2.11.6 Elementaarpinna muutumise kiirus

Lõpmata väiked vektorid $d\mathbf{X}_{(1)} = dX^L_{(1)}\mathbf{G}_L$ ja $d\mathbf{X}_{(2)} = dX^M_{(2)}\mathbf{G}_M$ † LK-s (või $d\mathbf{x}_{(1)} = dx^l_{(1)}\mathbf{g}_l$ ja $d\mathbf{x}_{(2)} = dx^m_{(2)}\mathbf{g}_m$ EK-s) määravad läbi vektorkorrutise elamentaarpinnad

$$d\mathbf{A} = d\mathbf{X}_{(1)} \times d\mathbf{X}_{(2)} \;\text{ ja }\; d\mathbf{a} = d\mathbf{x}_{(1)} \times d\mathbf{x}_{(2)}, \tag{2.335}$$

mida saab omakorda avaldada läbi kontravariantse baasi kujul

$$d\mathbf{A} = dA_K\mathbf{G}^K \;\text{ ja }\; d\mathbf{a} = da_k\mathbf{g}^k, \tag{2.336}$$

$$\begin{aligned} dA_K &= d\mathbf{A}\cdot\mathbf{G}_K = \left(dX^L_{(1)}\mathbf{G}_L \times dX^M_{(2)}\mathbf{G}_M\right)\cdot\mathbf{G}_K = \\ &= dX^L_{(1)}dX^M_{(2)}\underbrace{(\mathbf{G}_L\times\mathbf{G}_M)}_{\epsilon_{LMN}\mathbf{G}^N}\cdot\mathbf{G}_K = \ldots = \\ &= \epsilon_{LMN}\delta^N{}_K dX^L_{(1)}dX^M_{(2)} = \epsilon_{LMK}dX^L_{(1)}dX^M_{(2)}; \\ da_k &= \ldots = \epsilon_{lmk}dx^l_{(1)}dx^m_{(2)}. \end{aligned} \tag{2.337}$$

Elemantaarpinna vektorite $d\mathbf{A}$ ja $d\mathbf{a}$ kovariantsed komponendid saab omakorda avaldada *bivektorite* dA^{LM} ja da^{lm} kujul: ‡

$$\begin{aligned} dA^{LM} &= \epsilon^{KLM}dA_K, \quad & dA_K &= \frac{1}{2}\epsilon_{KLM}dA^{LM}, \\ da^{lm} &= \epsilon^{klm}da_k, & da_k &= \frac{1}{2}\epsilon_{klm}da^{lm}. \end{aligned} \tag{2.338}$$

Kuna $\epsilon_{LMK} = \epsilon_{KLM}$ ja $\epsilon_{lmk} = \epsilon_{klm}$, siis (2.337) ja (2.338) põhjal

$$dA^{LM} = 2dX^L_{(1)}dX^M_{(2)} \quad\text{ja}\quad da^{lm} = 2dx^l_{(1)}dx^m_{(2)}. \tag{2.339}$$

Deformatsiooni käigus $dX^K \to dx^k = x^k{}_{,K}dX^K$. Seega

$$da^{kl} = 2dx^k_{(1)}dx^l_{(2)} = 2x^k{}_{,K}dX^K_{(1)}x^l{}_{,L}dX^L_{(2)} \overset{(2.339)}{=} x^k{}_{,K}x^l{}_{,L}dA^{KL}. \tag{2.340}$$

Rakendades (2.338) saame minna tagasi elementaarpinna vektorite komponentidele

$$da_k = \frac{1}{2}\epsilon_{klm}x^l{}_{,L}x^m{}_{,M}dA^{LM} = \frac{1}{2}\epsilon_{klm}\epsilon^{KLM}x^l{}_{,L}x^m{}_{,M}dA_K. \tag{2.341}$$

Kuna saab näidata, et (vt. Eringeni ja Narasimhani õpikud)

$$\epsilon^{KLM}\epsilon_{klm}x^l{}_{,L}x^m{}_{,M} = 2JX^K{}_{,k} \;\text{ ja }\; J = \sqrt{\frac{g}{G}}j, \tag{2.342}$$

siis

$$da_k = JX^K{}_{,k}dA_K \;\text{ ja }\; dA_K = J^{-1}x^k{}_{,K}da_k. \tag{2.343}$$

Seega, materiaalne tuletis pinnaelemendist

$$\frac{D(da_k)}{Dt} = \frac{D(JX^K{}_{,k}dA_K)}{Dt} = \left[\frac{DJ}{Dt}X^K{}_{,k} + J\frac{D(X^K{}_{,k})}{Dt}\right]dA_K. \tag{2.344}$$

Materiaalne tuletis $\frac{DJ}{Dt} = Jv^m{}_{;m}$ kuid $\frac{D(X^K{}_{,k})}{Dt}$ määramiseks leiame materiaalse tuletise avaldisest $x^k{}_{,K}X^K{}_{,l} = \delta^k{}_l$: √

$$\begin{aligned} &x^k{}_{,K}\frac{D(X^K{}_{,l})}{Dt} = -\frac{D(x^k{}_{,K})}{Dt}X^K{}_{,l}\,\Big|\cdot X^L{}_{,k} \quad\Rightarrow \\ &\frac{D(X^K{}_{,l})}{Dt}x^k{}_{,K}X^L{}_{,k} = -\frac{D(x^k{}_{,K})}{Dt}X^K{}_{,l}X^L{}_{,k} \\ &\frac{D(X^K{}_{,l})}{Dt}\delta_K{}^L = -v^k{}_{;m}x^m{}_{,K}X^K{}_{,l}X^L{}_{,k} \\ &\frac{D(X^L{}_{,l})}{Dt} = -v^k{}_{;m}\delta^m{}_lX^L{}_{,k} = -v^k{}_{;l}X^L{}_{,k}. \end{aligned}$$

Tähistades indeksid ümber, saame kokkuvõttes

$$\frac{D(X^K{}_{,k})}{Dt} = -X^K{}_{,l}v^l{}_{;k} \tag{2.345}$$

Avaldisest (2.344) saame nüüd

$$\begin{aligned} \frac{D(da_k)}{Dt} &= \left[Jv^m{}_{;m}X^K{}_{,k} - JX^K{}_{,l}v^l{}_{;k}\right]dA_K = \\ &= v^m{}_{;m}\underbrace{JX^K{}_{,k}dA_K}_{=da_k} - v^l{}_{;k}\underbrace{JX^K{}_{,l}dA_K}_{=da_l}. \end{aligned}$$

Seega avaldub *pinnaelemendi materiaalne tuletis (muutumise kiirus)* kujul

$$\frac{D(da_k)}{Dt} = v^m{}_{;m}da_k - v^m{}_{;k}da_m. \tag{2.346}$$