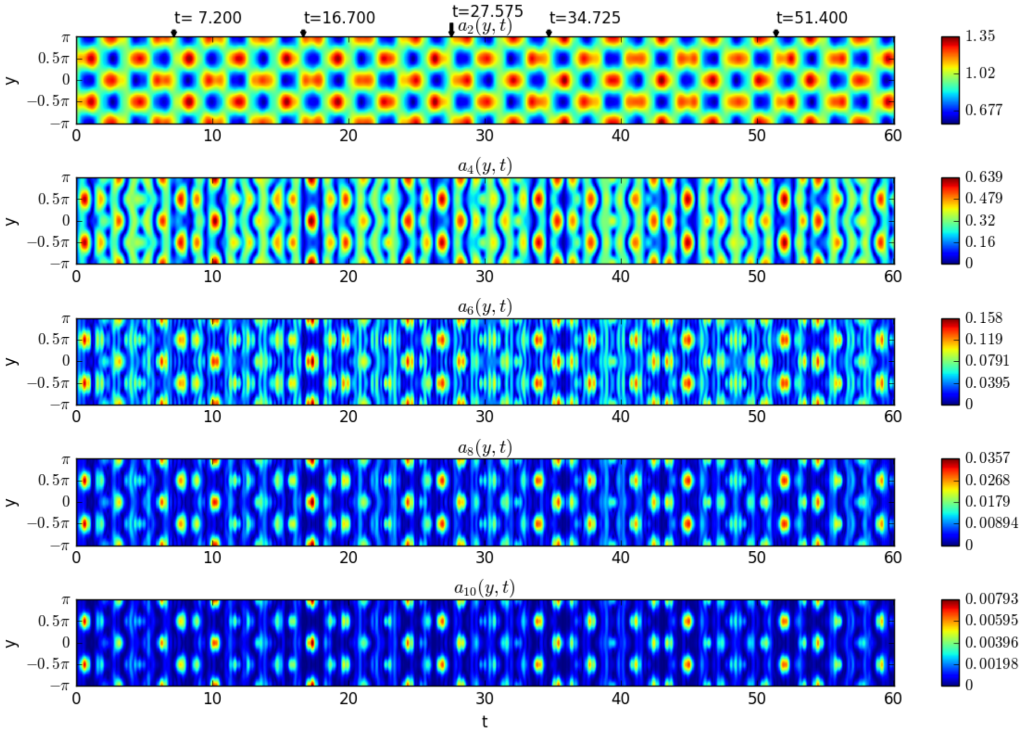

Figure 4.45: Spectral amplitude in case of the initial condition (3.9), parameters $a_x = 2$, $\beta = 0.3$, $a_y = 2$ and dispersion parameter $\alpha_2 = 0.1$, recurrence times shown with a small arrow

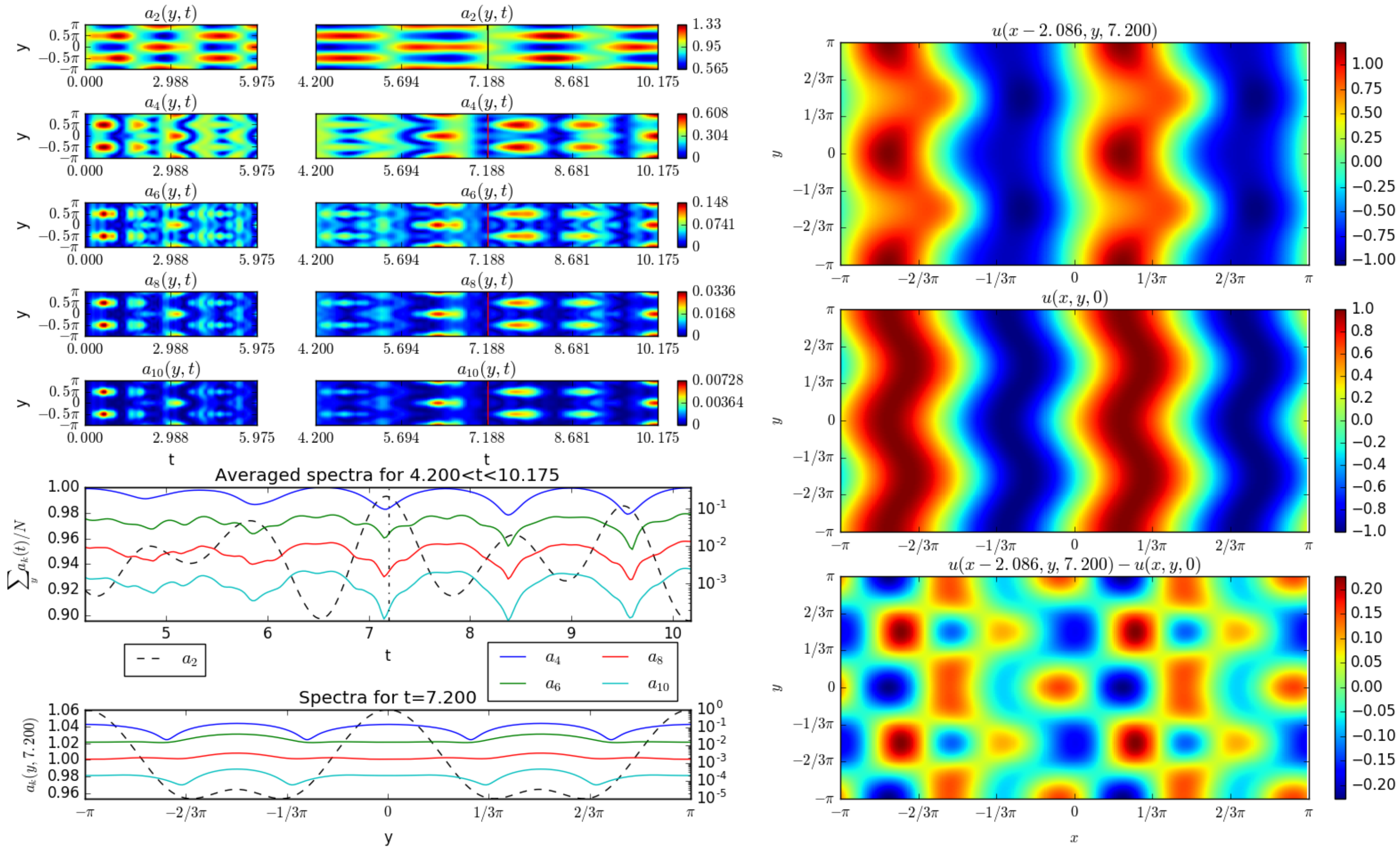

Figure 4.46: Recurrence in case of (3.9) and parameters $a_x = 2$, $\beta = 0.3$, $a_y = 2$ and dispersion parameter $\alpha_2 = 0.1$ at $t = 7.2$: full spectrum (upper left, time indicated by black/red line), averaged spectrum (middle left, time indicated by dotted line), spectra variance along y (lower left), wave profile at specified time (upper right), initial condition (middle right) and theri difference (lower right)

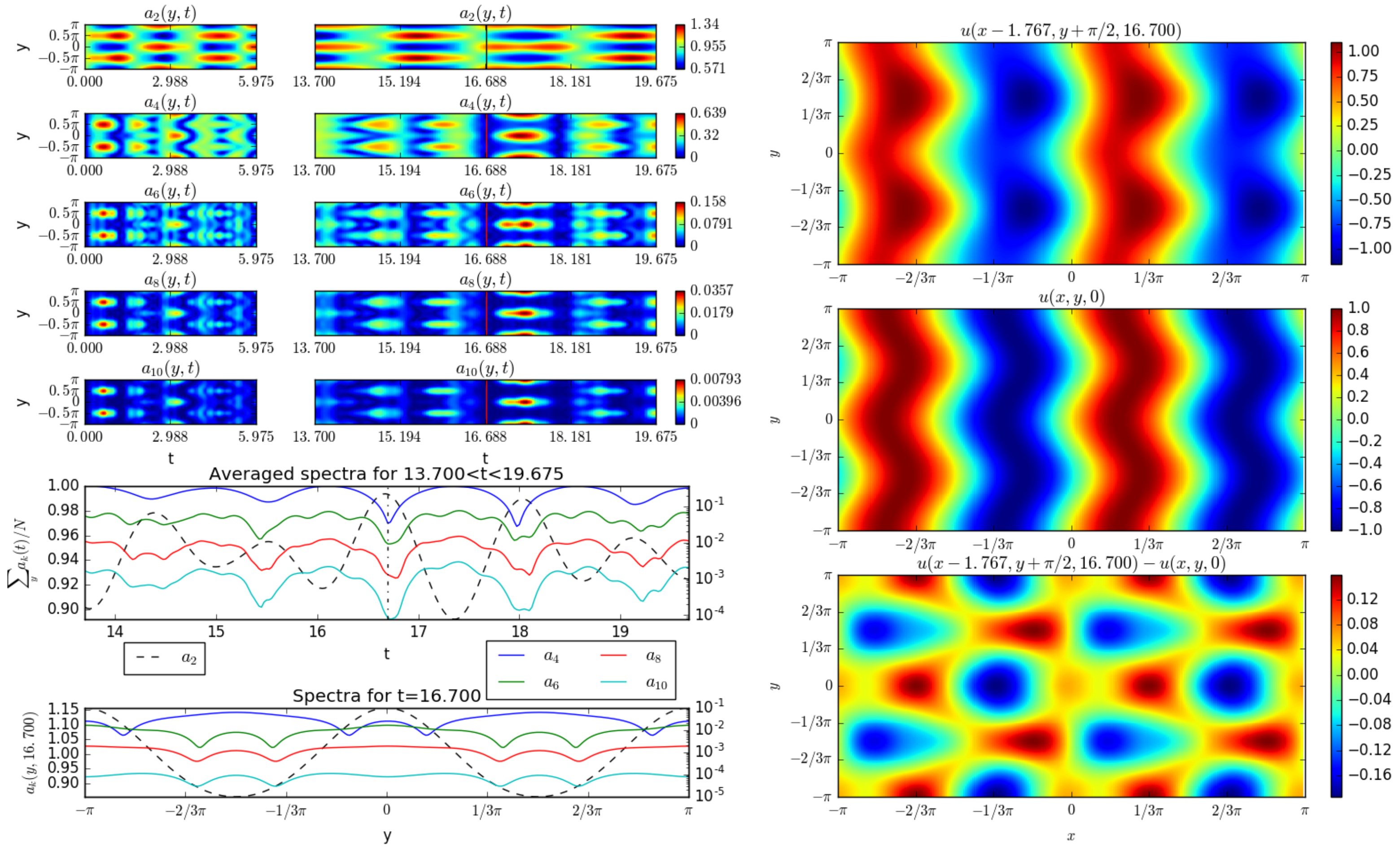

Figure 4.47: Recurrence in case of (3.9) and parameters $a_x = 2$, $\beta = 0.3$, $a_y = 2$ and dispersion parameter $\alpha_2 = 0.1$ at $t = 16.7$: full spectrum (upper left, time indicated by black/red line), averaged spectrum (middle left, time indicated by dotted line), spectra variance along y (lower left), wave profile at specified time (upper right), initial condition (middle right) and theri difference (lower right)

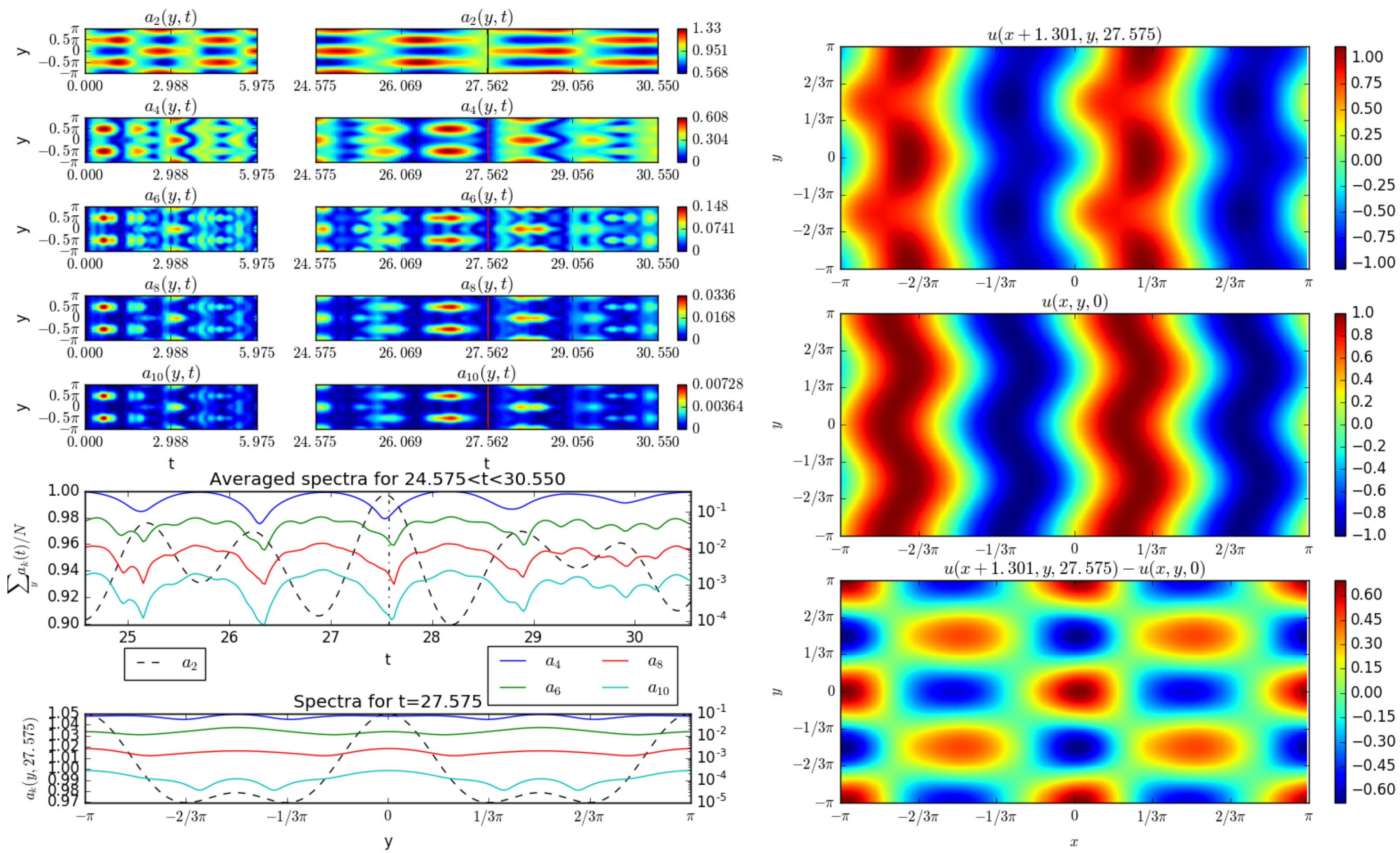

Figure 4.48: Recurrence in case of (3.9) and parameters $a_x = 2$, $\beta = 0.3$, $a_y = 2$ and dispersion parameter $\alpha_2 = 0.1$ at $t = 27.575$: full spectrum (upper left, time indicated by black/red line), averaged spectrum (middle left, time indicated by dotted line), spectra variance along y (lower left), wave profile at specified time (upper right), initial condition (middle right) and theri difference (lower right)

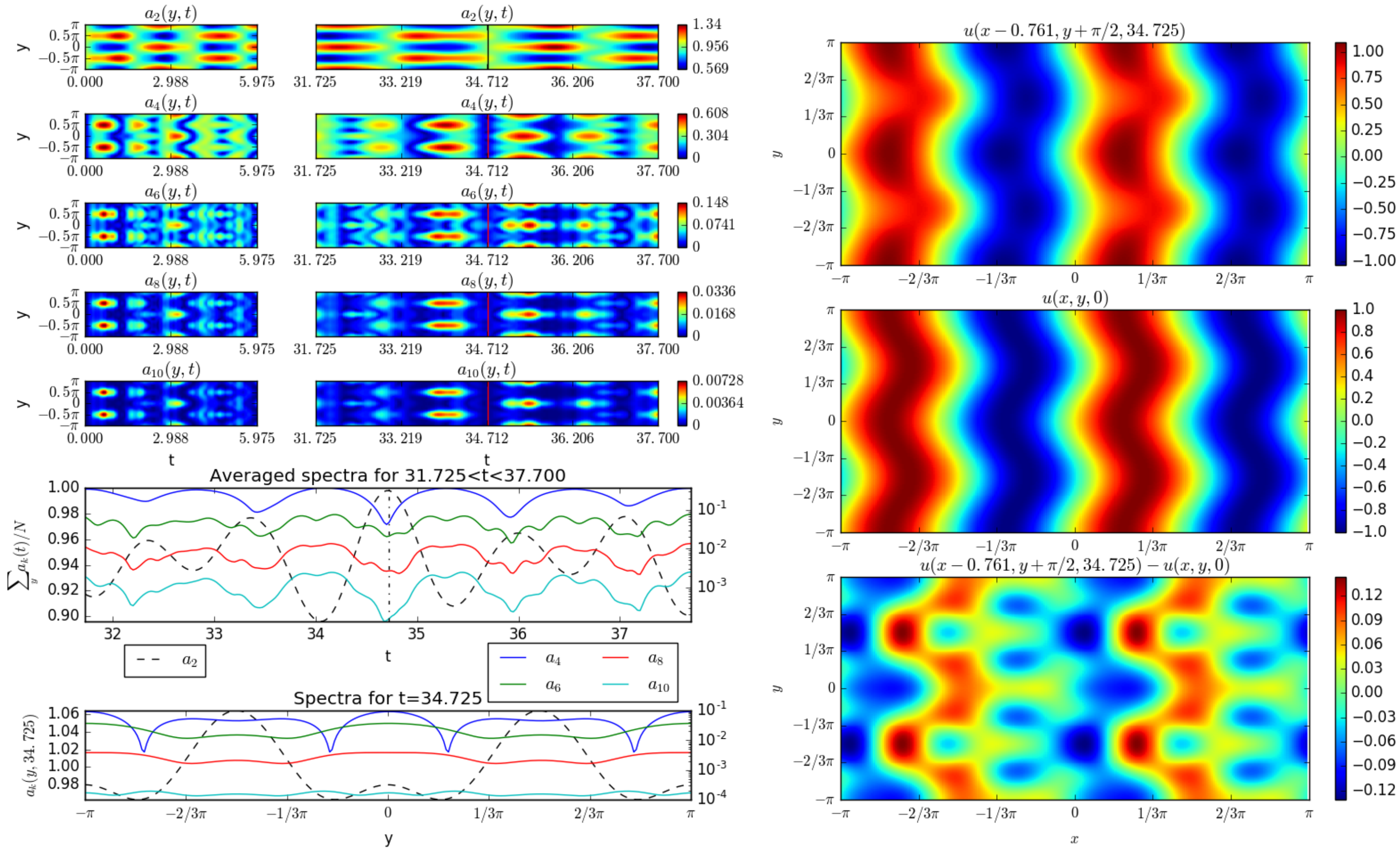

Figure 4.49: Recurrence in case of (3.9) and parameters $a_x = 2$, $\beta = 0.3$, $a_y = 2$ and dispersion parameter $\alpha_2 = 0.1$ at $t = 34.725$: full spectrum (upper left, time indicated by black/red line), averaged spectrum (middle left, time indicated by dotted line), spectra variance along y (lower left), wave profile at specified time (upper right), initial condition (middle right) and theri difference (lower right)

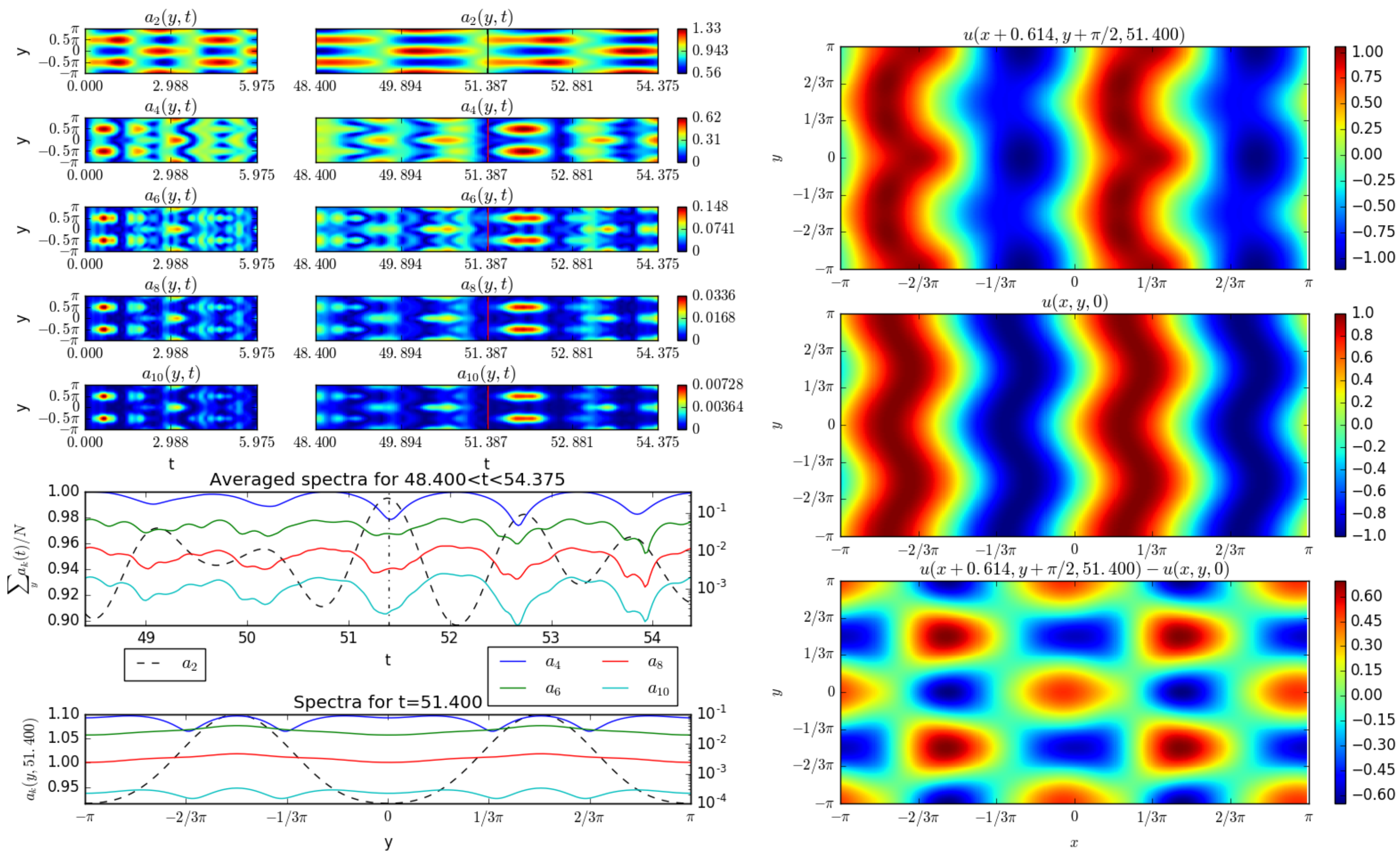

Figure 4.50: Recurrence in case of (3.9) and parameters $a_x = 2$, $\beta = 0.3$, $a_y = 2$ and dispersion parameter $\alpha_2 = 0.1$ at $t = 51.4$: full spectrum (upper left, time indicated by black/red line), averaged spectrum (middle left, time indicated by dotted line), spectra variance along y (lower left), wave profile at specified time (upper right), initial condition (middle right) and theri difference (lower right)

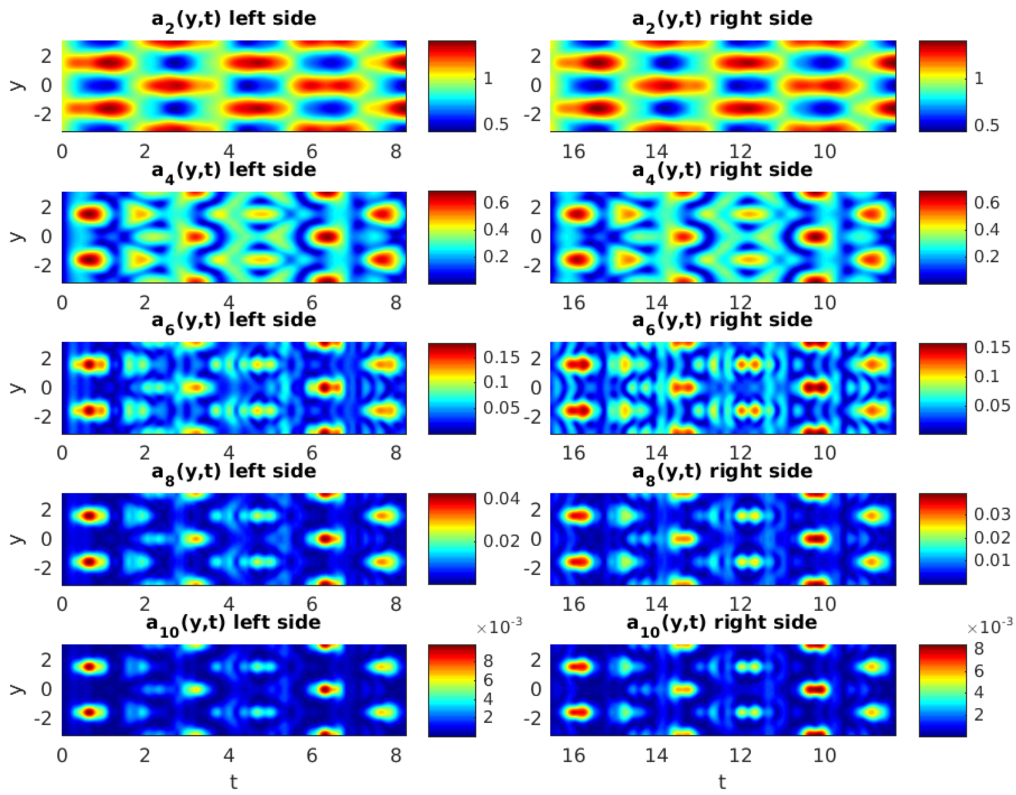

Figure 4.51: Temporal symmetries in case of parameters $a_x = 2$, $\beta = 0.4$, $a_y = 2$ around the time $8.275 < t^s < 8.3$

4.1.3 Temporal symmetries

In order to observe symmetry with respect to time we look at the KP equation with the dispersion parameter $\alpha_2 = 0.1$ and initial condition (3.9) with parameters $a_x = a_y = 2$ and $\beta = 0.4$. The spectral amplitudes in this case have already been shown in Fig. 4.13[5].

When taking a closer look at Fig. 4.13, it is evident that the spectral amplitudes around $8.275 < t < 8.3$ look as though they had been reflected off of that point. In Fig. 4.51 the left and the right side are plotted side by side with the temporal axes of the latter flipped for comparison. It is clear that there are slight differences between the two, but the difference is quite small. The maximum difference between the two spectra for the 2nd spectral amplitude is 4.9% and for the 4th spectral amplitude 10.4%. The following spectral amplitudes have a maximum difference of 20 – 30%. However, since their values are smaller in comparison to the 2nd and the 4th spectral amplitudes, they change the wave profile less.

[5]The example used in this subsection has an initial condition in which the odd number spectral amplitudes are extremely small which is why they have not been shown here

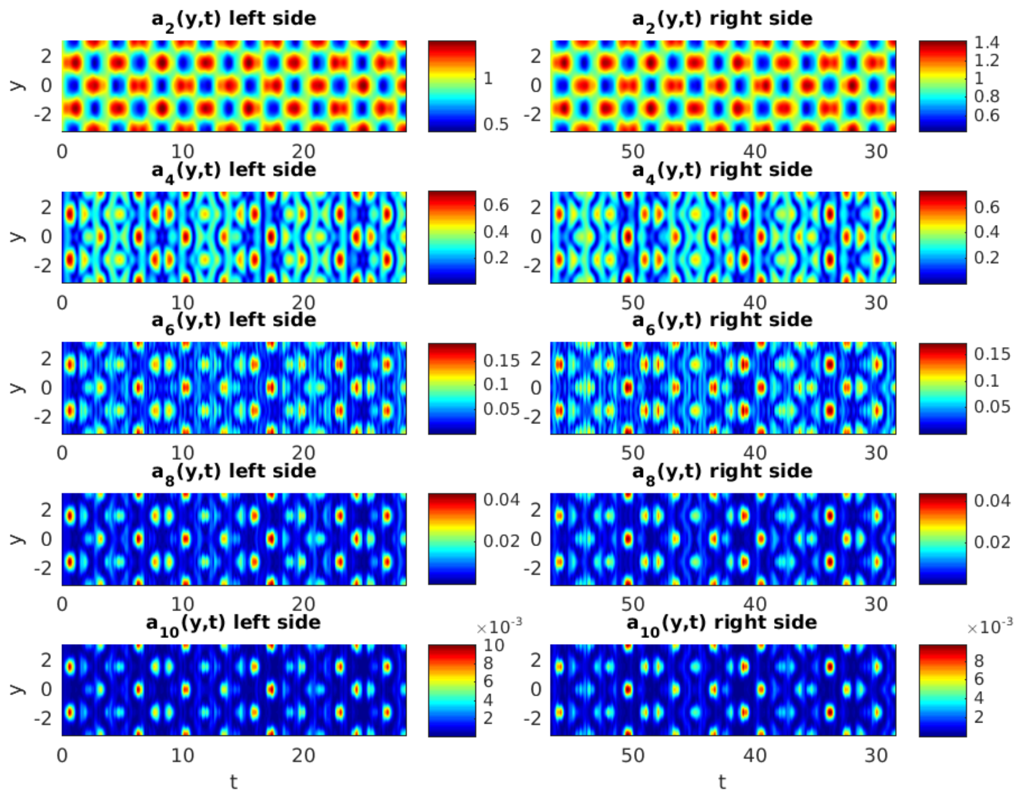

Figure 4.52: Temporal symmetries in case of parameters $a_x = 2$, $\beta = 0.4$, $a_y = 2$ around the time $28.4 < t^s < 28.425$

A similar point of temporal symmetry can be observed around a later time ($28.4 < t < 28.425$) in the same solution. Such a situation is depicted in Fig. 4.52. As with the previous case, it is clear that the two sides look very similar.

The above mentioned initial condition is not the only one where such temporal symmetries can be observed. It turns out changing only the strenght of the transverse pertubation to $\beta = 0.2$ or $\beta = 0.3$ will have the same effect. The spectra of the above mentioned solutions are shown in Figs. 4.39 and 4.45, respectively. The symmetries in the amplitude spectrum can be seen around $t^s \approx 22.875$ for the former (Fig. 4.53) and around $39.4 < t^s < 39.425$ for the latter (Fig. 4.54). In the case of $\beta = 0.2$ the maximum difference between the the two sides is around 1.9% for the 2nd spectral amplitudes, around 12.7% for the 4th spectral amplitude and around $20-30\%$ for the rest shown in Fig. 4.53. However, in case of $\beta = 0.3$ the symmetry is a little less exact with the maximum differences between the two sides at around 6% for the 2nd spectral amplitude, around 30.3% for the 4th spectral amplitude and around $30-40\%$ for the rest shown in Fig 4.54.

Thanks to the symmetries in time observed here, it is possible to predict the future (that is

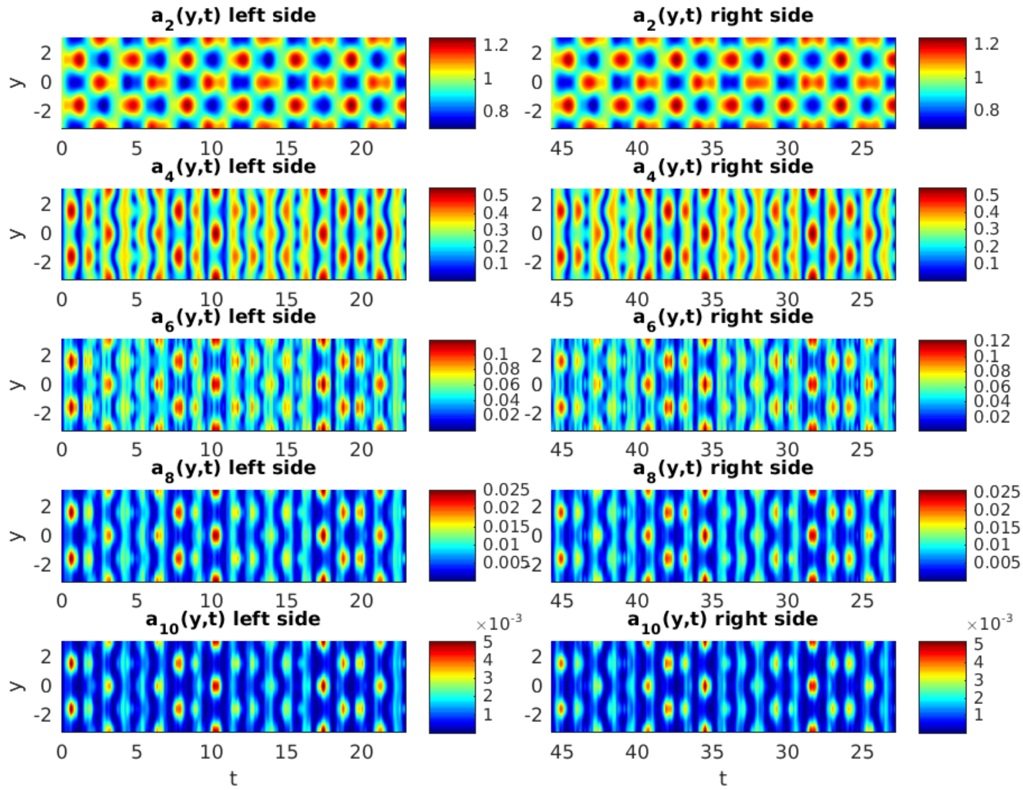

Figure 4.53: Temporal symmetries in case of parameters $a_x = 2$, $\beta = 0.2$, $a_y = 2$ around the time $t^s \approx 22.875$

find the rough wave profile from a time which the integration has not yet reached) or even the future (that is a time before the integration started). The latter should allow to start calculation from an already formed wave form and find a rough version of the initial condition with prior knowledge of the time that had passed. It is also important to note that even though the spectral amplitudes give a relatively good overview of the shape of the wave form, they do not provide all the information. Spectral amplitudes do not take into account the phases of their corresponding Fourier' coefficients. Therefore, even with the same (or extremely similar) amplitude spectrum, the wave profile will not always be the same. They can differ by phase. All this really means is that there is a shift with respect to the x axis (since we're looking at spectral amplitudes with respect to this axis). If we look at our given example's phase spectrum (Fig. 4.55), we can see similarities in that as well around the time $8.275 < t < 8.3$. Having a closer look at the phase spectrum (Fig. 4.56), it turns out that it is anti-symmetric ($\phi_k(y, t^s + \Delta t) \approx -\phi_k(y, t^s - \Delta t)$. The best match is only for the phase corresponding to the 2nd spectral amplitude, but the differences for higher spectral amplitudes are still local.

A similar situation can be described while looking at the phase around $28.4 < t < 28.425$ (Fig.

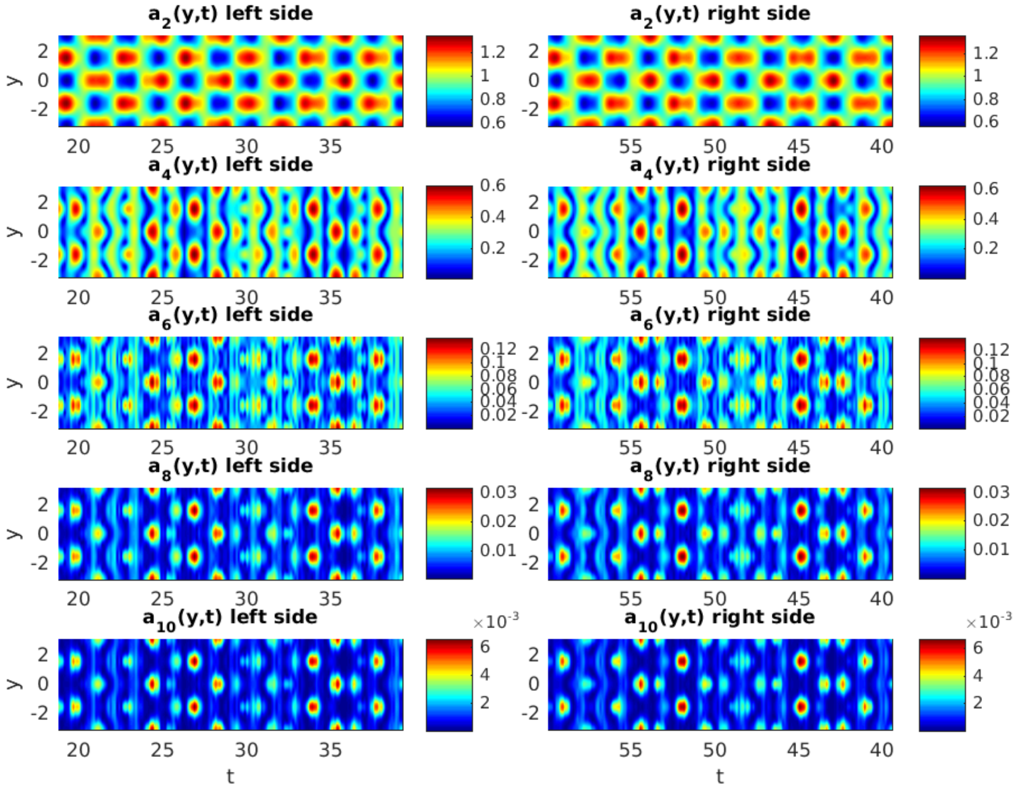

Figure 4.54: Temporal symmetries in case of parameters $a_x = 2$, $\beta = 0.3$, $a_y = 2$ around the time $39.4 < t^s < 39.425$

4.57). Even though at a glance the phase spectrum seems very different, it turns out that there is a (relatively) constant shift from being anti-phased between the two halves. It is evident from the fact that the difference of those phases revolve around a set value. In the current form, it is rather difficult to notice given that the phase has values $\phi_k(y,t) \in [-\pi, \pi]$ and because of the nature of phase it is possible to move from one extreme to another. This results in sudden changes within the spectrum which differ in location between the two sides. This is why it is easy to see the symmetry in amplitude spectrum – it does not have sudden jumps between extreme values.

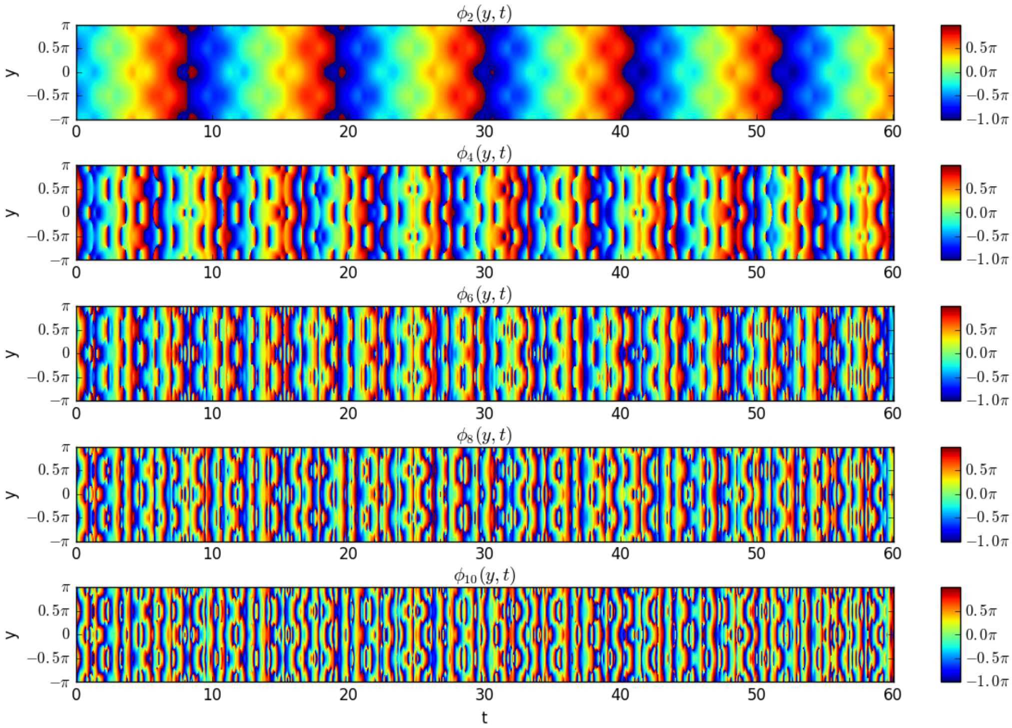

Figure 4.55: Phase spectrum in case of parameters $a_x = 2$, $\beta = 0.4$, $a_y = 2$

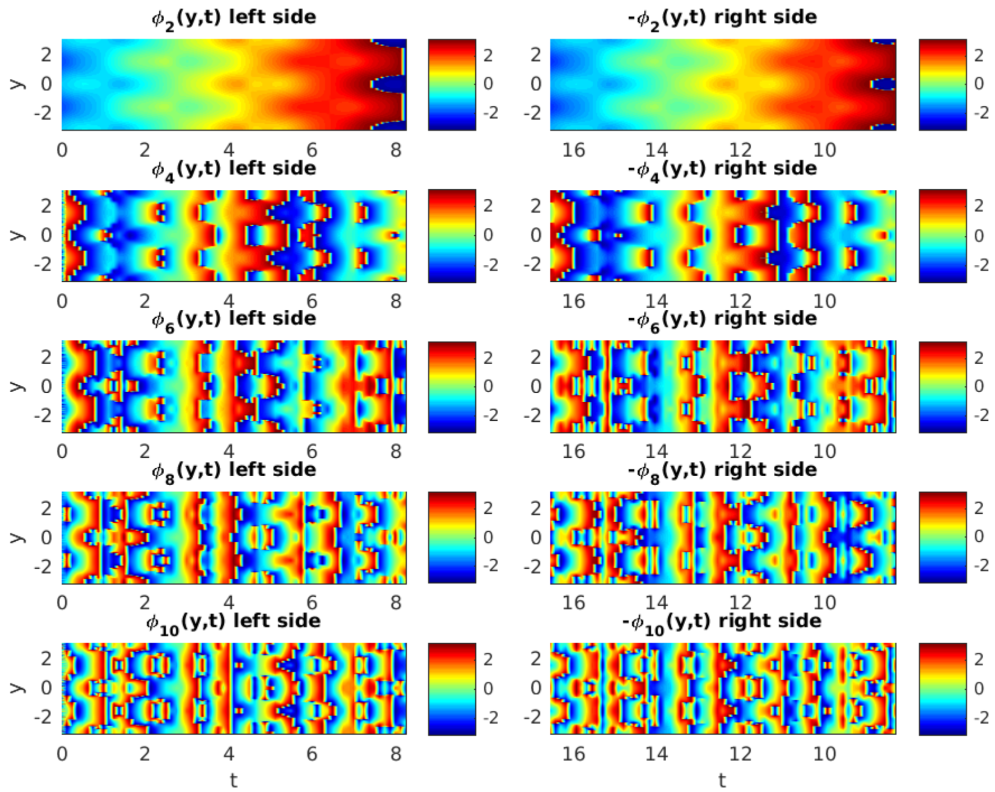

Figure 4.56: Temporal anti-symmetries in case of parameters $a_x = 2$, $\beta = 0.4$, $a_y = 2$ in the phase spectrum around the time $8.275 < t^s < 8.3$

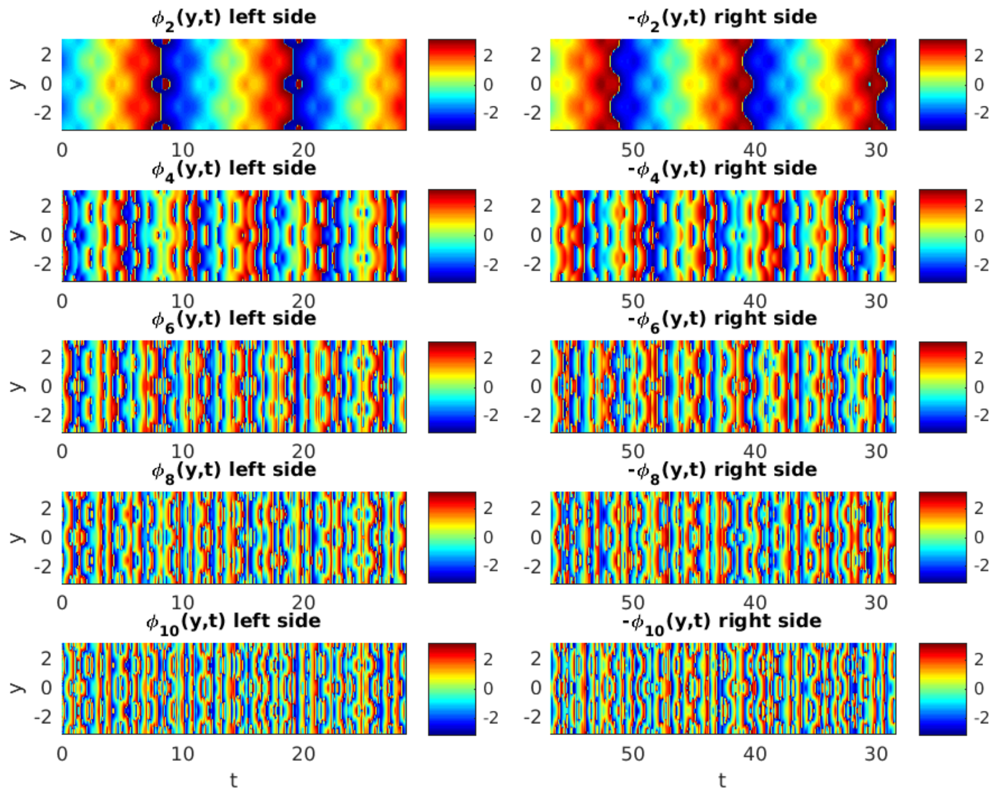

Figure 4.57: Temporal anti-symmetries in case of parameters $a_x = 2$, $\beta = 0.4$, $a_y = 2$ in the phase spectrum around the time $28.4 < t^s < 28.425$

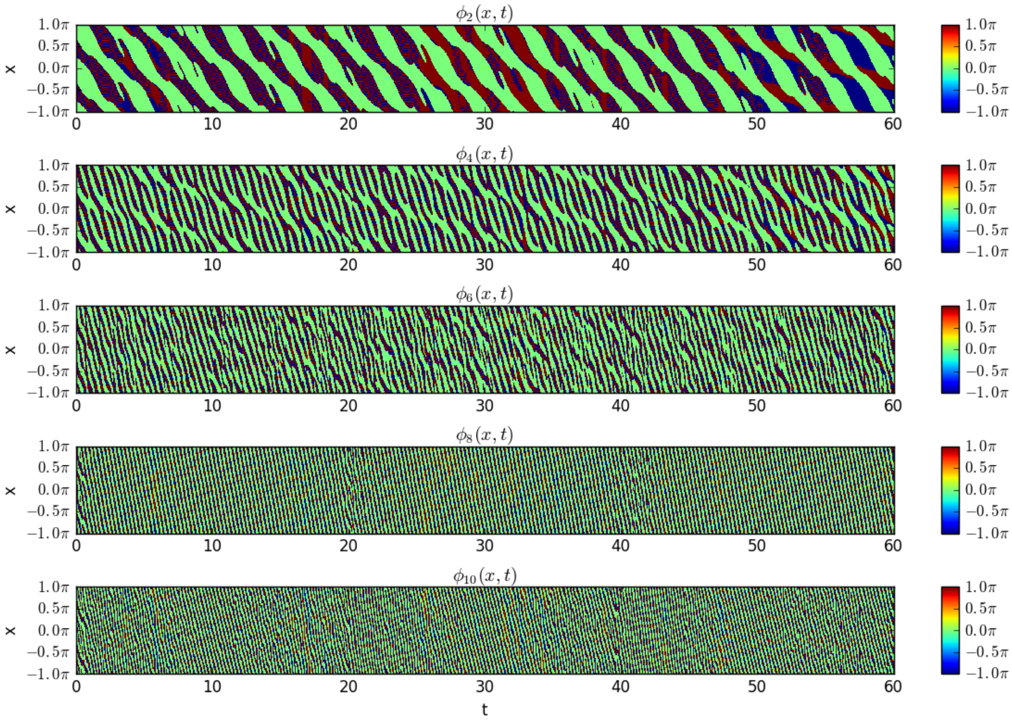

Figure 4.58: Phase spectrum along y axis in case of parameters $a_x = 2$, $\beta = 0.2$, $a_y = 2$ and $\alpha_2 = 1$

4.1.4 Number of solitons in a solution

Since the solutions in question within this work depend on two space variables and on one time variable, it is often difficult to understand how many solitons there are within a solution. In fact it can be unclear as to whether or not these solitons are moving along the y axis as well as the x axis. It is clear that movement along the x axis is indeed occuring looking at the phase spectrum of various solutions (e.g Fig. 4.55). Since the phase corresponding to the spectral amplitudes is changing, there is movement along the x axis. Not to mention the fact that recurrence states were often found with the wave out of phase with respect to the initial condition.

Now, another question arises as to whether or not the solitons move along the y axis. In order to see this, we once again introduce the phase spectrum. This time, however, we have taken the Fourier transform with respect to the y axis and will thus get the changes in that axis. One such phase spectrum is shown in Fig. 4.58, another in Fig. 4.59 and a third in Fig. 4.60. It must be noted that in these figures the vertical axis is x instead of y in the other spectrum figures. This is because having taken the Fourier transform in respect to the y axis we have the x axis left for the result to depend on. Using the figures mentioned above we can see that the phases are always either 0 (green) or $\pm\pi$ (red or blue). This indicates that the soliton is always at 0 phase or at π phase and is therefore in only 2 different positions along the y axis and never in between. Therefore it can be concluded that the solitons indeed do not move

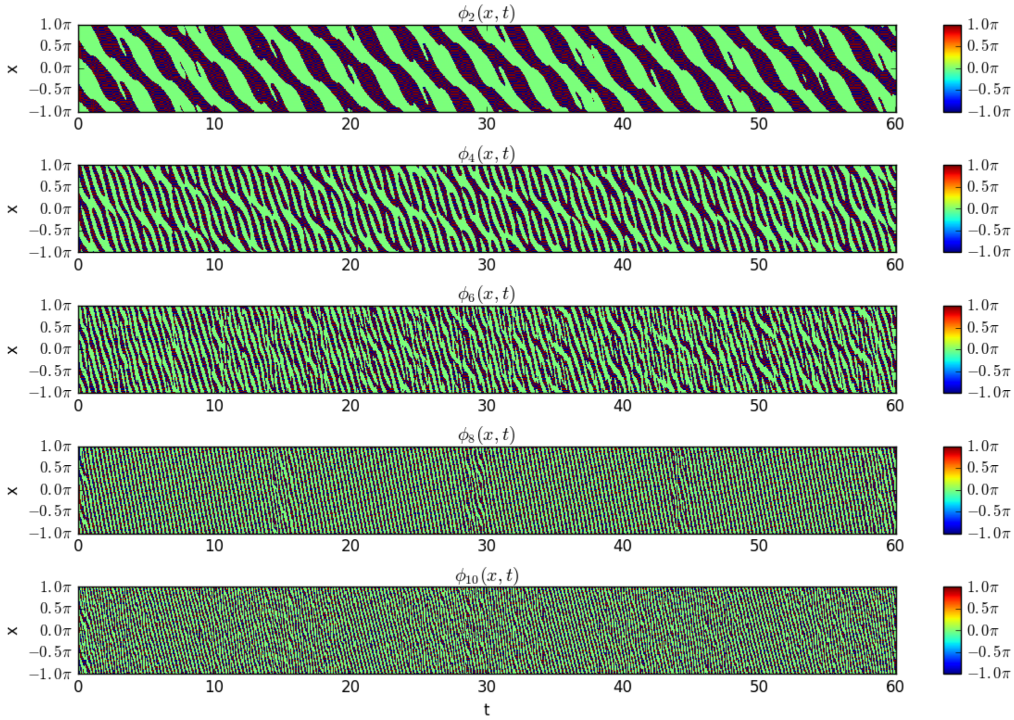

Figure 4.59: Phase spectrum along y axis in case of parameters $a_x = 2$, $\beta = 0.3$, $a_y = 2$ and $\alpha_2 = 1$

along the y axis. It must be noted that since an arbritrary nth spectral amplitude (and the phase it corresponds to) corresponds to the nth harmonic which has n periods within a 2π domain and therfore a phase shift of π means corresponds to a different shift number in space for different harmonics.

It is important to note that examples of only 3 cases were shown here, but similar phase spectrum can be seen in all cases considered within this work.

Having confirmed that the solitons within a solution moves only along the x axis, it is possible to fix specific points on the y axis and look at only one vector at a time. Each such vector can be treated as a solution to a KdV-like equation and thus tools that have been used to describe KdV equation solutions [**Salupere2003a**, 8, 15, 18] can be used here as well.

When picking a specific y value for the vector, it makes most sense to pick it where the solitons are at their maximum height. In all cases using the initial condition (3.9) with $a_y = 2$ (most of the solutions used within this work) the two values of interest are $y = 0$ and $y = \pi/2$. More peaks can be seen on the spectrum simply because there is essentialy 2 periods that have been calculated. The 2 values picked correspond to the 2 unique vectors within the solution, whereas $u(x,\pi,t) = u(x,0,t)$ and $u(x,-\pi/2,t) = u(x,\pi/2,t)$.

Now that we have successfully isolated 2 distinct vectors of a solution, we can use those vectors to draw the trajectories of maxima for those vectors. If we look for all local maxima, we

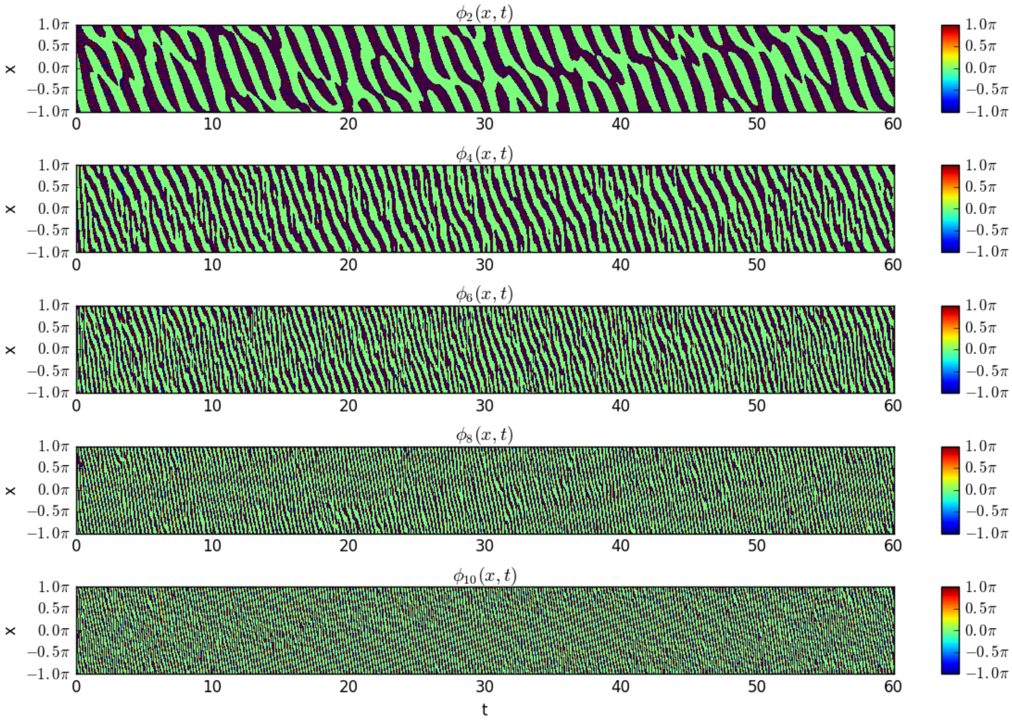

Figure 4.60: Phase spectrum along y axis in case of parameters $a_x = 1$, $\beta = 0.4$, $a_y = 2$ and $d_l = 1$

can also determine the amount of solitons observed at each instance in time and therefore also the maximum number of solitons observed within a solution's given y value. We have done exactly this in Figs. 4.61 and 4.62. In those figures it is clear that there are many solitons moving fast towards negative x. Those are the smaller maxima, the bigger maximum is moving slowly towards positive x instead. This can be seen by the fact that the place where the fast moving trajectories disappear is moving to the right in the figures.

This approach enabled counting the number of solitons in each different solutions computed. Table 4.2 presents those results as well as corresponding trajectories and spectral amplitudes. In table 4.2 N_s describes the number of solitons observed within a solution's $y = 0$ vector. Generally speaking, there will be 4 times as many solitons in the whole solution (there are 4 lines at $x = 0, \pi, \pm\pi$ on which solitons pop up, yet only 1 is shown in the figures and represented in the table). It must be noted that there is not too big a difference between Figs. 4.61 and 4.62. There are fast moving trajectories in both and holes in those trajectories move in the opposite direction. The difference between the two is therefore local and as such only $y = 0$ is shown for the rest of the solultions.

In Figs. 4.61 - 4.74 the time is selected such that around that time all the local maxima are visible for the longest around that point in time.

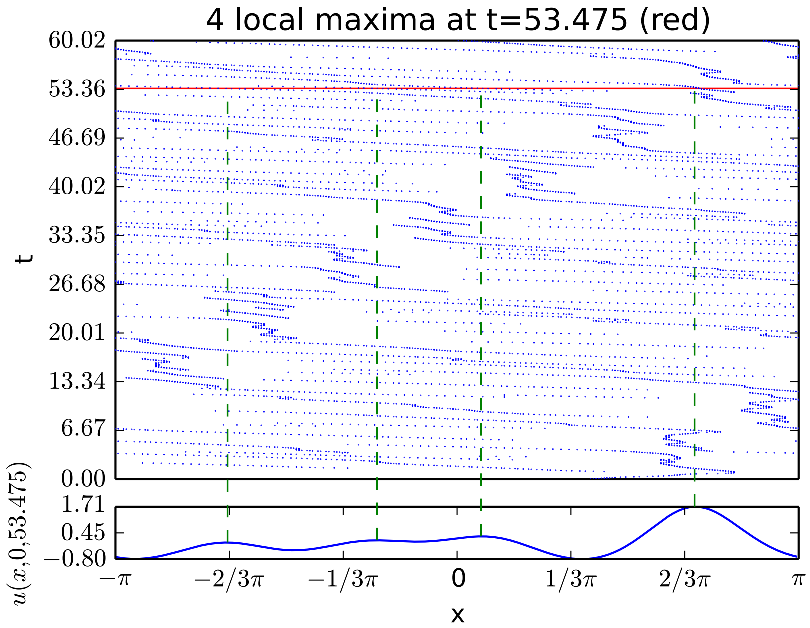

Figure 4.61: Trajectories in time for $y = 0$ in case of $a_x = 1$, $\beta = 0.4$, $a_y = 2$ and $\alpha_2 = 0.1$: the maximum amount of local maxima with an example in time

Table 4.2: Number of solitons in a given solution's $y = 0$ vector

Initial condition and parameters	Dispersion parameter	N_s	Trajectories Fig.	Spectra Fig.
(3.9), $a_x = 2$, $\beta = 0.4$, $a_y = 2$	$\alpha_2 = 1$	2	4.63	4.19
(3.9), $a_x = 2$, $\beta = 0.4$, $a_y = 2$	$\alpha_2 = 0.1$	4	4.64	4.13
	$\alpha_2 = 0.1$	4	4.61	4.3
	$\alpha_2 = 10^{-1.1}$	5	4.65	4.4
	$\alpha_2 = 10^{-1.2}$	5	4.66	4.5
	$\alpha_2 = 10^{-1.3}$	6	4.67	4.6
(3.9), $a_x = 1$, $\beta = 0.4$, $a_y = 2$	$\alpha_2 = 10^{-1.4}$	6	4.68	4.7
	$\alpha_2 = 10^{-1.5}$	6	4.69	4.8
	$\alpha_2 = 10^{-1.6}$	7	4.70	4.9
	$\alpha_2 = 10^{-1.7}$	8	4.71	4.10
	$\alpha_2 = 10^{-1.8}$	9	4.72	4.11
(3.9), $a_x = 2$, $\beta = 0.2$, $a_y = 2$	$\alpha = 0.1$	4	4.73	4.39
(3.9), $a_x = 2$, $\beta = 0.3$, $a_y = 2$	$\alpha = 0.1$	4	4.74	4.45

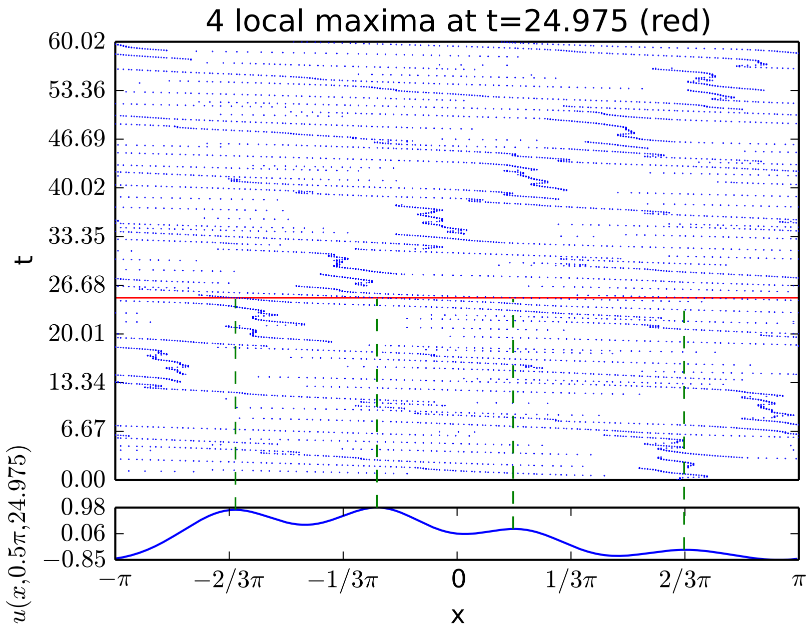

Figure 4.62: Trajectories in time for $y = \pi/2$ in case of $a_x = 1$, $\beta = 0.4$, $a_y = 2$ and $\alpha_2 = 0.1$: the maximum amount of local maxima with an example in time

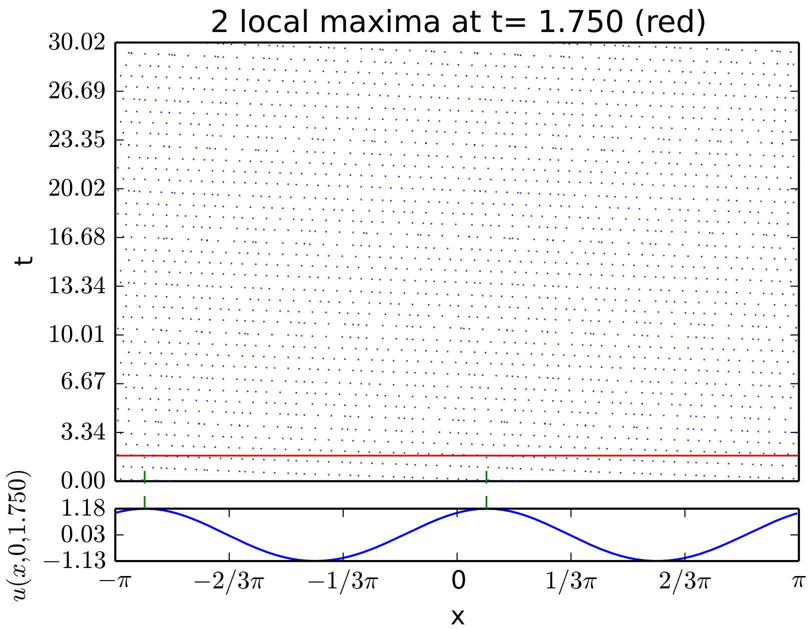

Figure 4.63: Trajectories in time for $y = 0$ in case of $a_x = 2$, $\beta = 0.4$, $a_y = 2$ and $\alpha_2 = 1$: the maximum amount of local maxima with an example in time

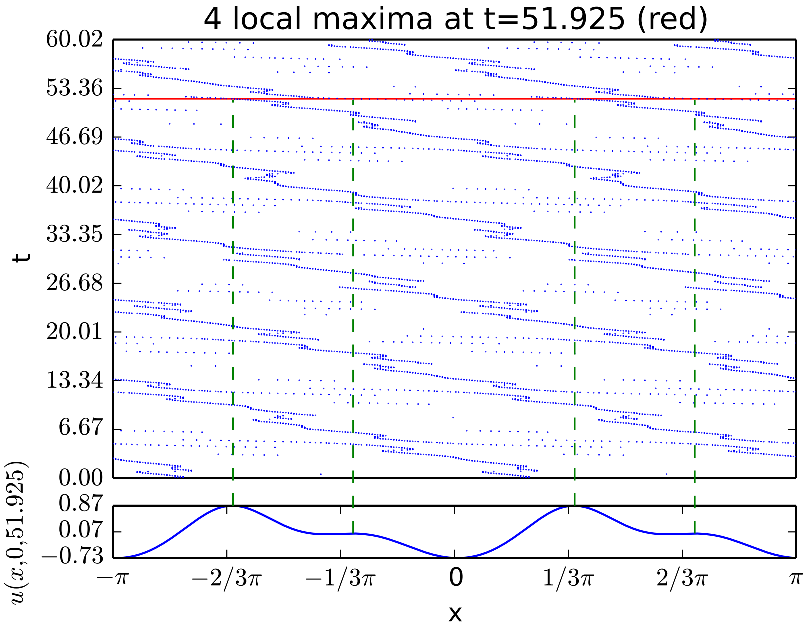

Figure 4.64: Trajectories in time for $y = 0$ in case of $a_x = 2$, $\beta = 0.4$, $a_y = 2$ and $\alpha_2 = 0.1$: the maximum amount of local maxima with an example in time

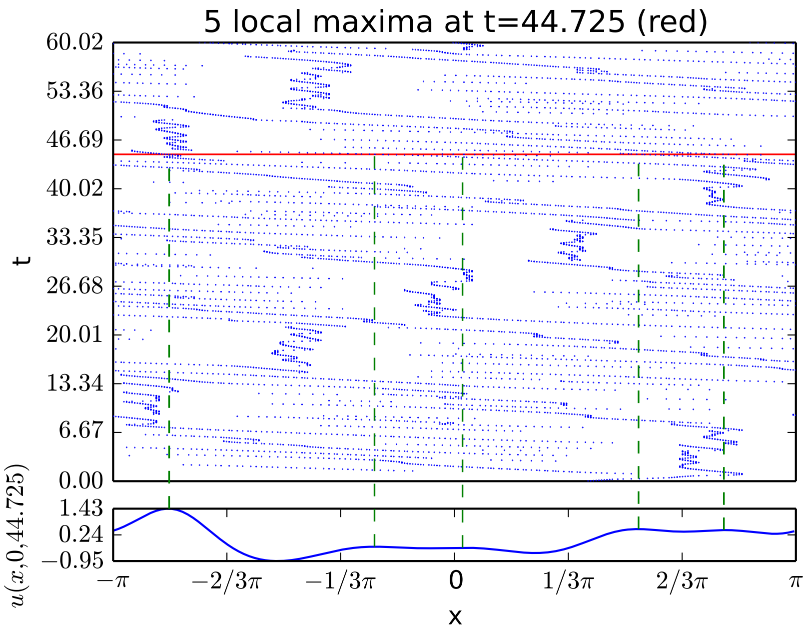

Figure 4.65: Trajectories in time for $y = 0$ in case of $a_x = 1$, $\beta = 0.4$, $a_y = 2$ and $\alpha_2 = 10^{-1.1}$: the maximum amount of local maxima with an example in time

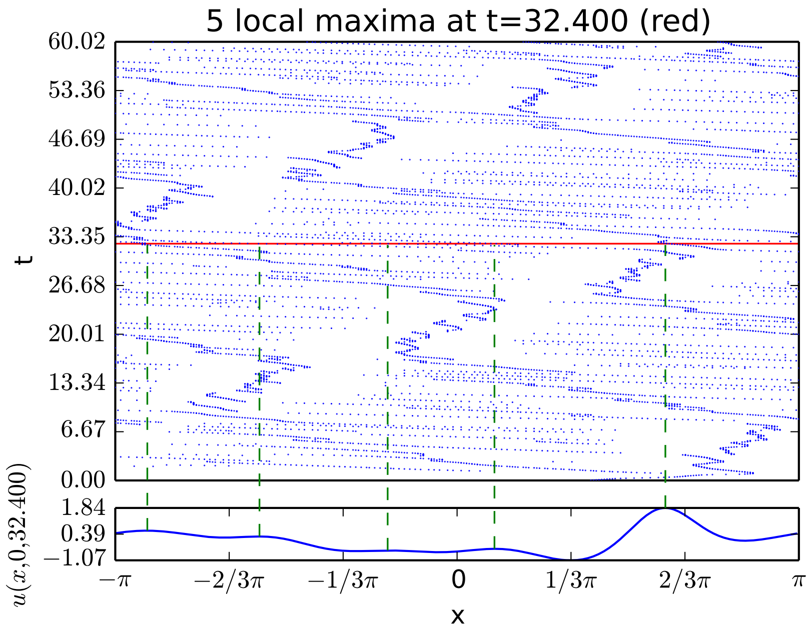

Figure 4.66: Trajectories in time for $y = 0$ in case of $a_x = 1$, $\beta = 0.4$, $a_y = 2$ and $\alpha_2 = 10^{-1.2}$: the maximum amount of local maxima with an example in time

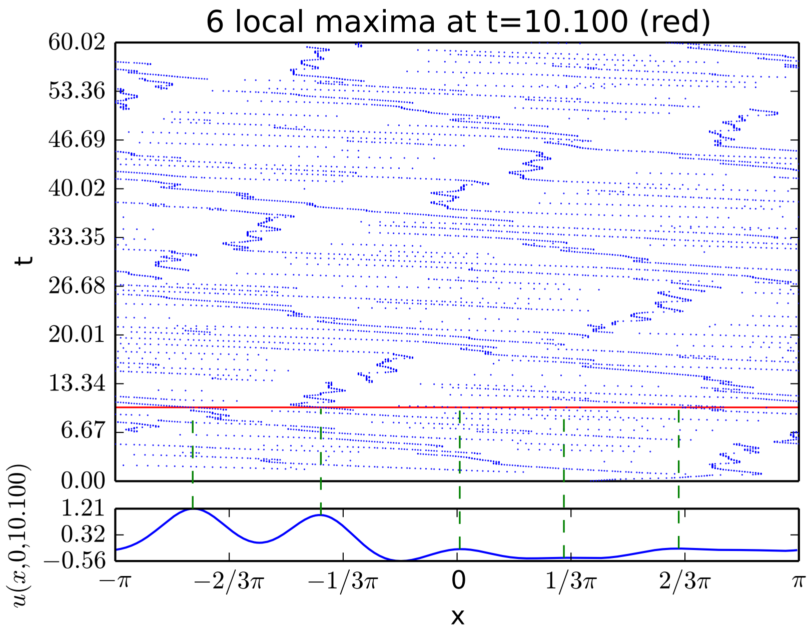

Figure 4.67: Trajectories in time for $y = 0$ in case of $a_x = 1$, $\beta = 0.4$, $a_y = 2$ and $\alpha_2 = 10^{-1.3}$: the maximum amount of local maxima with an example in time

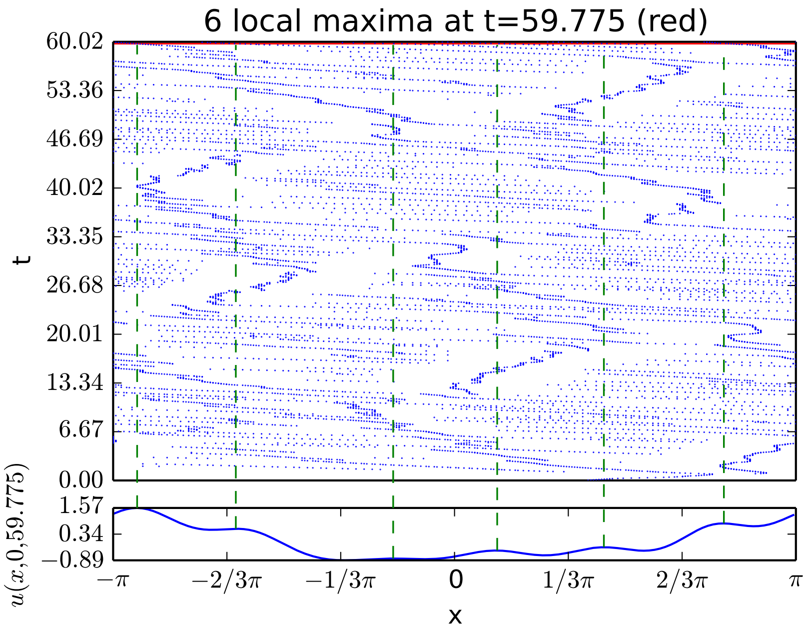

Figure 4.68: Trajectories in time for $y = 0$ in case of $a_x = 1$, $\beta = 0.4$, $a_y = 2$ and $\alpha_2 = 10^{-1.4}$: the maximum amount of local maxima with an example in time

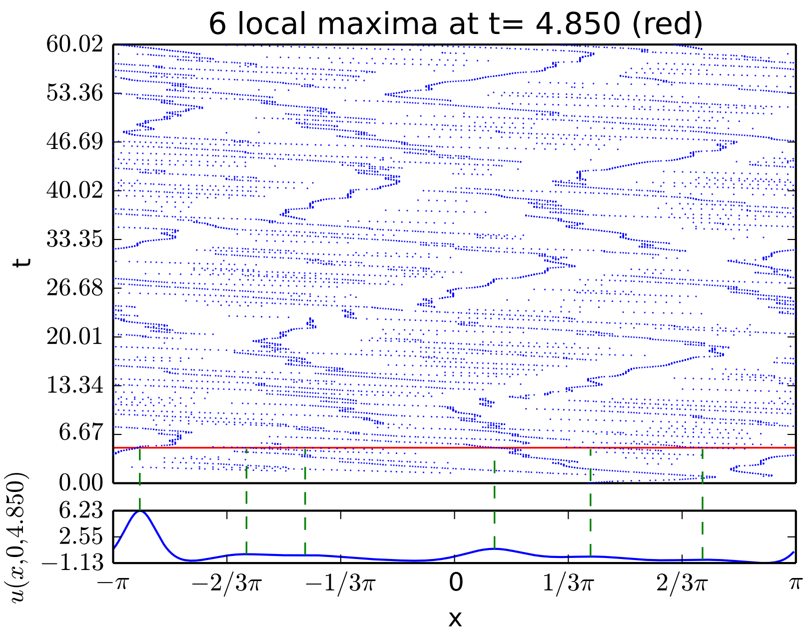

Figure 4.69: Trajectories in time for $y = 0$ in case of $a_x = 1$, $\beta = 0.4$, $a_y = 2$ and $\alpha_2 = 10^{-1.5}$: the maximum amount of local maxima with an example in time

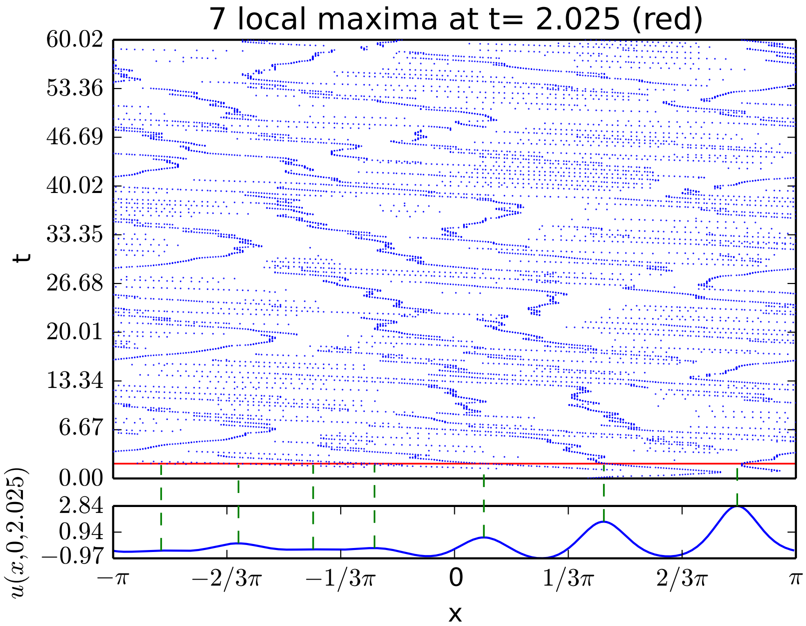

Figure 4.70: Trajectories in time for $y = 0$ in case of $a_x = 1$, $\beta = 0.4$, $a_y = 2$ and $\alpha_2 = 10^{-1.6}$: the maximum amount of local maxima with an example in time

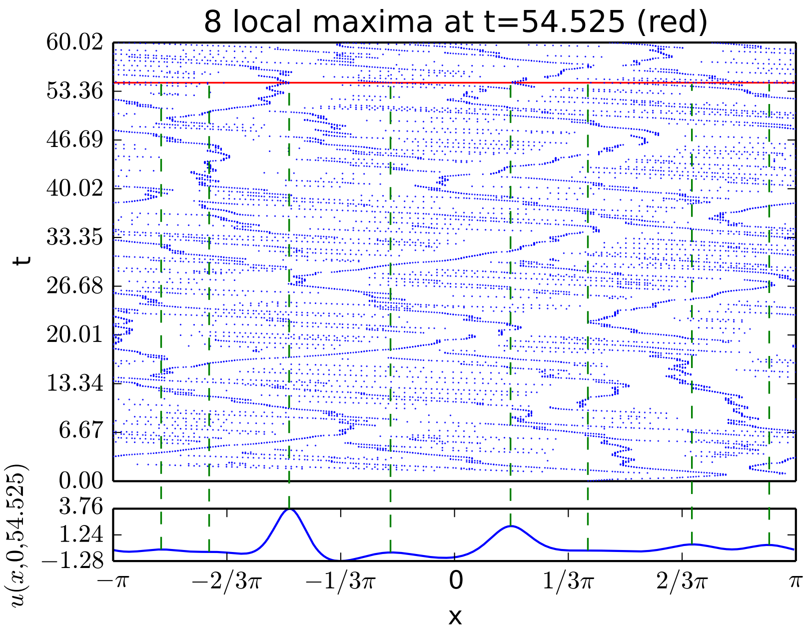

Figure 4.71: Trajectories in time for $y = 0$ in case of $a_x = 1$, $\beta = 0.4$, $a_y = 2$ and $\alpha_2 = 10^{-1.7}$: the maximum amount of local maxima with an example in time

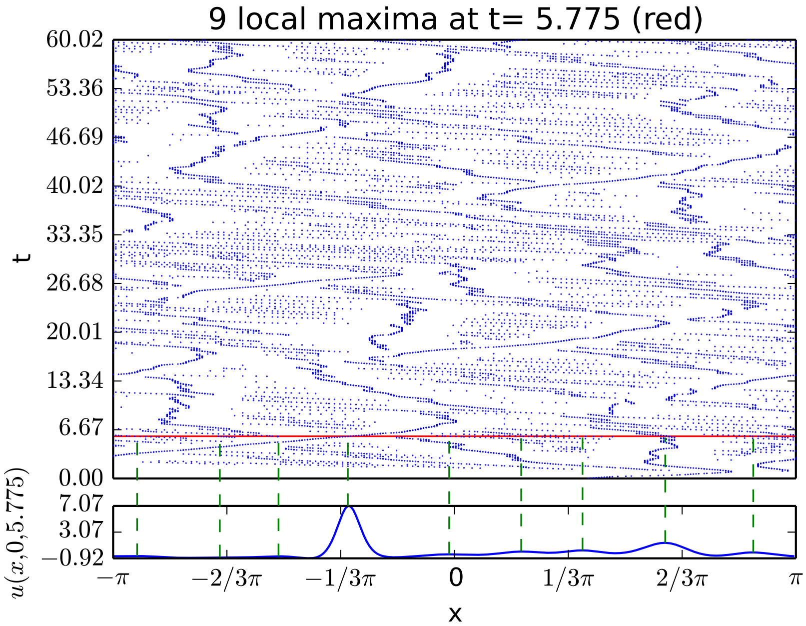

Figure 4.72: Trajectories in time for $y = 0$ in case of $a_x = 1$, $\beta = 0.4$, $a_y = 2$ and $\alpha_2 = 10^{-1.8}$: the maximum amount of local maxima with an example in time

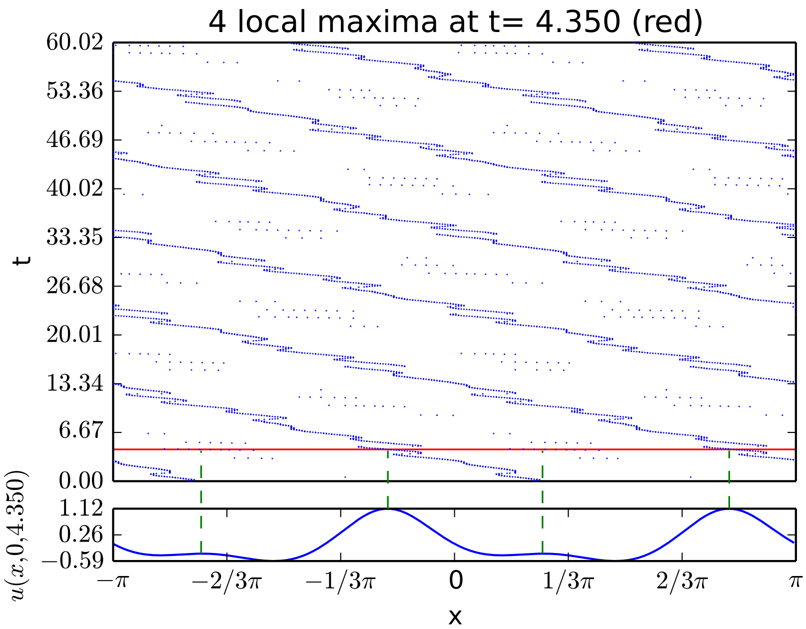

Figure 4.73: Trajectories in time for $y = 0$ in case of $a_x = 2$, $\beta = 0.2$, $a_y = 2$ and $\alpha_2 = 0.1$: the maximum amount of local maxima with an example in time

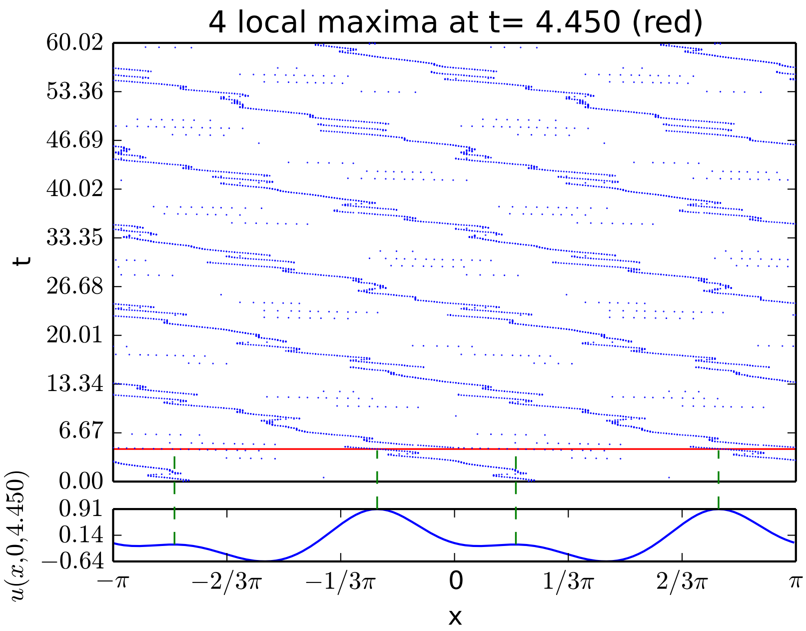

Figure 4.74: Trajectories in time for $y = 0$ in case of $a_x = 2$, $\beta = 0.3$, $a_y = 2$ and $\alpha_2 = 0.1$: the maximum amount of local maxima with an example in time

Table 4.3: All computations run

Initial condition and parameters	N	Dispersion parameter	t_f	Additional notes
(3.2), $x_0 = -25$	256	$\alpha_2 = 1$	30	20π periodic
(3.3), $x_0 = -25$, $\beta = 0.4$, $\delta = 0.2$	512	$\alpha_2 = 1$	1.3	20π periodic
(3.3), $x_0 = -25$, $\beta = 0.4$, $\delta = 0.1$	1024	$\alpha_2 = 1$	1.3	20π periodic
(3.3), $x_0 = -25$, $\beta = 0.4$, $\delta = 0.2$	1024	$\alpha_2 = 1$	7.7	20π periodic, still running
(3.8), $x_0 = -25$	512	$\alpha_2 = 1$	6.25	20π periodic
(3.8)[6], $x_0 = -25$, $\beta = 0.4$, $\delta = 0.2$	512	$\alpha_2 = 1$	3.3	20π periodic
(3.8)[7], $x_0 = -25$, $\beta = 0.4$, $\delta = 0.4$	512	$\alpha_2 = 1$	3.3	20π periodic
(3.11), $a = b = 2.5$	256	$\alpha_2 = 1$	7.5	
(3.9), $a_x = 2/3$, $\beta = 0.4$, $a_y = 2/3$	256	$\alpha_2 = 1$	30	6π periodic
(3.9), $a_x = 2$, $\beta = 0.4$, $a_y = 2$	256	$\alpha_2 = 1$	30	Periodicity
(3.9), $a_x = 2$, $\beta = 0.4$, $a_y = 2$	256	$\alpha_2 = 0.1$	60	Temporal symmetries
(3.9), $a_x = 1$, $\beta = 0.4$, $a_y = 2$	256	$\alpha_2 = 0.1$ $\alpha_2 = 10^{-1.1}$ $\alpha_2 = 10^{-1.2}$ $\alpha_2 = 10^{-1.3}$ $\alpha_2 = 10^{-1.4}$ $\alpha_2 = 10^{-1.5}$ $\alpha_2 = 10^{-1.6}$ $\alpha_2 = 10^{-1.7}$ $\alpha_2 = 10^{-1.8}$	60	
(3.9), $a_x = 2$, $\beta = 0.2$, $a_y = 2$	256	$\alpha = 0.1$	60	Temporal symmetries
(3.9), $a_x = 2$, $\beta = 0.3$, $a_y = 2$	256	$\alpha = 0.1$	60	Temporal symmetries

4.2 Computations run

This subsection attempts to list all the computations run on the High Performance Computing (HPC) farm within this work. The list is in table 4.3. Most of the computations were run with 2π periodic domain, if a computation has a different domain, it is written so in the notes.

4.3 Future possibilities

Currently the integration scheme does not work well with initial conditions where the periodic boundary condition is not satisfied. If this problem could somehow be handled, it would have a much wider area of applicability.

There has been several problems with initial conditions that do not satisfy the constraint (1.4). It is because the integration scheme then automatically moves makes the 0th spectral amplitude to 0. This results in situations where solitons move at a different speed to what was expected because of this phenomena. An initial idea to combat this was to use the integration constant to have the wave profile at a set level. This idea showed promise initially, but was not successful in the long run.

Additionally, it is clear that many more recurrences should be possible to be observed within the already found solutions.

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