ITT8040 Cellular Automata
Solutions to Assignment 2

Exercise 1

There are at least two ways to solve this exercise. One is as follows: consider a sequence $\{i_k\}_{k\geq 1}$ such that $c = \lim_{k\to\infty} c_{i_k}$ exists. Then, consider an increasing sequence $\{k_n\}_{n\geq 1}$ such that $e = \lim_{n\to\infty} e_{i_{k_n}}$ exists. For $i_n = i_{k_n}$ we have $\lim_{n\to\infty} c_{i_n} = c$ and $\lim_{n\to\infty} e_{i_n} = e$.

Another way goes as follows. For $i \geq 1$ define $\eta_i : \mathbb{Z}^d \to S \times S$ as $\eta_i(\vec{n}) = (c_i(\vec{n}), e_i(\vec{n}))$ for every $\vec{n} \in \mathbb{Z}^d$. By Proposition 4, there exists an increasing sequence $\{i_n\}_{n\geq 1}$ such that $\eta = \lim_{n\to\infty} \eta_{i_n}$ exists. Define then $c, e : \mathbb{Z}^d \to S$ by $\eta(\vec{n}) = (c(\vec{n}), e(\vec{n}))$ for every $\vec{n} \in \mathbb{Z}^d$: it can be seen that $\lim_{n\to\infty} c_{i_n} = c$ and $\lim_{n\to\infty} e_{i_n} = e$.

Exercise 2

The pattern 10101 can be seen to be an orphan, by the following argument:

Let f be the local update rule. We have $f(010) = f(100) = f(101) = 1$ and $f(000) = f(001) = f(011) = f(110) = f(111) = 0$. So, the middle 1 can only come from 010, 100, or 101. However, it cannot come from 100 or 101: otherwise, the first 0 in 10101 would need to come from 110, which contradicts such 0 having a 1 on its left ad $f(011) = f(111) = 0$. But the middle 1 cannot come from 010 either, as $f(100) = f(101) = 1$, so the digit on its right should be 1 instead of 0.

Exercise 4

As G commutes with translations, being able to evaluate $G(c)$ in $\vec{n}$ for arbitrary $c \in S^{\mathbb{Z}^d}$ and $\vec{n} \in \mathbb{Z}^d$, is the same as being able to evaluate $G(c)$ in $\vec{0}$ for arbitrary $c \in S^{\mathbb{Z}^d}$:

$$G(c)(\vec{n}) = (\tau_{\vec{n}} \circ G)(c)(\vec{0}) = G(\tau_{\vec{n}} \circ c)(\vec{0}) \,.$$

Let us follow the hint and try to adapt the proof of Proposition 7. Let us suppose, for the sake of contradiction, that G is not the global function of a cellular automaton: this is the same as saying that there is no neighborhood $N = \{\vec{n}_1, \ldots, \vec{n}_m\}$ and m-ary function $f : S^m \to S$ such that, for every configuration

$c \in S^{\mathbb{Z}^d}$, the value $G(c)(\vec{0})$ coincides with the value $f(c(\vec{n}_1), \ldots, c(\vec{n}_m))$. This, in turn, is the same as saying that, for every $i \geq 1$, there exist two configurations $c_i, e_i : \mathbb{Z}^d \to S$ such that c_i and e_i coincide on M_i, the Moore neighborhood of radius i, but $G(c_i)(\vec{0}) \neq G(e_i)(\vec{0})$.

Now, according to the solution to Exercise 1, there exist an increasing sequence $\{i_k\}_{k \geq 1}$ such that $c = \lim_{k \to \infty} c_{i_k}$ and $e = \lim_{k \to \infty} e_{i_k}$ both exist. Then we must have $c = e$: in fact, for every $\vec{n} \in \mathbb{Z}^d$ there exists $j \geq 1$ such that $\vec{n} \in M_i$ for every $i \geq j$, so also $c_{i_k}(\vec{n}) = e_{i_k}(\vec{n})$ for every $k \geq 1$ large enough, and consequently $c(\vec{n}) = e(\vec{n})$ by definition of c and e. By continuity, $\lim_{k \to \infty} G(c_{i_k}) = G(c) = G(e) = \lim_{k \to \infty} G(e_{i_k})$: which is impossible, because $G(c_{i_k})$ and $G(e_{i_k})$ always differ in $\vec{0}$.