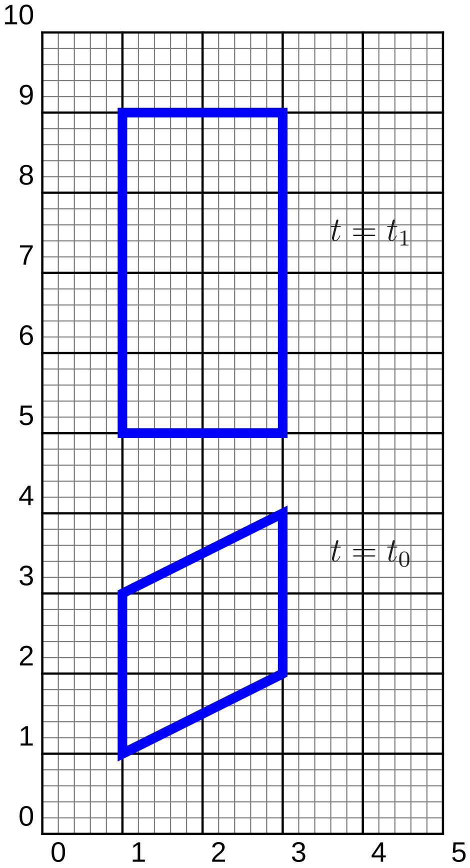
Näide 2.2.5 Pidev keskkond liigub tasapinnaliselt. On teada tema asend hetkel $t = t_0$ ja $t = t_1$. Leida deformatsioonitensorid ja keskkonna punktide siirded!

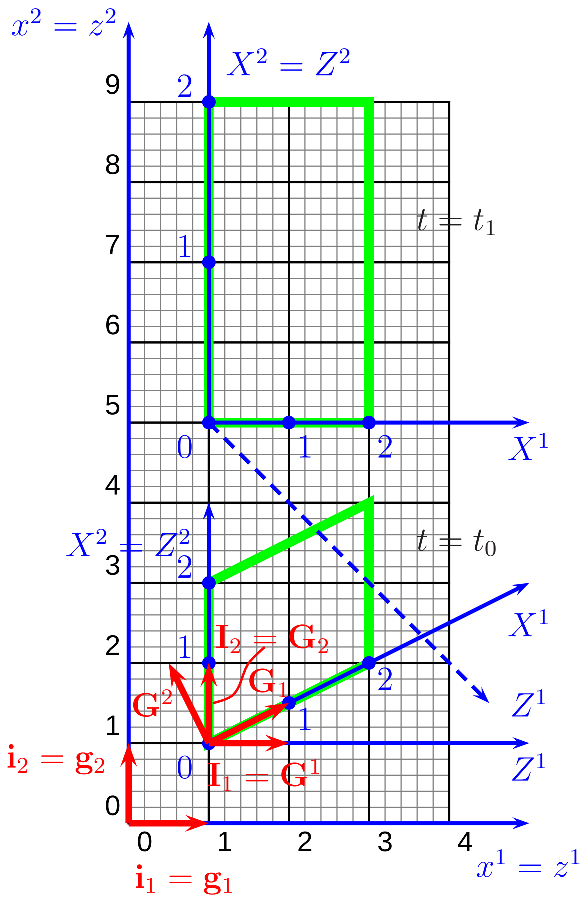


Joonis 1. Deformatsioon

Tegelikult tuleb:

- tuua sisse EDRK, EK, LDRK, LK;
- kirjeldada keskkonna liikumine;
- leida baasivektorid ja meetrilised tensorid;
- “uued baasivektorid” $\mathbf{c}_k$ ja $\mathbf{C}_K$ ning deformatsioonitensorid;
- siirdevektori $\mathbf{u}$ ko- ja kontravariantsed komponendid nii EK-s kui LK-s.

1) Valime EK $x^k = z^k$ (tasapinnaline ülesanne $\Rightarrow k = 1, 2$)



Joonis 2. Deformatsioon, koordinaadid ja baasivektorid

2) Vaatleme alghetke $t = t_0$

a) LDRK Z^K

$$\begin{cases} Z^1 = x^1 - 1, \\ Z^2 = x^2 - 1. \end{cases} \tag{1}$$

NB! LDRK ja EDRK ei ühti!

b) LK X^K — kaldteljestik mööda rööpküliku külgi:

$$\begin{cases} Z^1 = X^1, \\ Z^2 = 0.5X^1 + X^2. \end{cases} \tag{2}$$

Pöördteisendus

$$\begin{cases} X^1 = Z^1, \\ X^2 = Z^2 - 0.5Z^1. \end{cases} \tag{3}$$

c) EK ja LK vahelised seosed (koordinaatteisendused): (1)→(3) ja (2)→(1) $\Rightarrow$

$$\begin{cases} X^1 = x^1 - 1, \\ X^2 = -0.5x^1 + x^2 - 0.5. \end{cases} \tag{4}$$

$$\begin{cases} x^1 = X^1 + 1, \\ x^2 = 0.5X^1 + X^2 + 1. \end{cases} \tag{5}$$

d) Baasivektorid ja meetrilised tensorid.

EK: $z^k \equiv x^k \Rightarrow \mathbf{g}_k = \mathbf{g}^k = \mathbf{i}_k \Rightarrow g_{kl} = g^{kl} = \delta_{kl}$.

LK: kovariantne baas $\mathbf{G}_K = Z^M{}_{,K}\mathbf{I}_M$

$$\mathbf{G}_1 = 1 \cdot \mathbf{I}_1 + 0.5 \cdot \mathbf{I}_2, \qquad \mathbf{G}_2 = 0 \cdot \mathbf{I}_1 + 1 \cdot \mathbf{I}_2. \tag{6}$$

Kovariantne meetriline tensor $G_{KL} = \mathbf{G}_K \cdot \mathbf{G}_L = Z^M{}_{,K} Z^N{}_{,L} \delta_{MN}$

$G_{11} = \dots$

$G_{22} = \dots$

$G_{12} = G_{21} = \dots$

$$[G_{KL}] = \begin{bmatrix} 5/4 & 1/2 \\ 1/2 & 1 \end{bmatrix}. \tag{7}$$

Kontravariantne meetriline tensor

$$G^{KL} = \frac{\mathrm{cofactor(G_{KL})}}{G}, \qquad G = \begin{vmatrix} 5/4 & 1/2 \\ 1/2 & 1 \end{vmatrix} = \dots = 1. \tag{8}$$

$G^{11} = \dots \qquad G^{22} = \dots \qquad G^{21} = G^{12} = \dots$

$$\left[G^{KL}\right] = \begin{bmatrix} 1 & -1/2 \\ -1/2 & 5/4 \end{bmatrix}. \tag{9}$$

Kontroll:

$$[G_{KL}] \cdot \left[G^{LM}\right] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Kontravariantne baas: $\mathbf{G}^K = G^{KL}\mathbf{G}_L$.

$\mathbf{G}^1 = \dots \qquad \dots = \mathbf{I}_1$

$\mathbf{G}^2 = \dots \qquad \dots = -0.5\mathbf{I}_1 + \mathbf{I}_2$

Kontroll: $\mathbf{G}^K \cdot \mathbf{G}_L = \delta^K{}_L$

e) «Uued baasivektorid» $\mathbf{c}_k$ ja $\mathbf{C}_K$ ning deformatsioonitensorid c_{kl} (Cauchy DT), C_{KL} (Greeni DT), e_{kl} (Euleri DT) ja E_{KL} (Lagrange'i DT).

Vektor $\mathbf{c}_k = \mathbf{G}_K X^K{}_{,k}$. Kuna vaatleme hetke $t = t_0$, siis tuleb kasutada seoseid (4).

$\mathbf{c}_1 = \dots \qquad \dots = \mathbf{I}_1$

$\mathbf{c}_2 = \dots \qquad \dots = \mathbf{I}_2$

Kuna $\mathbf{i}_k = \mathbf{I}_K$, siis $c_{kl} = \mathbf{c}_k \cdot \mathbf{c}_l = \delta_{kl} = g_{kl}$, st. alghetkel ühtib Cauchy DT meetrilise tensoriga $g_{kl} = \delta_{kl}$.

Vektor $\mathbf{C}_K = \mathbf{g}_k x^k{}_{,K} = \mathbf{i}_k x^k{}_{,K}$. Kuna vaatleme hetke $t = t_0$, siis tuleb kasutada seoseid (5).

$\mathbf{C}_1 = \dots \qquad \dots = \mathbf{i}_1 + 0.5\mathbf{i}_2$

$\mathbf{C}_2 = \dots \qquad \dots = \mathbf{i}_2$

Kuna $\mathbf{i}_k = \mathbf{I}_K$, siis $\mathbf{G}_K = \mathbf{C}_K$ ja $C_{KL} = \mathbf{C}_K \cdot \mathbf{C}_L = G_{KL}$, st. Greeni DT ühtib meetrilise tensoriga G_{KL}.

Lagrange'i- ja Euleri DT $2E_{KL} = C_{KL} - G_{KL} = 0$ ja $2e_{kl} = g_{kl} - c_{kl} = 0$.

Tulemused on ootuspärased, sest alghetkel $t = t_0$ on keskkond loomulikus ehk deformeerumata olekus.

a) Hetkel $t = t_1$ on keskkonna deformeerunud kuju esitatav koordinaatteisendustega

$$\begin{cases} X^1 = x^1 - 1, \\ X^2 = 0.5x^2 - 2.5. \end{cases} \tag{10}$$

$$\begin{cases} x^1 = X^1 + 1, \\ x^2 = 2X^2 + 5. \end{cases} \tag{11}$$

LK ja LDKR on teinud deformatsiooni kaasa, kusjuures Z^K pole enam ristkoordinaadid ja X^K on muutunud ristkoordinaatideks.

Nende omavaheline suhe on aga jäänud samaks: tingimus $Z^2 = 0$ määrab Z^1 telje asukoha (sirge $X^2 = -0.5X^1$). Samaks on jäänud ka vektorite $\mathbf{G}_K$ ja $\mathbf{I}_K$ vaheline seos.

«Uued baasivektorid» $\mathbf{c}_k = \mathbf{G}_K X^K{}_{,k}$ ja $\mathbf{C}_K = \mathbf{g}_k x^k{}_{,K}$ leitakse nüüd avaldiste (10) ja (11) abil.

$\mathbf{c}_1 = \ldots \qquad \ldots = \mathbf{I}_1 + 0.5\mathbf{I}_2$

$\mathbf{c}_2 = \ldots \qquad \ldots = 0.5\mathbf{I}_2$

$\mathbf{C}_1 = \ldots \qquad \ldots = \mathbf{i}_1$

$\mathbf{C}_2 = \ldots \qquad \ldots = 2\mathbf{i}_2$

b) Cauchy DT $c_{kl} = \mathbf{c}_k \cdot \mathbf{c}_l = G_{KL} X^K{}_{,k} X^L{}_{,l}$ ja Greeni DT $C_{KL} = \mathbf{C}_K \cdot \mathbf{C}_L = g_{kl} x^k{}_{,K} x^l{}_{,L}$.

$c_{11} = \ldots, \qquad c_{22} = \ldots, \qquad c_{12} = c_{21} = \ldots.$

$C_{11} = \ldots, \qquad C_{22} = \ldots, \qquad C_{12} = C_{21} = \ldots.$

$$[c_{kl}] = \frac{1}{4}\begin{bmatrix} 5 & 1 \\ 1 & 1 \end{bmatrix}, \qquad [C_{KL}] = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}. \tag{12}$$

c) Euleri ja Lagrange'i DT:

$$2\,[e_{kl}] = [g_{kl} - c_{kl}] = \qquad = -\frac{1}{4}\begin{bmatrix} 1 & 1 \\ 1 & -3 \end{bmatrix} \tag{13}$$

$$2\,[E_{KL}] = [C_{KL} - G_{KL}] = \qquad = \begin{bmatrix} -1/4 & -1/2 \\ -1/2 & 3 \end{bmatrix} \tag{14}$$

Kontroll: $E_{KL} = e_{kl} x^k{}_{,K} x^l{}_{,L}$ ja $e_{kl} = E_{KL} X^K{}_{,k} X^L{}_{,l}$.

$E_{11} = \ldots$

$e_{11} = \ldots$

jne.

d) Siirdevektor $\mathbf{u}$

$$\mathbf{u} = \mathbf{p}_1 - \mathbf{p}_0 = \underbrace{\mathbf{x}(\mathbf{X}, t_1)}_{(11)} - \underbrace{\mathbf{x}(\mathbf{X}, t_0)}_{(5)} \tag{15}$$

$$\mathbf{u} = U^L \mathbf{G}_L = U_L \mathbf{G}^L = u^l \mathbf{g}_l = u_l \mathbf{g}^l \tag{16}$$

LK

$$U^1 = \underbrace{x^1(X^1, X^2, t_1)}_{(11)} - \underbrace{x^1(X^1, X^2, t_0)}_{(5)} = \qquad = 0$$

$$U^2 = \underbrace{x^2(X^1, X^2, t_1)}_{(11)} - \underbrace{x^2(X^1, X^2, t_0)}_{(5)} = \qquad = -0.5X^1 + X^2 + 4 \tag{17}$$

(U^1, U^2) — materiaalse punkti (X^1, X^2) siire.

EK

$\mathbf{u} = u^l \mathbf{g}_l = U^L \mathbf{G}_L | \cdot \mathbf{g}^k \Rightarrow u^k = U^L \mathbf{G}_L \cdot \mathbf{g}^k$

Meil $\mathbf{g}^k = \mathbf{g}_k = \mathbf{i}_k$, seega $u^k = U^L \mathbf{G}_L \cdot \mathbf{i}_k$

$$u^1 = U^L \mathbf{G}_L \cdot \mathbf{i}_1 = \qquad = 0,$$

$$u^2 = U^L \mathbf{G}_L \cdot \mathbf{i}_2 = \qquad = -x^1 + x^2 + 4. \tag{18}$$

(u^1, u^2) — hetkel $t = t_0$ ruumipunktis (x^1, x^2) asunud materiaalse punkti siire.

Siirdevektori kovariantsed komponendid: $U_K = G_{KL} U^L$

$$U_1 = \ldots \qquad = -0.25X^1 + 0.5X^2 + 2$$

$$U_2 = \ldots \qquad = -0.5X^1 + X^2 + 4 \tag{19}$$