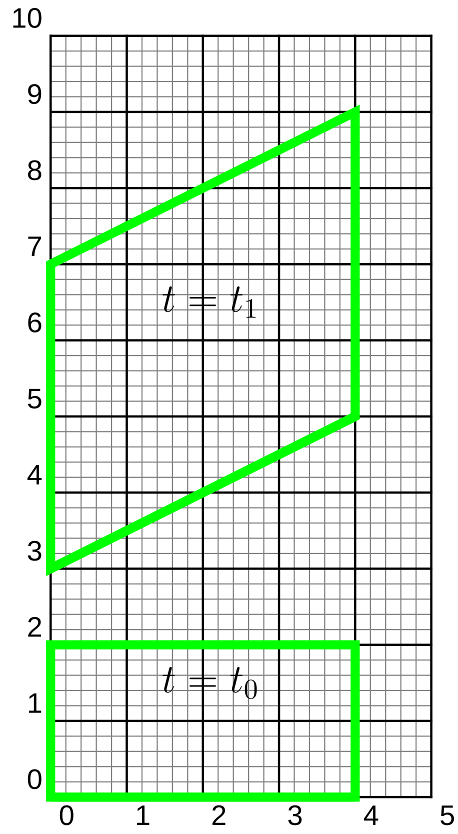
Näide K1: deformatsioonitensorite ja siirdevektorite leidmine

Pidev keskkond liigub tasapinnaliselt. Tema asend hetkel $t = t_0$ ja $t = t_1$ on kujutatud joonisel.

- Tuua sisse EDRK ja LDRK, ning vastavad baasivektorid.
- Kirjeldada keskkonna liikumine.
- Leida deformatsioonigradiendid.
- Leida "uued baasivektorid" $\mathbf{c}_k$ ja $\mathbf{C}_K$.
- Leida Cauchy, Greeni, Fingeri, Piola, Euleri ja Lagrange'i deformatsioonitensorid.
- Leida siirdevektori $\mathbf{u}$ komponendid nii EK-s kui LK-s.



Joonis 1. Deformatsioon

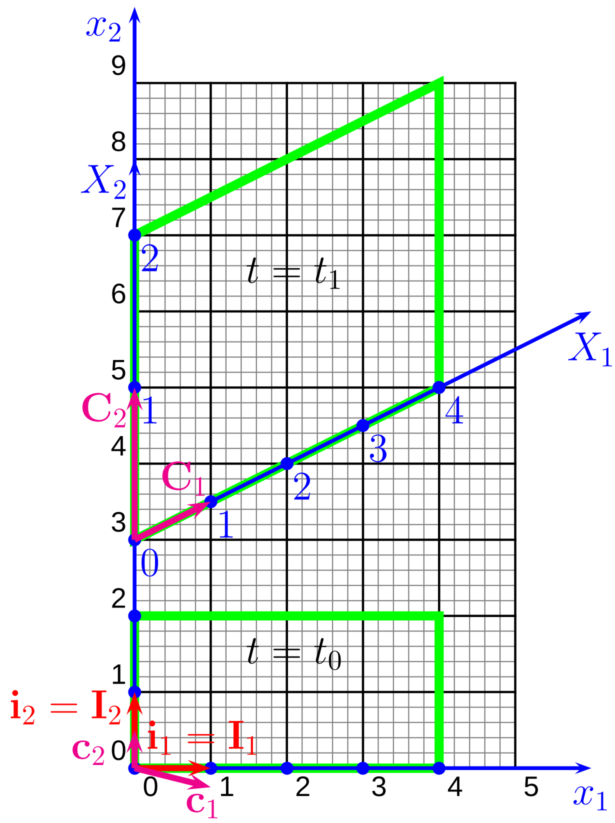
- Alghetkel $t = t_0$ langevad EK ja LK kokku, st. $x_k \equiv X_K$ ja $\mathbf{i}_k \equiv \mathbf{I}_K$, kui $k = K$.
- Hetkel $t = t_1$ on LK deformeerunud koos keskkonnaga — LK nullpunkt asub ruumipunktis $(0,3)$ ja ristkoordinaadid on muutunud kaldkoordinaatideks.
- Liikumisseadus:

$$\begin{cases} x_1 = X_1, \\ x_2 = 0,5X_1 + 2X_2 + 3 \end{cases} \quad (1)$$

$$\begin{cases} X_1 = x_1, \\ X_2 = -0,25x_1 + 0,5x_2 - 1,5. \end{cases} \quad (2)$$

- Kontroll:

$(X_1, X_2) = (2,2) \overset{(1)}{\rightarrow} (x_1, x_2) = (2,8)$,
$(X_1, X_2) = (4,2) \overset{(1)}{\rightarrow} (x_1, x_2) = (4,9)$,
$(x_1, x_2) = (2,8) \overset{(2)}{\rightarrow} (X_1, X_2) = (2,2)$,
$(x_1, x_2) = (4,9) \overset{(2)}{\rightarrow} (X_1, X_2) = (4,2)$.



Joonis 2. Deformatsioon, koordinaadid ja baasivektorid

- Deformatsioonigradiendid

$$[x_{k,K}] \overset{(1)}{=} \begin{bmatrix} 1 & 0 \\ 0,5 & 2 \end{bmatrix}, \qquad [X_{K,k}] \overset{(2)}{=} \begin{bmatrix} 1 & 0 \\ -0,25 & 0,5 \end{bmatrix}. \tag{3}$$

- Uued baasivektorid:

$$\mathbf{c}_k = X_{K,k}\mathbf{I}_K \qquad \text{ja} \qquad \mathbf{C}_K = x_{k,K}\mathbf{i}_k \tag{4}$$

$$\begin{cases} \mathbf{c}_1 = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots = \mathbf{I}_1 - 0,25\mathbf{I}_2, \\ \mathbf{c}_2 = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots = 0,5\mathbf{I}_2, \end{cases} \tag{5}$$

$$\begin{cases} \mathbf{C}_1 = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots = \mathbf{i}_1 + 0,5\mathbf{i}_2, \\ \mathbf{C}_2 = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots = 2\mathbf{i}_2. \end{cases} \tag{6}$$

- Baasivektorite muutumine: võrrelge joonist 2 ja loengukonspekti jooniseid lk. 3-19 ja 3-20.

- Materiaalses punktis $X_1 = X_2 = 0$ rakendatud vektorid $\mathbf{c}_1$ ja $\mathbf{c}_2$ deformeeruvad hetkeks $t = t_1$ EDRK baasivektoriteks $\mathbf{i}_1$ ja $\mathbf{i}_2$.

$$\begin{cases} (x_1, x_2) = (0;0) \to (x_1, x_2) = (0;3); \\ (x_1, x_2) = (1;-0,25) \to (x_1, x_2) = (1;3); \\ (x_1, x_2) = (0;0,5) \to (x_1, x_2) = (0;4); \end{cases} \tag{7}$$

- Materiaalse punkti kohavektor alghetkel $t = t_0$

$$\mathbf{P} = X_1\mathbf{I}_1 + X_2\mathbf{I}_2 = x_1\mathbf{i}_1 + x_2\mathbf{i}_2 \tag{8}$$

- Materiaalse punkti kohavektor hetkel $t = t_1$

$$\mathbf{P} = X_1\mathbf{C}_1 + X_2\mathbf{C}_2 \overset{(2),(6)}{=} \ldots\ldots\ldots\ldots\ldots\ldots\ldots = x_1\mathbf{i}_1 + (x_2 - 3)\,\mathbf{i}_2. \tag{9}$$

Kui $(x_1, x_2) = (4;9)$, siis

$$\mathbf{P} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots = 4\mathbf{i}_1 + 6\mathbf{i}_2 \tag{10}$$

esitab materiaalse punkti $(X_1, X_2) = (4;2)$ kohavektori $\mathbf{P}$ Euleri koordinaatides x_k hetkel $t = t_1$.

- Vektorite teisedamine.
  Eeldusel, et liikumissedus on esitatud lineaarse koordinaatteisendusena, kehtivad seosed:

$$x_k = x_{k,K} X_K, \quad X_K = X_{K,k} x_k, \quad \text{ehk} \quad [\mathbf{x}] = [x_{k,K}] \cdot [\mathbf{X}], \quad [\mathbf{X}] = [X_{K,k}] \cdot [\mathbf{x}], \tag{11}$$

kus x_k ja X_K on vastavalt vektorite $\mathbf{x}$ ja $\mathbf{X}$ komponendid. Siinjuures on $\mathbf{X}$ nn. deformeerumata vektor ehk LK-s esitatud vektor ja $\mathbf{x}$ on nn. deformeerunud vektor, mis on vaadeldaval juhul on rakendatud ruumipunkti $(x_1, x_2) = (0,3)$ ja on esitatud EK-s.

$$\begin{cases} \mathbf{X} = \begin{bmatrix} 4 \\ 0 \end{bmatrix} \quad \rightarrow \quad \mathbf{x} = \begin{bmatrix} 1 & 0 \\ 0,5 & 2 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}, \\ \mathbf{X} = \begin{bmatrix} 0 \\ 2 \end{bmatrix} \quad \rightarrow \quad \mathbf{x} = \ldots = \begin{bmatrix} 0 \\ 4 \end{bmatrix}, \quad \mathbf{X} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} \quad \rightarrow \quad \mathbf{x} = \ldots = \begin{bmatrix} 4 \\ 6 \end{bmatrix}, \end{cases} \tag{12}$$

$$\begin{cases} \mathbf{x} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} \quad \rightarrow \quad \mathbf{X} = \begin{bmatrix} 1 & 0 \\ -0,25 & 0,5 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}, \\ \mathbf{x} = \begin{bmatrix} 0 \\ 4 \end{bmatrix} \quad \rightarrow \quad \mathbf{X} = \ldots = \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} 4 \\ 6 \end{bmatrix} \quad \rightarrow \quad \mathbf{X} = \ldots = \begin{bmatrix} 4 \\ 2 \end{bmatrix}. \end{cases} \tag{13}$$

- Cauchy ja Greeni deformatsioonitensorid:

$$c_{kl} = \mathbf{c}_k \cdot \mathbf{c}_l, \qquad C_{KL} = \mathbf{C}_K \cdot \mathbf{C}_L. \tag{14}$$

$$c_{11} = \mathbf{c}_1 \cdot \mathbf{c}_1 = \frac{17}{16}, \qquad c_{12} = c_{21} = \mathbf{c}_1 \cdot \mathbf{c}_2 = -\frac{1}{8}, \qquad c_{22} = \mathbf{c}_2 \cdot \mathbf{c}_2 = \frac{1}{4}, \tag{15}$$

$$C_{11} = \mathbf{C}_1 \cdot \mathbf{C}_1 = \frac{5}{4}, \qquad C_{12} = C_{21} = \mathbf{C}_1 \cdot \mathbf{C}_2 = 1, \qquad C_{22} = \mathbf{C}_2 \cdot \mathbf{C}_2 = 4. \tag{16}$$

$$[c_{kl}] = \frac{1}{16} \begin{bmatrix} 17 & -2 \\ -2 & 4 \end{bmatrix}, \qquad [C_{KL}] = \frac{1}{4} \begin{bmatrix} 5 & 4 \\ 4 & 16 \end{bmatrix}, \tag{17}$$

- Kontroll

$$c_{kl} = X_{K,k} X_{K,l}, \qquad C_{KL} = x_{k,K} x_{k,L}, \tag{18}$$

Maatriksitena

$$[c_{kl}] = [X_{K,k}]^T \cdot [X_{K,l}] = \ldots = \begin{bmatrix} \cdots & \cdots \\ \cdots & \cdots \end{bmatrix} \tag{19}$$

$$[C_{KL}] = [x_{k,K}]^T \cdot [x_{k,L}] = \ldots = \begin{bmatrix} \cdots & \cdots \\ \cdots & \cdots \end{bmatrix} \tag{20}$$

- Fingeri ja Piola deformatsioonitensorid:

$$\overset{-1}{c}_{kl} = x_{k,K} x_{l,K}, \qquad \overset{-1}{C}_{KL} = X_{K,k} X_{L,k}, \tag{21}$$

Maatriksitena

$$\left[\overset{-1}{c}_{kl}\right] = [x_{k,K}] \cdot [x_{l,K}]^T = \ldots = \frac{1}{4}\begin{bmatrix} 4 & 2 \\ 2 & 17 \end{bmatrix}, \tag{22}$$

$$\left[\overset{-1}{C}_{KL}\right] = [X_{K,k}] \cdot [X_{L,k}]^T = \ldots = \frac{1}{16}\begin{bmatrix} 16 & -4 \\ -4 & 5 \end{bmatrix}. \tag{23}$$

- Kontroll

$$c_{kl} \overset{-1}{c}_{lm} = \delta_{km}, \qquad C_{KL} \overset{-1}{C}_{LM} = \delta_{KM}. \tag{24}$$

Maatriksitena

$$[c_{kl}] \cdot \left[\overset{-1}{c}_{lm}\right] = \ldots = \quad \begin{bmatrix} \ldots & \ldots \\ \ldots & \ldots \end{bmatrix} \tag{25}$$

$$[C_{KL}] \cdot \left[\overset{-1}{C}_{LM}\right] = \ldots = \quad \begin{bmatrix} \ldots & \ldots \\ \ldots & \ldots \end{bmatrix} \tag{26}$$

- Euleri ja Lagrange'i deformatsioonitensorid:

$$2e_{kl} = \delta_{kl} - c_{kl}, \qquad 2E_{KL} = C_{KL} - \delta_{KL}. \tag{27}$$

- Maatriksitena

$$[e_{kl}] = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots = \frac{1}{32}\begin{bmatrix} -1 & 2 \\ 2 & 12 \end{bmatrix} \tag{28}$$

$$[E_{KL}] = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots = \frac{1}{8}\begin{bmatrix} 1 & 4 \\ 4 & 12 \end{bmatrix} \tag{29}$$

- Siirdevektori komponendid EK-s ja LK-s

$$\text{Siirdevektor:} \qquad \mathbf{u} = \mathbf{p} - \mathbf{P}, \tag{30}$$

$$u_k = x_k - X_K(\mathbf{x}), \qquad U_K = x_k(\mathbf{X}) - X_K, \qquad k = K. \tag{31}$$

$$\begin{cases} u_1 = x_1 - X_1 \overset{(2)}{=} \ldots\ldots\ldots\ldots = 0 \\ \\ u_2 = x_2 - X_2 \overset{(2)}{=} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots = 0,25x_1 + 0,5x_2 + 1,5 \end{cases} \tag{32}$$

$$\text{Kontroll:} \qquad (x_1, x_2) = (4, 9) \rightarrow u_2 = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots = 7 \tag{33}$$

$$\begin{cases} U_1 = x_1 - X_1 \overset{(1)}{=} \ldots\ldots\ldots\ldots = 0 \\ \\ U_2 = x_2 - X_2 \overset{(1)}{=} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots = 0,5X_1 + X_2 + 3 \end{cases} \tag{34}$$

$$\text{Kontroll:} \qquad (X_1, X_2) = (4, 2) \rightarrow U_2 = \ldots\ldots\ldots\ldots\ldots\ldots\ldots = 7 \tag{35}$$

- Lisakontroll

$$\begin{cases} U_2 = u_2(\mathbf{X}) = \ldots \\ \\ u_2 = U_2(\mathbf{x}) = \ldots \end{cases} \tag{36}$$