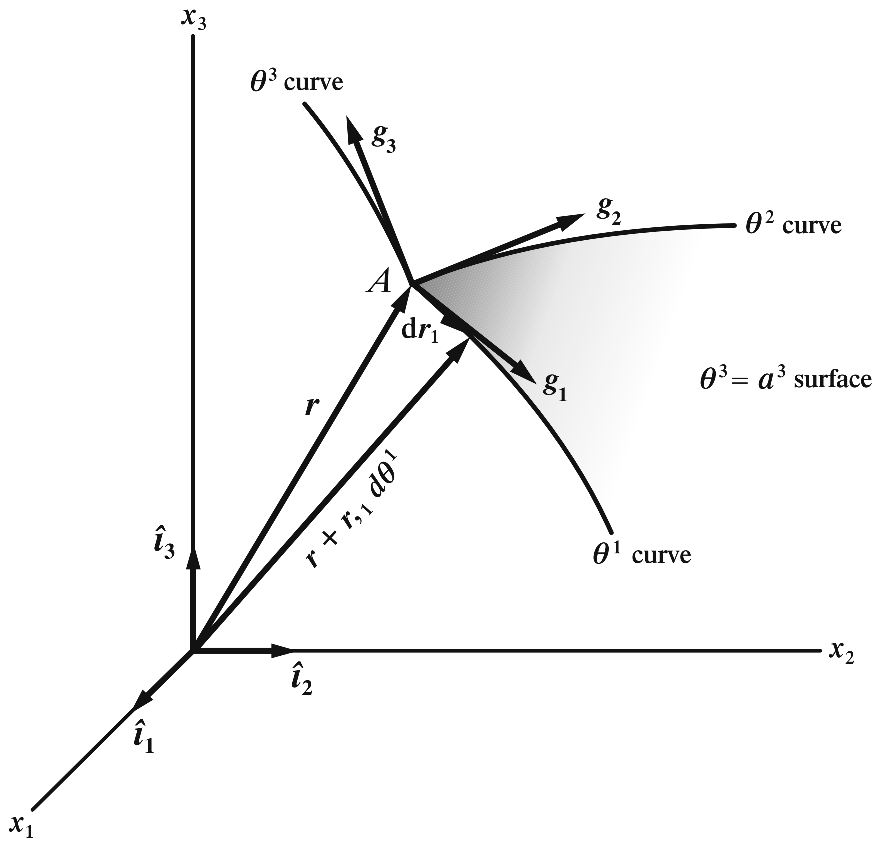

Figure 2.1 Curvilinear coordinate lines and surfaces

2.2 Curvilinear Coordinates, Base Vectors, and Metric Tensor

The position of point A in Figure 2.1 is given by the vector

$$\boldsymbol{r} = x_i \hat{\boldsymbol{\imath}}_i, \tag{2.1}$$

where x_i are rectangular coordinates and $\hat{\boldsymbol{\imath}}_i$ are unit vectors as shown.

Let θ^i denote arbitrary curvilinear coordinates. We assume the existence of equations which express the variables x_i in terms of θ^i and vice versa; that is,

$$x_i = x_i(\theta^1, \theta^2, \theta^3), \qquad \theta^i = \theta^i(x_1, x_2, x_3). \qquad (2.2), (2.3)$$

Also, we assume that these have derivatives of any order required in the subsequent analysis.

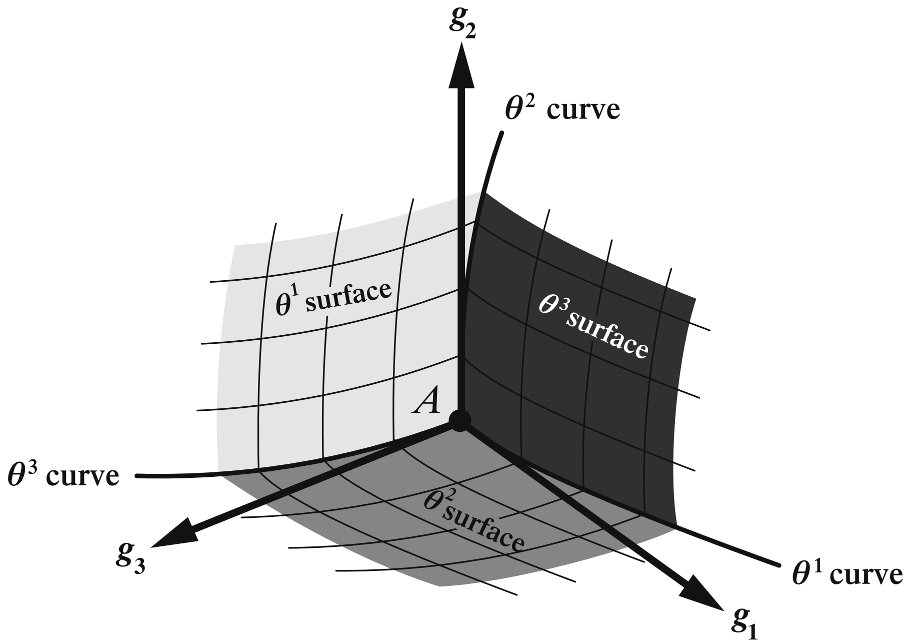

Figure 2.2 Network of coordinate curves and surfaces

Suppose that $x_i = x_i(a^1, a^2, a^3)$ are the rectangular coordinates of point A in Figure 2.1. Then $x_i(\theta^1, a^2, a^3)$ are the parametric equations of a curve through A; it is the θ^1 curve of Figure 2.1. Likewise, the θ^2 and θ^3 curves through point A correspond to fixed values of the other two variables. The equations $x_i = x_i(\theta^1, \theta^2, a^3)$ are parametric equations of a surface, the θ^3 surface shown shaded in Figure 2.1. Similarly, the θ^1 and θ^2 surfaces correspond to $\theta^1 = a^1$ and $\theta^2 = a^2$. At each point of the space there is a network of curves and surfaces (see Figure 2.2) corresponding to the transformation of equations (2.2) and (2.3).

By means of (2.2) and (2.3), the position vector $\boldsymbol{r}$ can be expressed in alternative forms:

$$\boldsymbol{r} = \boldsymbol{r}(x_1, x_2, x_3) = \boldsymbol{r}(\theta^1, \theta^2, \theta^3). \tag{2.4}$$

A differential change $d\theta^i$ is accompanied by a change $d\boldsymbol{r}_i$ tangent to the θ^i line; a change in θ^1 only causes the increment $d\boldsymbol{r}_1$ illustrated in Figure 2.1. It follows that the vector

$$\boldsymbol{g}_i \equiv \frac{\partial \boldsymbol{r}}{\partial \theta^i} = \frac{\partial x_j}{\partial \theta^i} \hat{\boldsymbol{\imath}}_j \tag{2.5}$$

is tangent to the θ^i curve. The tangent vector $\boldsymbol{g}_i$ is sometimes called a *base vector*.

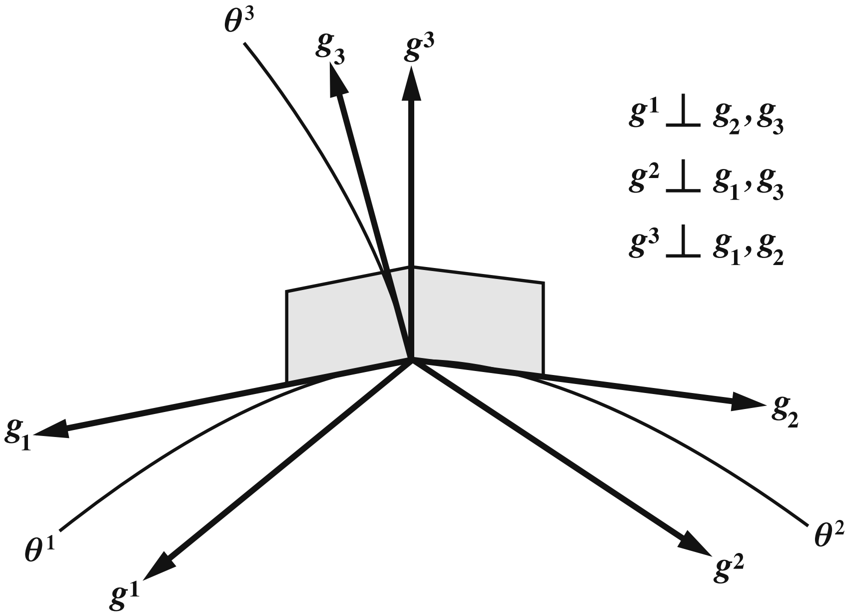

Figure 2.3 Tangent and normal base vectors

Let us define another triad of vectors $\boldsymbol{g}^i$ such that

$$\boldsymbol{g}^i \cdot \boldsymbol{g}_j \equiv \delta^i_j. \qquad (2.6)$$

The vector $\boldsymbol{g}^i$ is often called a *reciprocal* base vector. Since the vectors $\boldsymbol{g}_i$ are *tangent* to the *coordinate curves*, equation (2.6) means that the vectors $\boldsymbol{g}^i$ are *normal* to the *coordinate surfaces*. This is illustrated in Figure 2.3. We will call the triad $\boldsymbol{g}_i$ *tangent base vectors* and the triad $\boldsymbol{g}^i$ *normal base vectors*. In general they are not unit vectors.

The triad $\boldsymbol{g}^i$ can be expressed as a linear combination of the triad $\boldsymbol{g}_i$, and vice versa. To this end we define coefficients g^{ij} and g_{ij} such that

$$\boldsymbol{g}_i \equiv g_{ij}\boldsymbol{g}^j, \qquad \boldsymbol{g}^i \equiv g^{ij}\boldsymbol{g}_j. \qquad (2.7), (2.8)$$

From equations (2.6) to (2.8), it follows that

$$g_{ij} = g_{ji} = \boldsymbol{g}_i \cdot \boldsymbol{g}_j, \qquad g^{ij} = g^{ji} = \boldsymbol{g}^i \cdot \boldsymbol{g}^j, \qquad (2.9), (2.10)$$

$$g^{im}g_{jm} = \delta^i_j. \qquad (2.11)$$

The linear equations (2.11) can be solved to express g^{ij} in terms of g_{ij}, as follows:

$$g^{ij} = \frac{\text{cofactor of element } g_{ij} \text{ in matrix } [g_{ij}]}{|g_{ij}|}. \qquad (2.12)$$